Log-Normal continuous cascades: aggregation properties and estimation. Application to financial time-series

E. Bacry, A. Kozhemyak and J.F. Muzy

Abstract

Log-normal continuous random cascades form a class of multifractal processes that has already been successfully used in various fields. Several statistical issues related to this model are studied. We first make a quick but extensive review of their main properties and show that most of these properties can be analytically studied. We then develop an approximation theory of these processes in the limit of *small intermittency* $\lambda^2 \ll 1$, i.e., when the degree of multifractality is small. This allows us to prove that the probability distributions associated with these processes possess some very simple aggregation properties accross time scales. Such a control of the process properties at different time scales, allows us to address the problem of parameter estimation. We show that one has to distinguish two different asymptotic regimes: the first one, referred to as the "low frequency regime", corresponds to taking a sample whose overall size increases whereas the second one, referred to as the "high frequency regime", corresponds to sampling the process at an increasing sampling rate. We show that, the first regime leads to convergent estimators whereas, in the high frequency regime, the situation is much more intricate : only the intermittency coefficient λ^2 can be estimated using a consistent estimator. However, we show that, in practical situations, one can detect the nature of the asymptotic regime (low frequency versus high frequency) and consequently decide whether the estimations of the other parameters are reliable or not. We finally illustrate how both our results on parameter estimation and on aggregation properties, allow one to successfully use these models for modelization and prediction of financial time series.

Index Terms

Scaling Phenomena, Self-similarity, Multifractal scaling, Intermittency, Parameter estimation, Financial time series.

I. Introduction

DAta displaying multi-scaling behavior are observed in various fields of applied and fundamental sciences: the velocity field of fully developed turbulent flows [1], financial time-series [2], [3], the telecommunication traffic load in high speed networks [4], medical time-series [5], [6], geological shapes [7] are only few of numerous examples. The paradigm of multifractal processes are multiplicative cascades originally introduced by the russian school [8] for modelling the energy cascade in fully developed turbulence and further studied by Mandelbrot [9], [10]. Very recently, continuous versions of these processes have been defined: they share exact multifractal scaling with discrete cascades but they display continuous scaling and possess stationary increments [11]–[14]. Despite the huge number of mathematical studies devoted to discrete (e.g., [15]–[19]) or continuous random cascades [12], [14], [20]–[22], only very few works considered standard statistical problems associated with these processes (see however [23]–[27]). Our goal in this paper is to adress several statistical issues related to multifractal processes.

The self-similarity of a process $X(t)$ [1] can be characterized by the power-law behavior of the q-order moments of its increments as functions of the scale

$$\forall q \in \mathbf{R}, \quad \mathbb{E}\left[|X(t)|^q\right] \simeq C_q t^{\zeta_X(q)}, \; \forall t \leq T, \tag{1}$$

where T is referred to as the "integral scale", it actually corresponds to a decorrelation scale. In the case the so-obtained "scaling exponents" $\zeta_X(q)$ are not depending linearly on q but is a concave function of q, the process is said to be a multifractal process. Since scaling of moments of different orders do not behave homogeously as the time scale is changed, the probability distribution of the increments of a multifractal process strongly depends on the scale of the increments. For random cascade models, one can show that the scaling exponent, as a function of q, corresponds to the cumulant generating function of a log-infinitely divisible law W,

$$\zeta_X(q) = \ln \mathbb{E}\left[W^q\right]. \tag{2}$$

Given the infinitely divisible law W, the continuous cascade process is entirely defined as soon as its variance σ^2 (i.e., a simple multiplicative factor) is fixed, as well as its integral scale T (i.e., the "size" of the cascade, or the decorrelation scale). In this paper, we will exclusively focus on continuous cascades with log-normal scaling exponents.

E.Bacry and A. Kozhemyak are in the Centre de Mathématiques Appliquées, Ecole Polytechnique, 91128 Palaiseau, France.

J.F. Muzy is in the Laboratoire Sciences Pour l'Environnement, CNRS, Université de Corse, UMR 6134, 20250 Corté, France.

[1] We consider exclusively processes $X(t)$ with stationary increments and such that $X(0) = 0$. In that respect, $\forall t'$, $X(t' + t) - X(t')$, has the same distribution than $X(t)$.

Among the whole class of log infinitely divisible cascade models, log normal cascades have the advantage of being fully determined by a single parameter λ^2, corresponding basically to the variance of $\ln W$. Thus, as shown by Eq. (2), this parameter λ^2 rules the non linearity of the scaling exponents (i.e., the "degree" of multifractality of the process). It is referred to as the the intermittency coefficient. Moreover, log-normal continuous cascades have a very simple alternative construction and, as we will see in the next section of this paper, most of their properties can be expressed under closed formulae. Though, they correspond to a particular log-infinitely divisible law, they are rich enough to raise challenging statistical questions such as estimator convergence in the context of long-memory correlation. Our approach relies upon an approximation that allows us to precisely control the aggregation properties of the process. In fact, we show that in the small intermittency limit ($\lambda^2 \ll 1$), a log-normal continuous cascade has increments that are "close" to be log-normal, at each scale, in a sense that will be precisely defined in the sequel. This approximation framework allows us to develop a method to estimate the process parameters. In that context, we are lead to introduce two distinct situations for asymptotic regimes: the first one, referred to as the "low frequency regime", corresponds to the classical notion of infinite observation scale at a fixed sampling rate while the second, referred to as the "high frequency regime", corresponds to sampling the process over a fixed observation scale at an increasing sampling rate. More precisely, if τ is the sampling rate and L is the observation scale, the observed samples corresponds to the values

$$\{X(n\tau)\}_{n \in [0,N[}, \quad \text{where } N = \frac{L}{\tau}. \tag{3}$$

Both asymptotic regimes corresponds to $N \to +\infty$, however, whereas the low-frequency regime corresponds to τ fixed and $L \to +\infty$, the high-frequency regime corresponds to L fixed and $\tau \to 0$. From an experimental point of view, the first regime corresponds to the case where $\frac{L}{T} \gg 1$ and $\frac{T}{\tau} \simeq 1$ whereas the second regime corresponds to the case where $\frac{L}{T} \simeq 1$ and $\frac{T}{\tau} \gg 1$. We show that the properties of the parameter estimators are fundamentally different depending on the nature of the asymptotic regime (and allow, by the way, to test what the effective nature of the regime is). In the last section, we apply all these results on parameter estimation for the calibration of a multifractal model to account for volatility dynamics in financial time series. Moreover, making extensive use of the aggregation properties, we show that the so-obtained model provides highly performant methods to forecast risk.

The paper is organized as follows: in section II we recall the main definition of log-normal cascades at the heart of this study and we state its main properties. We study both the case of the Multifractal Random Measure (MRM), a non-decreasing process, and the case of the Multifractal Random Walk (MRW), a Brownian motion subordinated by the MRM. The aggregation properties of the model are discussed extensively in section III where we develop our small intermittency approximation theory. In this section, we first introduce a Gaussian process : the "renormalized magnitude" $\Omega(t)$ that will be involved in all the following approximations. We show that, in the case $\lambda^2 \ll 1$, in some sense to be defined, the variations of the MRM or of the MRW are closely related to those of $\Omega(t)$. Whereas subsection III-C states a convergence theorem of the logarithm of the MRM towards Ω in the limit $\lambda^2 \to 0$, the other subsections establish different MRM/MRW moment approximation theorems (when $\lambda^2 \ll 1$) as functions of $\Omega(t)$ moments. In section IV, we show how these approximations can be used to calibrate the model. The estimation issues are discussed within both low-frequency and high-frequency asymptotic regimes. We first show, in Section IV-A, that, in the low-frequency regime, the "Generalized Moments Method" (GMM) leads to convergent estimators whereas, as shown in Section IV-B, in the high-frequency regime, the situation is more intricate. Indeed, in this regime, the integral scale T is shown to be a "fake" parameter and σ^2 cannot be estimated. However, in experimental situation, the order of the GMM estimation of the fake parameter T is proved to give some hints about the nature of the asymptotic regime and consequently about the reliability of the estimation of T and σ^2. In Section IV-C, we exhibit a GMM type estimator of λ^2 that is proved to be consistent. Numerical experiments illustrate all the estimation results. In Section V we apply our results concerning parameter estimation tp the calibration of a model for asset price fluctuations in financial markets. We then show how the aggregation properties can be successfully used to perform conditional risk forecasting. Conclusions and prospects are provided in section VI while Appendices contain additional technical results.

II. Definitions and main properties of log-normal continuous cascade models

In this section we will first focus on the definition of Multifractal Random Measures (MRM) originally introduced in [14]. We will denote $M[t_0, t_1]$ the measure of the interval $[t_0, t_1]$ and $M(t)$ the non decreasing process $M(t) = M[0, t]$. We propose two different approaches to define log-normal MRM. The first one relies upon the construction of some temporal Gaussian process which covariance mimics the observed "ultrametric" covariance of discrete Mandelbrot cascades while the second one involves random measures in a 2D half-plane. The direct construction has the advantage of being simpler and easy to implement while the second construction can be easliy extended to other infinitely divisible laws than the Gaussian law. Multifractal Random Walks (MRW) can be easily obtained from MRM by compounding a self-similar stochastic process with $M(t)$.

A. Direct definition

Let the measure $M_{l,T}(dt)$ be defined by

$$M_{l,T}(dt) = e^{2\omega_{l,T}(t)}dt . \tag{4}$$

in the sense that for all Lebesgue measurable set I one has, $M_{l,T}(I) = \int_I e^{2\omega_{l,T}(t)}dt$. The process $\omega_{l,T}(t)$ is Gaussian and stationary and is defined by its mean and covariance function :

$$\mathbb{E}\left[\omega_{l,T}(t)\right] = -\lambda^2 \left(\ln\left(\frac{T}{l}\right) + 1\right) \tag{5}$$

and

$$\rho_{l,T}(\tau) = \mathbb{C}\mathrm{ov}\left[\omega_{l,T}(t), \omega_{l,T}(t+\tau)\right] = \begin{cases} \lambda^2\left(\ln\left(\frac{T}{l}\right) + 1 - \frac{\tau}{l}\right), & \text{if } 0 \le \tau < l, \\ \lambda^2 \ln\left(\frac{T}{\tau}\right), & \text{if } l \le \tau < T, \\ 0, & \text{if } T \le \tau < +\infty, \end{cases} \tag{6}$$

where the parameters T is the integral scale and λ^2 is the intermittency coefficient. Note that the fact that expression (6) represents of definite positive function is proven in the next section. Using Kahane Chaos theory [28], on can prove that the weak limit

$$M_T(dt) = \lim_{l \to 0^+} M_{l,T}(dt), \tag{7}$$

exists and is non trivial as long as $\lambda^2 < 1/2$.

Let us remark that the process $\{\omega_{l,T}(t)\}_t$ can be represented as a stochastic integral of a Kernel against the Wiener White noise $dB(t)$, like Mandelbrot-Van Ness fractional Brownian motion representation:

$$\omega_{l,T}(t) = -\mathbb{E}\left[\omega_{l,T}(t)\right] + \int_{-\infty}^t K_{l,T}(t-u)dB(u), \tag{8}$$

where the kernel $K_{l,T}$ (that can be chosen to be causal) satisfies the convolution equation:

$$K_{l,T} * K_{l,T}(t) = \mathbb{C}\mathrm{ov}\left[\omega_l(0), \omega_l(t)\right]. \tag{9}$$

A simple Fourier transform of this equation together with expression (6) allows one to show that the process $\omega_{l,T}$ can be seen as a kind of fractional Brownian motion in the marginal limit $H \to 0$. Indeed, the kernel $K_{l,T}(t)$ in previous equation behaves, in the range $l \ll t \ll T$, like:

$$K_l(t) \sim \frac{K_0}{\sqrt{t}}. \tag{10}$$

For this reason, as emphasized in ref. [13], the MRM measure can be loosely defined as the exponential of an $1/f$ noise.

B. Alternative definition : continuous cascades

The previous construction is hard to extend to other laws than the Gaussian law. A more general construction that allows one to build continuous cascades with arbitrary log-infinitely divisible statistics has been proposed by Bacry and Muzy [14]. It amounts in building the process $\omega_{l,T}(t)$ from a 2d representation.

We distribute a non centered gaussian white noise $\mathcal{P}$ of variance λ^2 on the half plane $\{(t,l); t \in \mathbb{R}, l \in \mathbb{R}^{+*}\}$ using the density measure $\mu(dt, dl) = l^{-2}dtdl$. Consequently, for any measurable set $\mathcal{A}$ of the half-plane, $\mathcal{P}(\mathcal{A})$ is a gaussian random variable whose Laplace transform is of the form

$$\mathbb{E}\left[e^{q\mathcal{P}(\mathcal{A})}\right] = e^{\psi(q)\mu(\mathcal{A})}. \tag{11}$$

If we choose the mean of the white noise $\mathcal{P}$ such that for any $\mathcal{A} \in \mathcal{S}^+$, one has $\mathbb{E}[\mathcal{P}(A)] = -\mathbb{V}\mathrm{ar}\left[\mathcal{P}(\mathcal{A})\right] = -2\lambda^2\mu(\mathcal{A})$, then

$$\psi(q) = 2\lambda^2 q^2 - 2\lambda^2 q. \tag{12}$$

Then if we define, for all l and T such that $0 < l < T$, the cone-like domain: $\mathcal{A}_{l,T}(t)$ as:

$$\mathcal{A}_{l,T}(t) = \left\{(t', l'); l \le l', |t' - t| \le \frac{1}{2}\min(l', T)\right\}, \tag{13}$$

the gaussian process $\omega_{l,T}(t)$ defined by Eqs (5) and (6) has the following representation (see fig. 1)

$$\omega_{l,T}(t) = \frac{1}{2}\mathcal{P}(\mathcal{A}_{l,T}(t)), \tag{14}$$
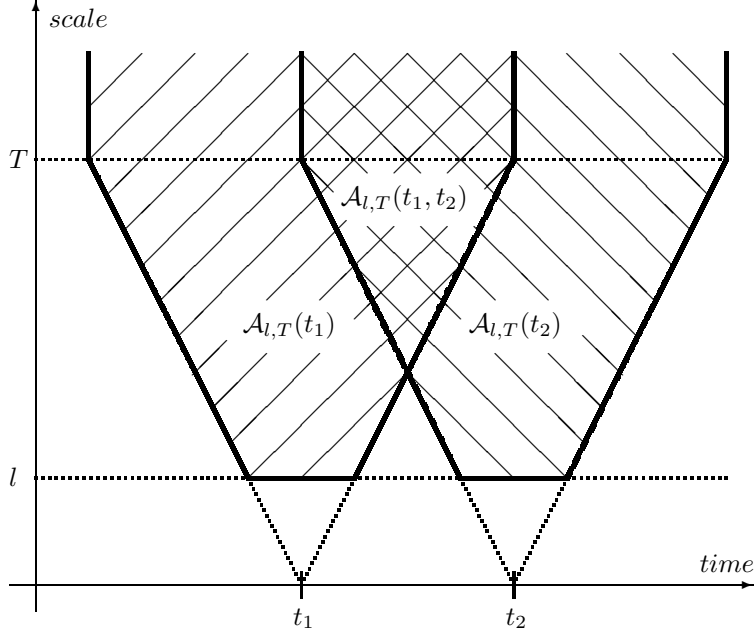
Fig. 1. Cone-like domains $\mathcal{A}_{l,T}(t_1)$, $\mathcal{A}_{l,T}(t_2)$ et $\mathcal{A}_{l,T}(t_1, t_2)$ used in the definition of continuous cascades (see definition 13). The parameter T is the integral scale and the parameter l is the small scale cut-off. The limit MRM is obtained in the limit $l \to 0$.

C. Properties

Many properties of the MRM measure involves the following concave function

$$\zeta_M(q) = q - \psi(q) = (1 + 2\lambda^2)q - 2\lambda^2 q^2 \; , \tag{15}$$

which satisfies $\zeta_M(0) = 0$ and $\zeta_M(1) = 1$. The non degeneracy condition $\lambda^2 < 1/2$, can be rewritten in terms of $\zeta_M(q)$:

Proposition 1 (Non degeneracy of M [14]):

$$\lambda^2 < \frac{1}{2} \Longleftrightarrow \zeta'_M(1) > 0 \Longrightarrow M \text{ is non degenerated and } \forall t, \; \mathbb{E}\left[M[0, t]\right] = t \tag{16}$$

The limit measure M possesses exact scale invariance properties that are directly resulting from the invariance of the function $\rho_{l,T}(\tau)$ as respect to time dilation. Indeed, the covariance function $\rho_{l,T}(\tau)$ in Eq. (6) satisfies the following invariance properties

(i) $\rho_{sl,sT}(s\tau) = \rho_{l,T}(\tau)$, $\forall s > 0$,
(ii) $\rho_{l,T}(s\tau) = \rho_{l,sT}(\tau) - \ln\lambda$, $\forall \tau \le sT, s \in [0,1]$,
(iii) $\rho_{sl,T}(s\tau) = \rho_{l,T}(\tau) - \ln\lambda$, $\forall \tau \le T, s \in [0,1]$,

It follows that the MRM measure M satisfies 3 scale invariance properties

Proposition 2 (Scale invariance properties [14]):

(i) Global scale-invariance

$$\{M_{sT}[0, st]\}_t \overset{\mathcal{L}}{=} s\{M_T[0, t]\}_t, \; \forall s \in \mathbb{R}^+ \tag{17}$$

(ii) Integral scale invariance property

$$\{M_T[0, t]\}_{0 \le t \le sT} \overset{\mathcal{L}}{=} W_s\{M_{sT}[0, t]\}_{0 \le t \le sT}, \; \forall s \in [0, 1] \tag{18}$$

(iii) Stochastic scale-invariance property

$$\{M_T[0, st]\}_{0 \le t \le T} \overset{\mathcal{L}}{=} W_s\{M_T[0, t]\}_{0 \le t \le T}, \; \forall s \in [0, 1] \tag{19}$$

with, in the last two equations, $W_s = se^{\Omega_s}$, where Ω_s is a gaussian variable independant of M and defined by $\mathbb{E}\left[\Omega_s\right] = -\mathbb{V}\text{ar}\left[\Omega_s\right]/2 = 2\lambda^2 \ln s$

From (iii) one can easily deduce that q-order moments verify an exact scale invariance property. Actually, one can show that

Proposition 3 (Finiteness of positive order moments [14]): Let $\lambda^2 < 1/2$. Let $q > 0$. If $\zeta_M(q) > 1$ then $\mathbb{E}\left[M_T[0, t]^q\right] < +\infty$. Conversely, let $q > 1$, if $M_T \ne 0$ then $\mathbb{E}\left[M_T[0, t]^q\right] < +\infty \Rightarrow \zeta_M(q) \ge 1$.

Moreover, a straightforward computation shows that

$$\zeta_M(q) = \frac{\log \mathbb{E}[W_s]}{\ln s}, \tag{20}$$

where W_s is defined in Proposition 2(iii) One thus gets

Proposition 4 (Exact scale invariance of q-order moments [14]): Let $\lambda^2 < 1/2$.

$$\forall q \in \mathbf{R}, \quad \mathbb{E}[M_T[0,t]^q] = K_q t^{\zeta_M(q)}, \ \forall t \le T, \tag{21}$$

where the prefactor K_q has an analytic formula in the case $q = n \in \mathbf{N}^*$:

$$K_n = \sigma^{2n} T^{2n(n-1)\lambda^2} \prod_{k=0}^{n-1} \frac{\Gamma(1 - 2(k+1)\lambda^2)\Gamma(1 - 2k\lambda^2)^2}{\Gamma(2 - 2(p+k-1)\lambda^2)\Gamma(1 - 2\lambda^2)}. \tag{22}$$

Moreover one can prove that all the negative order moments exist

Proposition 5 (Finiteness of negative order moments (condition (C4) in [21])): Let $\lambda^2 < 1/2$. Then, $\forall q < 0$ we have $\mathbb{E}[M_T[0,t]^q] < +\infty$.

D. The Multifractal Random Walk model

As said previously, a large class of multifractal stochastic processes can be associated with a given MRM. The simplest way is probably the approach initiated by Mandelbrot and Taylor [29] that consists in compounding a self-similar stochastic process which increments are stationnary with the non decreasing function $M_T[0,t]$, where M_T is a MRM as build in previous section. Another approach, inspired from econometrics, is to consider the measure $M_T(dt)$ as a stochastic variance associated with a Brownian motion [13], [30]. In this paper, for the sake of simplicity and concision, we will exclusively consider processes with stationnary and uncorrelated increment constructed from the standard Brownian motion. We define a Multifractal Random Walk (MRW) as follows: Let M_T be a (log-normal) MRM and consider $B(t)$ a Brownian motion[2] independent of M_T. The MRW $X_T(t)$ is simply defined as, for all $t \ge 0$:

$$X_T(t) = B\Big(M_T[0,t]\Big) \tag{23}$$

An alternative construction is obtained by considering the stochastic integral of the measure $M_{l,T}$ as respect to Wiener measure $dB(u)$ (independent of $M_{l,T}$) and then take the (weak) limit $l \to 0$:

$$X_T(t) = \lim_{l \to 0^+} \int_0^t e^{\omega_{l,T}(u)} dB(u). \tag{24}$$

Let us note that the equivalence between these two definitions is proven in [14]

The properties of the MRW directly result from those of the MRM and the self-similarity of Brownian motion.

Proposition 6 (Scale invariance properties [14]):

(i) Global scale-invariance

$$\{X_{sT}(st)\}_t \stackrel{\mathcal{L}}{=} s\{X_T(t)\}_t, \ \forall s \in \mathbf{R}^+ \tag{25}$$

(ii) Integral scale invariance property

$$\{X_T(t)\}_{0 \le t \le sT} \stackrel{\mathcal{L}}{=} W_s\{X_{sT}(st)\}_{0 \le t \le sT}, \ \forall s \in [0,1] \tag{26}$$

(iii) Stochastic scale-invariance property

$$\{X_T(st)\}_{0 \le t \le T} \stackrel{\mathcal{L}}{=} W_s\{X_T(t)\}_{0 \le t \le T}, \ \forall s \in [0,1] \tag{27}$$

with, in the last two equations, $W_s = s^{1/2} e^{\Omega_s/2}$, where Ω_s is a gaussian variable independant of M and defined by $\mathbb{E}[\Omega_s] = -\mathbb{V}\mathrm{ar}[\Omega_s]/2 = 2\lambda^2 \ln s$

Proposition 7 (Finiteness of positive order moments [14]): Let $\lambda^2 < 1/2$. Let

$$\zeta_X(q) = \zeta_M(\frac{q}{2}) = \frac{q}{2}(1 + 2\lambda^2) - \frac{\lambda^2}{2} q^2 . \tag{28}$$

Let $q > 0$. If $\zeta_X(q) > 1$ then $\mathbb{E}[X_T(t)^q] < +\infty$. Conversely, let $q > 1$, if $\mathbb{E}[X_T(t)^q] < +\infty \Rightarrow \zeta_X(q) \ge 1$.

---

[2]Note that a simple way to introduce long-range correlations in the MRW model would be to replace the Brownian motion $B(t)$ by a fractional Brownian motion $B_H(t)$.

A straightforward computation shows that

$$\zeta_X(q) = \frac{\log \mathbb{E}\left[W_s\right]}{\ln s}, \tag{29}$$

where W_s is defined in Proposition 6(iii). One thus gets

Proposition 8 (Exact scale invariance of q-order moments [14]): Let $\lambda^2 < 1/2$.

$$\forall q \in \mathbb{R}, \quad \mathbb{E}\left[X_T(t)^q\right] = \tilde{K}_q t^{\zeta_X(q)}, \ \forall t \leq T, \tag{30}$$

where the prefactor $\tilde{K}_q$ has an analytic formula in the case $q = n \in \mathbb{N}$: $\tilde{K}_n = (2n-1)!! \, K_n$, where K_n is given by (22).

Proposition 9 (Finiteness of negative order moments (condition (C4) in [21])): Let $\lambda^2 < 1/2$. Then, $\forall q < 0$ we have $\mathbb{E}\left[X_T(t)^q\right] < +\infty$.

E. Discrete time representation of a MRW - Monte Carlo simulation

Notation 1: For the sake of simplicity, in the following, if $Y(t)$ is a sochastic process, we will use the notation

$$\delta_\tau Y(t) = Y(t) - Y(t - \tau). \tag{31}$$

Moreover we recall that if $M(dt)$ is a measure, $M(t)$ refers to the non decreasing process

$$M(t) = M[0, t], \tag{32}$$

and, consequently

$$\delta_\tau M(t) = M(t) - M(t - \tau) = M[t - \tau, t]. \tag{33}$$

Let fix $\tau > 0$. We want to simulate the discrete time process $\{X_T(n\tau)\}_n$. Approximated Monte Carlo simulation of this discrete time process can be obtained using Eqs (4), (5) and (6). One first fixes l small enough ($l = \frac{\tau}{128}$ will be sufficient for the purpose of this paper) such that $\frac{\tau}{l}$ is an integer. The Gaussian stationary discrete time process $\{\omega_{l,T}(n\tau)\}_n$ can be simulated using the analytical formulae of its mean (5) and of its autocovariance (6).

Thus one can easily simulate the measure $\tilde{M}_{l,T}(dt)$ that is uniform on each interval of the form $[kl, (k+1)l]$ with the density $e^{2\omega_{l,T}(kl)}$. For $n \geq 0$, one has

$$\tilde{M}_{l,T}(nl) = \tilde{M}_{l,T}[0, nl] = \sum_{k=0}^{n-1} e^{2\omega_{l,T}(kl)} l. \tag{34}$$

From these simulations, one can easily simulate the process $\{\tilde{X}_{l,T}(t)\}_{t \geq 0}$ which is linear on each interval of the form $[kl, (k+1)l]$, and which satisfies

$$\tilde{X}_{l,T}(nl) = \sum_{k=1}^{n} \epsilon[k] \sqrt{l} e^{\omega_{l,T}(k)}, \tag{35}$$

where $\epsilon[k]$ is a gaussian white noise which is independant of $\tilde{M}$. The convergence of the linear-wise process $M_{l,T}(t)$ towards $M(t)$ when $l \to 0$, and consequently the convergence of the linear-wise process $X_{l,T}(t)$ towards $X_T(t)$, are proved in [14]. Thus, simulations of the discrete-time process $\{X_{l,T}(n\tau)\}_n$ can be seen as a good approximations of simulations of the discrete-time process $\{X_T(n\tau)\}_n$.

III. AGGREGATION PROPERTIES

A. Introduction

One of the nice features of standard Brownian motion is its stability as respect to time aggregation: At each scale, the increment probability distributions remain Gaussian. Propositions 2(iii) and 6(iii) state that both the log-normal MRM and MRW processes have stochastic scale-invariance property. This means that, in some sense, they possess stable properties when changing the time scale. However, this property is of poor practical interest because it does not provide the probability law at a given time scale τ but simply indicates how this law changes as τ varies. In the log-normal continuous cascade models, the multifractality, i.e., the non-linearity of the moment scaling exponent $\zeta_M(q)$, is fully characterized by the intermittency coefficient λ^2. Empirically this exponent is often found to be close to zero: For instance, the commonly reported value of λ^2 for energy dissipation field Eulerian Turbulence and for the volatility fluctuations associated with financial asset returns are respectively $\lambda^2 \simeq 0.2$ and $\lambda^2 \simeq 0.02$. It is therefore natural to study the properties of the log-normal MRM measure in the limit $\lambda^2 \ll 1$. In this section, we will see how, in this regime, the law of the MRM can be well approximated by the law of an explicit log normal process based on the so-called (normal) *renormalized magnitude* process $\Omega(t)$.

In this section, we show that, in this regime, the variations of the MRM or of the MRW are closely related to those of an explicit log normal process based on the so-called (normal) *renormalized magnitude* process $\Omega(t)$. Whereas subsection III-C states a convergence theorem of the logarithm of the MRM towards Ω in the limit $\lambda^2 \to 0$, the other subsections establish

different MRM/MRW moment approximation theorems as functions of $\Omega(t)$ moments. All along these sections, the moment approximation will be made on the following criterium

Notation 2: Let $\{X_\lambda(t)\}_t$ and $\{Y_\lambda(t)\}_t$ be two processes that depend on the parameter λ^2. Let $M_{X_\lambda}(t_1, ... t_n)$ be a given generalized moment of the process $\{X_\lambda(t)\}_t$. Let $M_{Y_\lambda}(t_1, ... t_n)$, the corresponding generalized moment of the process $\{Y_\lambda(t)\}_t$. Let us consider the Taylor series (for λ^2 around 0) of these moments. In the case the zero orders as well as the first following non trivial orders of these Taylor series are identical for any finite generalized moment, we will write

$$X_\lambda(t) \stackrel{\lambda}{\simeq} Y_\lambda(t). \tag{36}$$

B. The renormalized magnitude $\Omega(t)$

Let us define the process $\Omega(t)$ that is at the heart of our approximation theory: let $\omega_{l,T}(t)$ be the Gaussian process defined in Eqs (5) and (6) or Eq. (14). We define the Gaussian process $\Omega_l(t)$ as

$$\Omega_l(t) = \frac{1}{\lambda} \int_0^t \left(\omega_{l,T}(s) - \mathbb{E}\left[\omega_l\right] \right) ds, \tag{37}$$

The renormalized magnitude process $\Omega(t)$ is defined as the weak limit of $\Omega_l(t)$:

Theorem 1: The process $\{\Omega_l(t)\}_t$ admits a weak limit when l goes to 0:

$$\Omega(t) = \lim_{l \to 0^+} \Omega_l(t) \tag{38}$$

Proof: The proof of this theorem is a direct consequence of Lemma 1 (Convergence of the finite dimensional laws) and Lemma 2 (tightness) of Appendix A. ∎

In the sequel, if $I = [t - \tau, t]$ is some interval, $\Omega(I)$ will stand for the variation of the renormalized magnitude over this interval:

$$\Omega(I) = \delta_\tau \Omega(t) = \Omega(t) - \Omega(t - \tau). \tag{39}$$

The exact expression of the covariance of the renormalized magnitude can be simply computed using Lemma 1 of Appendix A:

Proposition 10: Let $\tau > 0$ and $h \geq \tau$. For all t, one has:

- if $h + \tau \leq T$,

$$\mathbb{C}ov\left[\frac{\delta_\tau \Omega(t)}{\tau}, \frac{\delta_\tau \Omega(t + h)}{\tau}\right] = \ln\left(\frac{T e^{3/2}}{h}\right) + f\left(\frac{h}{\tau}\right), \tag{40}$$

where the function $f(u)$ reads

$$f(u) = \begin{cases} -\frac{(u+1)^2}{2} \ln\left(1 + \frac{1}{u}\right) - \frac{(u-1)^2}{2} \ln\left(1 - \frac{1}{u}\right), & \text{if } u \geq 2, \\ -2 \ln(2), & \text{if } u = 1, \\ 0, & \text{if } u = 0, \end{cases} \tag{41}$$

- if $h \geq T + \tau$,

$$\mathbb{C}ov\left[\delta_\tau \Omega(t), \delta_\tau \Omega(t + h)\right] = 0. \tag{42}$$

Let us note that in the case $\tau \ll h < T + \tau$, Eqs (40) and (41) simplify a lot. Indeed, in this case, the function $f(u)$ becomes $f(u) = -3/2 + \mathcal{O}(1/u)$, one thus gets

Corollary 1: Let $\tau \ll h < T + \tau$, then for all t, one has:

$$\mathbb{C}ov\left[\frac{\delta_\tau \Omega(t)}{\tau}, \frac{\delta_\tau \Omega(t + h)}{\tau}\right] = \ln\left(\frac{T}{h}\right) + \mathcal{O}(\tau/h), \tag{43}$$

We are now ready to formulate the main approximation results one can obtain in the limit of small intermittency $\lambda^2 \to 0$.

C. Convergence in law towards the renormalized magnitude

One can prove an asymptotic theorem concerning the logarithm of the measure of an interval. More precisely, one has the following result:

Theorem 2: Let $I_1, \ldots, I_n$, be n arbitrary intervals. When λ^2 goes to zero we have the following convergence in law:

$$\left(\frac{1}{2\lambda} \ln\left(\frac{M(I_1)}{|I_1|}\right), \ldots, \frac{1}{2\lambda} \ln\left(\frac{M(I_n)}{|I_n|}\right)\right) \xrightarrow{\mathcal{L}} \left(\frac{\Omega(I_1)}{|I_1|}, \ldots, \frac{\Omega(I_n)}{|I_n|}\right). \tag{44}$$

Proof: From Proposition 12 of Appendix B and Proposition 13 of Appendix C, it results that, for all n,

$$\lim_{\lambda \to 0} \mathbb{E}\left[\prod_{j=1}^{n} \frac{1}{2\lambda} \ln\left(\frac{M(I_j)}{|I_j|}\right)\right] = \mathbb{E}\left[\frac{\Omega(I_1)}{|I_1|}, \ldots, \frac{\Omega(I_n)}{|I_n|}\right]. \tag{45}$$

A simple multidimensional generalization of the Theorem 4.5.5 in [31] allows one to deduce the convergence in law from the convergence of the generalized moments [32]. ∎

The following corollary on the successive incremements of the measure is a direct consequence of the previous theorem:

Corollary 2: If $\tau > 0$, then

$$\left\{\frac{1}{2\lambda} \ln\left(\frac{\delta_\tau M(t)}{\tau}\right)\right\}_t \xrightarrow{\mathcal{L}} \left\{\frac{\delta_\tau \Omega(t)}{\tau}\right\}_t. \tag{46}$$

D. Approximation of the moments of the logarithm of the measure

The following result will be particularly useful for the estimation of log-normal MRM as discussed in section IV below.

Theorem 3 (Convergence of the magnitude generalized moments): At scale $\tau > 0$ the process $\{2\lambda\delta_\tau\Omega(t)/\tau\}_t$ reproduces the Taylor series (in λ^2), up to the first non trivial order, of any finite generalized moment of the logarithm of the log-normal MRM increments (see Eq. (36) for precision on the following notation):

$$\ln\left(\frac{\delta_\tau M(t)}{\tau}\right) \overset{\lambda}{\simeq} 2\lambda\frac{\delta_\tau\Omega(t)}{\tau}. \tag{47}$$

Proof: This result is a direct consequence of Proposition 12 of Appendix B and Proposition 13 of Appendix C. ∎

This theorem allows one to obtain approximations of the mean and of the covariance function of logarithm of the MRM increments:

Theorem 4 (Magnitude mean and covariance approximations): For all $\tau > 0$ and $h \geq 0$ and t, one has

$$\mathbb{C}ov\left[\ln\left(\frac{\delta_\tau M(t)}{\tau}\right), \ln\left(\frac{\delta_\tau M(t+h)}{\tau}\right)\right] = 4\lambda^2\mathbb{C}ov\left[\frac{\delta_\tau\Omega(t)}{\tau}, \frac{\delta_\tau\Omega(t+h)}{\tau}\right] + o(\lambda^2), \tag{48}$$

where the covariance of the increments of the renormalized magnitude is provided by Lemma 1 in Appendix A. Moreover, as far as $h + \tau \leq T$, then the term $o(\lambda^2)$ no longer depends on the integral scale T and depends on τ only through the ratio h/τ and goes to 0 when $\tau \to 0$ (with h fixed). Moreover

$$\mathbb{E}\left[\ln\left(\frac{\delta_\tau M(t)}{\tau}\right)\right] = -2\lambda^2 \ln\left(\frac{Te^{3/2}}{\tau}\right) + o(\lambda^2), \tag{49}$$

where the term $o(\lambda^2)$ depends neither on T nor on τ

Proof: The relationship (48) directly results from previous theorem. We simply have to show that for $h + \tau \leq T$, $o(\lambda^2)$ depends only on h/τ and goes to 0 when $\tau \to 0$.

By using the invariance properties (17) and (18) of the Proposition 2 we get the equality in law:

$$\{\delta_\tau M_T(t)\}_{\tau \leq t \leq h} \overset{\mathcal{L}}{=} W_{h/T}\{\delta_\tau M_{h/\tau}(t/\tau)\}_{\tau \leq t \leq h}, \tag{50}$$

where $W_{h/T}$ is a log-normal random variable that satisfies

$$\mathbb{V}ar\left[\ln(W_{h/T})\right] = 4\lambda^2 \ln\left(\frac{T}{h}\right). \tag{51}$$

From (50), one can easily prove that the difference $\mathbb{C}ov\left[\ln(|\delta_\tau M_T(t)|), \ln(|\delta_\tau M_T(t+h)|)\right] - \mathbb{V}ar\left[\ln(W_{h/T})\right]$ depends only on λ^2 and h/τ. The fact that it goes to 0 when τ goes to 0 comes from a straightforward argument using the cone representation. Moreover, thanks to Lemma 1 in Appendix A, one obtains, if $h + \tau \leq T$,

$$4\lambda^2\mathbb{C}ov\left[\frac{\delta_\tau\Omega(t)}{\tau}, \frac{\delta_\tau\Omega(t+h)}{\tau}\right] = 4\lambda^2 \int_{t-\tau}^{t} \frac{du}{\tau} \int_{t+h-\tau}^{t+h} \frac{dv}{\tau} \ln\left(\frac{h}{|u-v|}\right) + \mathbb{V}ar\left[\ln(W_{h/T})\right]. \tag{52}$$

By choosing the new variables $u' = u/\tau$ and $v' = v/\tau$, we can show that the above integral depends only on h/τ and goes to 0 when $\tau \to 0$ (with h fixed). By inserting this expression in Eq. (48), we thus conclude that the terms $o(\lambda^2)$ in this equation depends only on h/τ.

A similar computation allows us to deduce Eq. (49). ∎

E. Approximation of the moments of the measure

As far as the generalized moments of the measure itself ($\delta_\tau M(t)$) are concerned, Theorem 3 suggests that they could be well reproduced to the first non trivial order by the moments of the process $\tau e^{2\lambda \frac{\delta_\tau \Omega(t)}{\tau}}$. It is easy to see that this cannot be true. Indeed, the mean of the two previous processes are different simply because the expectation of the exponential of a random variable is not the exponential of its expectation. It is therefore necessary to slightly modify the process $\frac{\delta_\tau \Omega(t)}{\tau}$ by changing its mean value.

Theorem 5: Let $\tau > 0$. The process $\left\{ \tau e^{2\lambda \delta_\tau \Omega(t)/\tau - 2\lambda^2 \mathbb{V}\mathrm{ar}[\delta_\tau \Omega(t)/\tau]} \right\}_t$ reproduces the Taylor series (in λ^2), up to the first non trivial order, of any finite generalized moment of the log-normal MRM increments (see Eq. (36) for precision on the following notation):

$$\delta_\tau M(t) \overset{\lambda}{\simeq} \tau e^{2\lambda \frac{\delta_\tau \Omega(t)}{\tau} - 2\lambda^2 \mathbb{V}\mathrm{ar}\left[\frac{\delta_\tau \Omega(t)}{\tau}\right]}. \tag{53}$$

Proof: The n-points moment of the r.h.s. process can be written as:

$$m(t_1, \ldots, t_n) = \tau^n e^{-2n\lambda^2 \mathbb{V}\mathrm{ar}\left[\frac{\delta_\tau \Omega}{\tau}\right]} \mathbb{E}\left[e^{2\lambda \sum_{i=1}^n \frac{\delta_\tau \Omega(t_i)}{\tau}} \right] = \tau^n e^{2\lambda^2 \mathbb{V}\mathrm{ar}\left[\sum_{i=1}^n \frac{\delta_\tau \Omega(t_i)}{\tau}\right]}. \tag{54}$$

If one considers the Taylor series expansion of this expression and replaces the variance of $\sum_{i=1}^n \delta_\tau \Omega(t_i)$ by its expression (provided by Lemma 1 in Appendix A), one gets:

$$m(t_1, \ldots, t_n) = \tau^n + 4\tau^n \int_{t_n}^{t_n+\tau} du_1 \cdots \int_{t_n}^{t_n+\tau} du_n \sum_{1 \le j \le k \le n} \rho(u_j - u_k) + o(\lambda^2), \tag{55}$$

where ρ is defined by (99). Using Lemma 4 in Appendix B, it follows

$$m(t_1, \ldots, t_n) = \mathbb{E}\left[\delta_\tau M(t_1), \ldots, \delta_\tau M(t_n) \right] + o(\lambda^2), \tag{56}$$

which leads to the expected result. ∎

F. Approximation of the MRW process

The log-normal MRW process being defined by a Brownian motion subordinated with a log-normal MRM measure, it is obvious that the generalized moments of its increments and their logarithm are related to those of the MRM measure. In previous sections we have obtained an approximation of these MRM generalized moments. The Theorems 3, 4 and 5 naturally extend to MRW increments. The following theorems are direct consequences from these theorems.

Theorem 6: Let $\tau > 0$ and $\{\epsilon[n]\}_n$ a gaussian white noise of variance σ^2.
The discrete time process $\left\{ \tau^{1/2} \epsilon[n] e^{\delta_\tau \Omega(n\tau)/\tau - \lambda^2 \mathbb{V}\mathrm{ar}[\delta_\tau \Omega/\tau]} \right\}_n$ reproduces the Taylor series (in λ^2), up to the first non trivial order, of any finite generalized moment of the increments of a MRW process $X(t)$ (see Eq. (36) for precision on the following notation):

$$\delta_\tau X(n\tau) \overset{\lambda}{\simeq} \tau^{1/2} \epsilon[n] e^{\lambda \frac{\delta_\tau \Omega(n\tau)}{\tau} - \lambda^2 \mathbb{V}\mathrm{ar}\left[\frac{\delta_\tau \Omega}{\tau}\right]}. \tag{57}$$

Moreover the first non trivial order is of order λ^2.

Theorem 7: Let $\tau > 0$ and $\{\epsilon[n]\}_n$ a gaussian white noise of variance σ^2.
The discrete time process $\left\{ \ln(\tau^{1/2}) + \ln(|\epsilon[n]|) + \lambda \delta_\tau \Omega(n\tau)/\tau \right\}_n$ reproduces the Taylor series (in λ^2), up to the first non trivial order, of any finite generalized moment of the absolute increments of a MRW process $X(t)$ (see Eq. (36) for precision on the following notation):

$$\ln|\delta_\tau X(n\tau)| \overset{\lambda}{\simeq} \frac{1}{2}\ln(\tau) + \ln(|\epsilon[n]|) + \lambda \frac{\delta_\tau \Omega(n\tau)}{\tau}. \tag{58}$$

As in previous theorem, the first non trivial order is of order λ^2.

Theorem 8: For all $\tau > 0$ et $h \ge 0$, one has, for all t

$$R_\tau(h) = \mathbb{C}\mathrm{ov}\left[\ln(|\delta_\tau X(t)|), \ln(|\delta_\tau X(t+h)|) \right] = \frac{\pi^2}{8}\delta(h) + \lambda^2 \mathbb{C}\mathrm{ov}\left[\frac{\delta_\tau \Omega(t)}{\tau}, \frac{\delta_\tau \Omega(t+h)}{\tau} \right] + o(\lambda^2), \tag{59}$$

where the covariance of the increments of the renormalized magnitude is provided by Lemma 1 in Appendix A. Moreover, as far as $h + \tau \le T$, then the term $o(\lambda^2)$ no longer depends on the integral scale T and depends on τ only through the ratio h/τ and goes to 0 when $\tau \to 0$ (with h fixed). Moreover, one has

$$\mathbb{E}\left[\ln(|\delta_\tau X(n\tau)|) \right] = -\frac{\gamma + \ln(2)}{2} - \lambda^2 \ln\left(\frac{Te^{3/2}}{\tau} \right) + o(\lambda^2), \tag{60}$$

where the term $o(\lambda^2)$ depends neither on T nor on τ and γ is the Euler constant.

In the case $\tau \ll h < T + \tau$, Eq. (43) gives an approximation of the auto covariance of the renormalized magnitude. It can be used to get an approximation of the auto covariance $R_\tau(h)$, i.e.,

Corollary 3: Let $\tau \ll h < T + \tau$, then for all t, one has:

$$R_\tau(h) = \mathbb{C}ov\big[\ln(|\delta_\tau X(t)|), \ln(|\delta_\tau X(t+h)|)\big] = \lambda^2 \ln\left(\frac{T}{h}\right) + \lambda^2 \mathcal{O}(\tau/h) + o(\lambda^2). \tag{61}$$

where the term $o(\lambda^2)$ no longer depends on the integral scale T and depends on τ only through the ratio h/τ and goes to 0 when $\tau \to 0$ (h fixed).

IV. PARAMETER ESTIMATION

We have seen that a log-normal cascade model is mainly defined by 2 parameters (apart from the variance parameter σ^2 which is a simple multiplicative factor for the MRW): the integral scale T and the intermittency coefficient λ^2. Among the huge literature devoted to multifractal models and multifractal analysis, there are only very few papers that focus on issues related to parameter estimation or related statistical questions (see however [23]–[27]).

A simple method to estimate λ^2 would consist in performing a regression of the empirical $\zeta(q)$ function estimated from the scaling behavior of the empirical moments. However this method is far from being robust, the variance of this estimator converges very slowly (of the type $N^{-1+\alpha}$ with $\alpha > 0$, see [23]). This method is however sufficient to establish the pertinence of the approximation $\lambda^2 \ll 1$ in many empirical situations like the analysis of turbulence or financial time series [11], [30]. The starting point of our approach of parameter estimation is therefore to assume that we are in the small intermittency regime $\lambda^2 \ll 1$ and that the results of section III can be used.

Let $N = \frac{L}{\tau}$ be the total number of samples available, where L is the observation scale and τ the sampling period. Consequently, the observed samples corresponds to the values

$$\{X_T(n\tau)\}_{n \in [0,N[}, \quad \text{where } N = \frac{L}{\tau}. \tag{62}$$

The estimation problem must be studied in the asymptotic regime $N \to +\infty$. However, this limit can be achieved in two different ways. The first one, referred to as *low-frequency regime*, corresponds to the case where τ is fixed and $L \to +\infty$. In the second one, referred to as *high frequency regime*, L is fixed but the and $\tau \to 0$. From a numerical point of view, $L \gg T$, corresponds to the low-frequency regime whereas $\tau \ll T$ corresponds to the high-frequency regime. In both cases $N = L/\tau \to +\infty$. In the particular case where one has both $L \gg T \gg \tau$, the effective asymptotic can be considered to be the high (resp. low) frequency regime if $\frac{L}{T} \ll \frac{T}{\tau}$ (resp. $\frac{L}{T} \gg \frac{T}{\tau}$). For discussions on *mixed regime* for which $L \to +\infty$ and $\tau \to 0$ at the same time, we refer the reader to [33]–[36].

A. *GMM in the low frequency regime, $L \to +\infty$*

The first application of GMM to estimate multifractal models can be found in econometric literature. More precisely, Calvet and Fisher [27], [37] used this method to estimate the parameters of a simple cascade model where the random weights follow a binomial law. Their work has been further developed by Lux [25], [26].

It is easy to see that the three parameters λ^2, T and σ^2, are directly related to some moments associated with MRW increments or their logarithm. It is therefore natural to use a GMM to estimate these parameters. GMM was initially proposed by Hansen [38] and can be described as follows:

Let us consider the process $\{Z_\tau^{(\theta)}[k]\}_k$ of the logarithms of absolute increments of some MRW process at size τ:

$$Z_\tau^{(\theta)}[k] = \ln|\delta_\tau X[k]|. \tag{63}$$

This process is characterized by $p = 3$ parameters:

$$\theta = \{\ln(\sigma), \lambda^2, \ln T\}. \tag{64}$$

Given some observation $\{Z_\tau^{(\theta_0)}[k]\}_k$, let us denote $f(Z_\tau^{(\theta_0)}[k], \theta)$ the *moment function* of dimension $r > p$, which satisfies the following *moment condition*:

$$\mathbb{E}\left[f(Z_\tau^{(\theta_0)}[k], \theta)\right] = 0, \text{ if and only if } \theta = \theta_0. \tag{65}$$

In our case, it is natural to choose the variance of the process $\{e^{2Z_\tau^{(\theta_0)}[k]}\}_k$ in order to estimate σ^2 and the empirical covariance of $Z_\tau^{(\theta_0)}$ at various time lags in order to estimate λ^2 and T. This leads us to consider

$$f(Z_\tau^{(\theta_0)}[k], \theta) = \begin{pmatrix} \exp(2Z_\tau^{(\theta_0)}[k] \\ (Z_\tau^{(\theta_0)}[k] - \mu_\theta)(Z_\tau^{(\theta_0)}[k-h_1] - \mu_\theta) \\ \vdots \\ (Z_\tau^{(\theta_0)}[k] - \mu_\theta)(|Z_\tau^{(\theta_0)}[k-h_K] - \mu_\theta) \end{pmatrix} - \begin{pmatrix} \sigma^2\tau \\ C_\theta[h_1] \\ \vdots \\ C_\theta[h_K] \end{pmatrix}, \tag{66}$$

where

$$\mu_\theta = \mathbb{E}\left[Z_\tau[k]\right] \tag{67}$$

and

$$C_\theta[h] = \mathbb{C}ov\left[Z_\tau^{(\theta)}[k], Z_\tau^{(\theta)}[k-h]\right], \tag{68}$$

and $h_1, \ldots, h_K$ are K different positive lags. Let us note that, a first order (in λ^2) analytical expression of $C_\theta[h] = R_\tau(h\tau)$ is provided by Eq. (59).

The moment condition (65) can be approximated by using the empirical mean:

$$g_N(\theta) = \frac{1}{N}\sum_{k=1}^N f(Z_\tau^{(\theta_0)}[k], \theta). \tag{69}$$

The GMM estimator is then simply defined by

$$\widehat{\theta} = argmin_\theta\left(g_N^T W_N g_N\right), \tag{70}$$

where W_N is a sequence of weighting matrices that converges, when $N \to +\infty$ towards some matrix positive definite W_∞. Hansen has established the following result:

Theorem 9 (Hansen [38]): If the following hypotheses hold:

- The process $\{Z_\tau^{(\theta_0)}[k]\}_k$ is ergodic,
- The series $\{f(Z_\tau^{(\theta_0)}[k], \theta)\}_k$ satisfies a central limit theorem, i.e.,

$$\frac{1}{\sqrt{N}}\sum_{k=1}^N f(Z_\tau^{(\theta_0)}[k], \theta) \to \mathcal{N}(0, V_\theta), \tag{71}$$

where the matrix V_θ is defined as:

$$V_\theta = \lim_{M \to +\infty}\sum_{k=-M}^M \mathbb{E}\left[f(Z_\tau^{(\theta_0)}[k], \theta)f(Z_\tau^{(\theta_0)}[k], \theta)^T\right]. \tag{72}$$

- The $(r \times p)$ matrix $Dg_N = \frac{\partial g_N}{\partial \theta}$ has full rank (p) and converges towards

$$Df = \mathbb{E}\left[\frac{\partial f(Z_\tau^{(\theta_0)}[k], \theta)}{\partial \theta}\right], \tag{73}$$

then, the GMM estimator $\widehat{\theta}$ is consistent and verifies

$$\sqrt{N}\left(\widehat{\theta} - \theta\right) \to \mathcal{N}(0, \Sigma), \tag{74}$$

where

$$\Sigma = \left(Df^T W_\infty Df\right)^{-1} Df^T W_\infty V_{\theta_0} W_\infty Df\left(Df^T W_\infty Df\right)^{-1}. \tag{75}$$

Moreover, the estimator $\widehat{\theta}$ is optimal if $W_\infty = V_{\theta_0}^{-1}$, as defined in Eq. (72). In that case the asymptotic covariance of the estimator is

$$\Sigma_{opt} = \left(Df^T V_{\theta_0}^{-1} Df\right)^{-1}. \tag{76}$$

In practice [39], it is obviously difficult to use the optimal weighting matrix $W_\infty = V_{\theta_0}^{-1}$ since one does not know the vector θ_0. One usually proceeds using the following iterative algorithm:

1) Choose some arbitrary initial weighting matrix W_N, such as $\mathbb{I}d_N$,
2) Compute the GMM estimator (70) using this matrix W_N,
3) Replace the weighting matrix by $W_N = V_{\widehat{\theta}}^{-1}$, where $\widehat{\theta}$ is the obtained estimated parameter vector.
4) Repeats step 2 and 3 until successive estimates are sufficiently close one to each other.

Confidence intervals for $\widehat{\theta}$ can be obtained using Eq. (74).

One can easily show that the hypothesis of Theorem 9 hold in the case the moment function is defined by (66). However, there is one major problem for implementing the corresponding GMM method : we do not have any analytical expressions neither of μ_θ (Eq. (67)) nor $C_\theta[h]$ (Eq. (68)), nor V_θ (Eq. (72)) . Actually Eqs (59) and (60) of Theorem 8 give analytical approximations (up to a $o(\lambda^2)$ term) to both μ_θ and $C_\theta[h]$. Let us note that these very same equations also allow to derive an analytical expression (up to a $o(\lambda^2)$ term) of V_θ [32]. It is tempting to use these approximations in the moment function (66) and for the weighting matrix and try to use the exact same GMM algorithm. This is exactly the framework of the so-called

GMM estimation in a misspecified model (see [39]). The model is considered as *misspecified* since the moment function no longer satisfies (65). Instead, one has

$$\mathbb{E}\left[f^*(Z_\tau^{(\theta_0)}[k],\theta)\right] = r(\theta), \text{ for all } k, \text{ and } ||r(\theta)|| > 0 \text{ for all } \theta, \tag{77}$$

where f^* corresponds to the moment function (66) in which we have substituted μ_θ and $C_\theta[h]$ by their approximations. Now, if we suppose that there exists θ^* such that

$$\mathbb{E}\left[f^*(Z_{\theta_0},\theta^*\right]^T W_\infty \mathbb{E}\left[f^*(Z_{\theta_0},\theta^*\right] < \mathbb{E}\left[f^*(Z_{\theta_0},\theta\right]^T W_\infty \mathbb{E}\left[f^*(Z_{\theta_0},\theta\right], \quad \forall \theta \neq \theta^*, \tag{78}$$

using the results of [39], one can show that the so-obtained *approximated* GMM gives a consistent asymptotically gaussian estimator of θ^*. Moreover, we expect

$$\theta^* = \theta + o(\lambda^2). \tag{79}$$

In order, to illustrate this estimation method, we have run a Monte-Carlo test on MRW realizations. The results are shown in Table I. Each MRW was simulated on a discrete time grid of period $\tau = 1$ and of various size L using the algorithm described in Section II-E[3]. The number K of different lags h_k used in the moment function (66) is $K = 43$ and the lags h_k are approximately logarithmically distributed between 1 and 150.

For each set of parameters, we simulated 10000 realizations of such MRW and ran the misspecified GMM algorithm on each of these realizations. For each parameter $(\ln(\sigma), \lambda^2, \ln(T))$ we computed associated GMM estimators $(\widehat{\ln(\sigma)}, \widehat{\lambda^2}, \widehat{\ln(T)})$. We then computed for each of them the so-obtained bias (the Bias column), the mean square error (MSE column) and we ran the Kolmogorov-Smirnov [40] test for testing the normality of the estimations. The corresponding p-values for this test are indicated in the KS column. Thus, for instance, a 5% level test is satisfied if the p-value is greater than 0.05.

Clearly the effective value of σ will slightly affect the performance of the GMM algorithm, since it just corresponds to a multiplicative factor. Thus, in all the numerical experiments we arbitrarily set it to $\sigma = 1$ (i.e., $\ln(\sigma) = 0$). The global scale invariance property (17) shows that changing the value of the parameter T amounts to changing the number of samples $L/\tau = L$ (since $\tau = 1$) of the realizations. Consequently, the realizations only depend on the ratio L/T, i.e., the number of integral scales in a realization. We arbitrarily choose to fix T and have L varying. In this section, we only adress the low frequency regime $T \ll L$. We choose $T = 200$ (i.e., $\ln(T) \simeq 5.298...$) and L among $\{2048, 4096, 8192, 16384, 65536\}$, i.e., the number of integral scales L/T varies from 10 to more than 320. We are thus left with only λ^2 as a "free" parameter. We used two different values for λ^2 : 0.02 and 0.04. Thus, two different sets of parameters were used : the first set (top half of Table I) corresponds to $\sigma = 1$, $\lambda^2 = 0.02$ and $T = 200$ and the second set (bottom half of Table I) corresponds to $\sigma = 1$, $\lambda^2 = 0.04$ and $T = 200$. Let us note that adding some more lags h_k (i.e., increasing K) does not significantly improve the results (see the line corresponding to size $L = 16384*$ in Table I which corresponds to $K = 69$ instead of $K = 43$).

For all parameters, Table I shows clearly that the MSE is entirely dominated by the variance (the bias contribution is negligeable). Let us discuss the results obtained for the estimation of each parameter one after the other.

- $\widehat{\log \sigma}$: Clearly, the theoretical GMM asymptotics for the parameter $\ln(\sigma)$ is reached as soon as $N = 2048$. This is indicated both by the fact that the MSE decreases as $1/\sqrt{L}$ and that the Kolmogorov-Smirnov normality test has a very high p-value as soon as $N \geq 2048$ (for $N = 2048$, the p-value is almost 40% when $\lambda^2 = 0.02$ and almost 25% when $\lambda^2 = 0.04$).
- $\widehat{\lambda^2}$: For λ^2 the situation is somewhat different. Though the estimation is surprisingly good even for the shortest realizations in the sense that the MSE is very small, the normal asymptotics cannot be considered to be reached when $N \leq 16384$, i.e., when the number of integral scales L/T is smaller than 80.
- $\widehat{\ln T}$: Here the GMM asymptotics for parameter $\ln(T)$ is the slowest. Though the MSE is small for $N \geq 16384$, the normal asymptotic can hardly be considered to be reached even for $N = 65536$.

Let us note that, in any case, the $o(\lambda^2)$ term in (79) due to the mispecification of the model hardly shows up in these results. Indeed, we expect a bias of the order of λ^4 (i.e., the "next" order after λ^2), thus of the order of $4e - 04$ for the top half of the Table ($\lambda^2 = 0.02$) and 1.610^{-3} for the bottom half ($\lambda^2 = 0.04$). Except for the case $\lambda^2 = 0.04$ and for the estimation of the parameter λ^2 (for which the bias saturates around 210^{-5}), there does not seem to be any trace of this term : L is not large enough. Even when the bias saturates around $210-5$ (for an MSE of 0.0015), in order this saturation value to dominate the MSE, L should be of the order of 10^{17}! Thus, though the model is theoretically mispecified, from a practical point of view, it can be considered as well specified.

As a conclusion to this section, we can state that the GMM estimations are reliable in the low frequency regime, however, except for σ the normal asymptotic confidence intervals should not be used. Monte Carlo simulations should be performed to get confidence intervals.

---

[3]As explained in this section, we chose $l = 128$. Let us note that increasing l does not significantly change the numerical results

$\tau = 1$	$\ln\sigma = 0$ ($\sigma = 1$)			$\lambda^2 = 0.02$			$\ln T \simeq 5.298..$ ($T = 200$)		
L	Bias	MSE	KS	Bias	MSE	KS	Bias	MSE	KS
2048	-5e-03	0.070	0.39	5e-04	0.0072	3e-08	-0.013	1.15	2e-72
4096	-2e-03	0.049	0.49	3e-04	0.0048	1e-03	-0.026	0.76	1e-33
8192	-6e-04	0.034	0.67	1e-04	0.0032	2e-03	-0.015	0.50	9e-16
16384	-8e-04	0.024	0.56	2e-05	0.0022	0.08	-0.009	0.34	6e-8
16384*	-9e-04	0.024	0.54	-2e-05	0.0022	0.08	0.005	0.35	5e-09
65536	-2e-04	0.012	0.49	6e-06	0.0011	0.45	-0.002	0.17	0.01
$\tau = 1$	$\ln\sigma = 0$ ($\sigma = 1$)			$\lambda^2 = 0.04$			$\ln T \simeq 5.298..$ ($T = 200$)		
L	Bias	MSE	KS	Bias	MSE	KS	Bias	MSE	KS
2048	-1e-02	0.110	0.24	7e-04	0.0095	9e-05	-0.130	0.88	5e-32
4096	-5e-03	0.072	0.34	4e-04	0.0064	0.01	-0.054	0.59	4e-18
8192	-3e-03	0.050	0.48	2e-05	0.0044	0.06	-0.027	0.41	3e-6
16384	-2e-03	0.035	0.52	-2e-05	0.0031	0.08	-0.014	0.28	2e-5
65536	-4e-04	0.018	0.42	-4e-05	0.0015	0.40	-0.002	0.14	0.05

TABLE I

GMM ESTIMATION OF MRW PARAMETERS. EACH LINE CORRESPONDS TO GMM ESTIMATION AS EXPLAINED IN SECTION IV-A ON 10000 REALIZATIONS OF DISCRETE-TIME MRW WITH $\tau = 1$ AND OF SIZE L. L VARIES FROM 2048 TO 65536. THE LAGS h_k USED FOR GMM ESTIMATION IN EQ. (66) ARE CHOSEN SUCH THAT $K = 43$ AND APPROXIMATELY LOGARITHMICALLY DISTRIBUTED BETWEEN 1 AND 150 EXCEPT FOR THE LINE CORRESPONDING TO $N = 16384*$ FOR WHICH MORE LAGS WERE TAKEN ($K = 69$). THE MRW WERE MONTE-CARLO GENERATED USING THE ALGORITHM DESCRIBED IN SECTION II-E. TWO SETS OF PARAMETERS WERE USED : $\sigma = 1$, $\lambda^2 = 0.02$, $T = 200$ FOR THE TOP HALF AND $\sigma = 1$, $\lambda^2 = 0.04$, $T = 200$ FOR THE BOTTOM HALF.

B. GMM estimation in the high frequency regime $\tau \to 0$ - Estimation of the nature of the asymptotic regime

In many practical situations (e.g., when dealing with financial time series) the data are sampled at some high frequency $\tau \ll T$ over a time period L that is smaller than (or of the order of) the integral scale T. As already explained, in that case, the right asymptotic regime to consider is the high frequency regime $\tau \to 0$. Let us try to understand how behaves the previously described GMM procedure in that context. As we have already pointed out, the moment function (66) involved in the GMM has two types of components : the first component corresponds to the empirical variance of the increments of the MRW process itself and basically allows one to estimate σ^2 while all the other components correspond to the empirical covariance of the logarithm of the same increments and allow one to estimate T and λ^2.

According to Eq. (26), the log-normal MRW process $\{X_T(t)\}_{t\leq L}$ satisfies the following equality in law:

$$\{X_T(t)\}_{t\leq L} \stackrel{\mathcal{L}}{=} \{W_{L/T} X_L(t)\}_{t\leq L}, \tag{80}$$

where $W_{L/T}$ is a log-normal random variable which law is given in Proposition 6 and which is independent of the MRW process $\{X_L(t)\}_{t\leq L}$ which integral scale is equal to the observation scale L. Given some sample of length L of the MRW process, the variable $W_{L/T}$ takes a fixed value and can be considered as a simple multiplicative factor that simply changes the variance of the process. Consequently, the estimation problem of both σ^2 and T is ill-posed. It is fundamentally impossible to estimate independently the integral scale T and the variance σ^2 of the process since they both appear as a multiplicative factor of the whole process. T is no longer a "true" parameter of the model it can be arbitrarily fixed. Moreover, even if we knew the true value of T, there is no chance for the GMM variance estimator $\widehat{\ln\sigma}$ to converge to 0 in the asymptotic limit $\tau \to 0$ since it is easy to show that the empirical variance of the increments, itself, does converge in the limit $\tau \to 0$ towards a random variable (see e.g. [14]). Hence, in this regime, the first hypothesis upon which GMM relies, namely the ergodicity of $\{Z_\tau[n]\}_n$ is not satisfied.

Since the value of T is the key to decide in which asymptotic regime one is ($L \gg T$ for the low frequency regime and $T \gg \tau$ for the high frequency regime), it is of fundamental interest to understand how the GMM estimation of T behaves in the high frequency regime $\tau \to 0$. Actually, the GMM estimation of $\ln T$ (and of λ^2) basically consists in fitting the empirical covariance of the logarithm of the increments of the MRW process. Thus, it is natural to study the mean of this empirical covariance, in the high frequency regime $\tau \to 0$.

Proposition 11: Let us consider the fixed observation scale $L \leq T$ where τ is the sampling scale and L/τ is the number of samples of the MRW process. We introduce the empirical covariance $\hat{R}_\tau[n]$ and the empirical mean $\hat{\mu}_{\tau,M}$:

$$\hat{R}_\tau[n] = \frac{\tau}{L} \sum_{k=1}^{L/\tau-n-1} \left(\ln|\delta_\tau X(k\tau)| - \hat{\mu}_{\tau,L/\tau}\right)\left(\ln|\delta_\tau X((k+n)\tau)| - \hat{\mu}_{\tau,L/\tau}\right), \quad \hat{\mu}_{\tau,M} = \frac{1}{M}\sum_{k=1}^{M}\ln|\delta_\tau X(k\tau)|. \tag{81}$$

Let $h > 0$, then the expectation of the empirical covariance $\hat{R}_\tau[h/\tau]$, in the high frequency asymptotic $\tau \to 0$, is

$$\lim_{\tau\to 0}\mathbb{E}\left[\widehat{R_\tau}[h/\tau]\right] = \lambda^2\left[\ln\left(\frac{L}{he^{3/2}}\right) - \frac{h}{L}\ln\left(\frac{L}{he^{3/2}}\right) + \frac{h^2}{L^2}\ln\left(\frac{h}{L}\right) + \frac{(L-h)^2}{2L^2}\ln\left(1-\frac{h}{L}\right)\right] - \left(1-\frac{h}{L}\right)\frac{\pi^2}{8L}. \tag{82}$$

$\tau = 1$	$\ln \sigma = 0 \ (\sigma = 1)$			$\lambda^2 = 0.02$			$\ln T \simeq 9.704.. \ (T = 16384)$		
L	Bias	MSE	KS	Bias	MSE	KS	Bias	MSE	KS
8192	-0.04	0.27	.55	1e-04	0.003	0.26	-1.98	2.44	0

TABLE II

GMM ESTIMATION OF MRW PARAMETERS IN THE CASE $T/\tau \gg 1$ AND $L \leq T$. EACH LINE CORRESPONDS TO GMM ESTIMATION AS EXPLAINED IN SECTION IV-A ON 10000 REALIZATIONS OF DISCRETE-TIME MRW WITH $\tau = 1$ AND OF SIZE $L = 8192$. THE LAGS h_k USED FOR GMM ESTIMATION IN EQ. (66) ARE CHOSEN SUCH THAT $K = 43$ AND APPROXIMATELY LOGARITHMICALLY DISTRIBUTED BETWEEN 1 AND 150. THE MRW WERE MONTE-CARLO GENERATED USING THE ALGORITHM DESCRIBED IN SECTION II-E

Proof: One can easily prove the following general relation that gives the expectation of the empirical correlation function of a given process :

$$\mathbb{E}\left[\hat{R}_\tau[h/\tau]\right] - R_\tau(h) = -\mathbb{V}ar\left[\hat{\mu}_{\tau,L/\tau}\right] - \frac{h}{L}\left(R_\tau(h) + \mathbb{V}ar\left[\hat{\mu}_{\tau,L/\tau}\right] - 2\mathbb{C}ov\left[\hat{\mu}_{\tau,h/\tau}, \hat{\mu}_{\tau,L/\tau}\right]\right) \tag{83}$$

According to Eq. (61) of Corollary 3, under the condition $L \leq T$, the covariance function $R_\tau(h)$ for $h > 0$, in the high frequency asymptotic $\tau \to 0$, is given by

$$\lim_{\tau \to 0} R_\tau(h) = \lambda^2 \ln\left(\frac{T}{h}\right). \tag{84}$$

Moreover, from the definition of the empirical mean, one can write the following equations

$$\lim_{\tau \to 0} \mathbb{V}ar\left[\hat{\mu}_{\tau,L/\tau}\right] = \lambda^2 \int_0^L \frac{du}{L} \int_0^L \frac{dv}{L} \ln\left(\frac{T}{|u-v|}\right) + \frac{\pi^2}{8L} = \lambda^2 \ln\left(\frac{Te^{3/2}}{L}\right) + \frac{\pi^2}{8L}, \tag{85}$$

and

$$\lim_{\tau \to 0} \mathbb{C}ov\left[\hat{\mu}_{\tau,h/\tau}, \hat{\mu}_{\tau,L/\tau}\right] = \lambda^2 \int_0^h \frac{du}{h} \int_0^L \frac{dv}{L} \ln\left(\frac{T}{|u-v|}\right) + \frac{\pi^2}{8L}$$

$$= \lambda^2 \ln\left(\frac{Te^{3/2}}{L}\right) - \lambda^2 \frac{h}{2L} \ln\left(\frac{h}{L}\right) + \lambda^2 \frac{(L-h)^2}{2Lh} \ln\left(1 - \frac{h}{L}\right) + \frac{\pi^2}{8L}. \tag{86}$$

Inserting these last three equations in Eq. (83) leads to the expected result. ∎

Let us remark that $\lim_{\tau \to 0} \mathbb{E}\left[\widehat{R_\tau}[h/\tau]\right]$ does not depend on the integral scale T. This is not surprising considering the remark we just made at the beginning of this section. Now, the leading term of Eq. (82), when $L \gg h$ is

$$\lim_{\tau \to 0} \mathbb{E}\left[\widehat{R_\tau}[h/\tau]\right] \simeq \lambda^2 \ln\left(\frac{Le^{-3/2}}{h}\right). \tag{87}$$

Identifying this equation with Eq. (61), shows that we expect

(i) the estimator $\widehat{\lambda^2}$ to be unbiased and

(ii) the mean of the estimator $\widehat{\ln T}$ to be of the order of $\mathbb{E}\left[\widehat{\ln T}\right] \simeq \ln(Le^{-3/2}) = \ln(L) - 3/2$, independently of the "true" integral scale T value.

These results are illlustrated in Table II which displays the output of the GMM estimators described in the previous section. The estimations where computed using a realization of size $L = 8192$ ($\tau = 1$) of a MRW process with parameters $\sigma = 1$, $\lambda^2 = 0.02$ and $T = 16384$. The choice $T > L \gg \tau$ clearly corresponds to the high frequency regime. This table uses the same format as Table I : for each parameter ($\ln \sigma$, λ^2 and $\ln T$), the bias, the mean square error (MSE) and the p-value of the Kolmogorov Smirnov normality test (KS) are computed using a Monte-Carlo method with 10000 realizations.

Let us discuss the results obtained for the estimation of each parameter one after the other.

- $\widehat{\ln \sigma}$: As expected, the estimator of $\ln \sigma$ has both a very high bias and mean square error (it does not converge!).
- $\widehat{\ln T}$: the estimator of $\ln T$ is biased, its mean is found to be $\mathbb{E}\left[\widehat{\ln T}\right] \simeq 9.704 - 1.98 = 7.724$ which is very close to the expected order $\ln(L) - 3/2 \simeq 7.51$. This can be used as a way to detect the fact that we are in the high frequency regime.
- $\widehat{\lambda^2}$: On the contrary, the estimation of λ^2 is excellent, the bias and the MSE are of the same order as the ones obtained in Table I. In the following section we prove that an estimator of λ^2 based on the regression of the empirical covariance function of the logarithm of the increments of a MRW is an unbiased and consistent estimator.

C. A convergent estimator of λ^2 in the high frequency regime $\tau \to 0$

We consider the sequence of the absolute increments of an MRW

$$Z_\tau[k] = \ln|\delta_\tau X[k]|. \tag{88}$$

According to Theorem 8 and Proposition 10, for any integer $n > 0$ such that $n\tau < T$, one has

$$R_\tau(n\tau) = \mathbb{Cov}\big[Z_\tau[k], Z_\tau[k+n]\big] = \lambda^2 \left(\ln\left(\frac{Te^{3/2}}{n\tau}\right) - f(n)\right) + o(\lambda^2), \tag{89}$$

where $o(\lambda^2)$ depends only on λ^2 and n and where the function $f(n)$ is defined by Eq. (41). It follows that, if n and n' are two different integers such that $0 < n\tau < T$ and $0 < n'\tau < T$, the difference $R_\tau(n\tau) - R_\tau(n'\tau)$ does not depend neither on the integral scale T nor on the sampling scale τ. This naturally leads us to the estimation of λ^2 relying upon a simple regression:

$$\widehat{\lambda^2} = \frac{\widehat{R}_\tau[n] - \widehat{R}_\tau[n']}{g(n) - g(n')}, \tag{90}$$

where the empirical covariance $\widehat{R}_\tau$ is defined by (81) and the function $g(n)$ by

$$g(n) = f(n) + \ln(n). \tag{91}$$

We then have the following theorem

Theorem 10: Let n, n' two different integers such that $0 < n\tau < T$ and $0 < n'\tau < T$. In the high frequency asymptotic regime $\tau \to 0$, the estimator defined by (90) is biased with an asymptotic bias of the order of $o(\lambda^2)$. Moreover it is consistent and its variance decreases as

$$\mathbb{V}ar\big[\widehat{\lambda^2}\big] = \mathcal{O}\left(\frac{\ln(N)}{N}\right), \tag{92}$$

where $N = L/\tau$.

Proof: The proof for the first assertion is straightforward : the estimator (90) has a bias of the form

$$\mathbb{E}\left[\widehat{\lambda^2}\right] - \lambda^2 = \mathcal{O}\left(\frac{1}{N}\right) + o(\lambda^2). \tag{93}$$

The hard part of this theorem is to prove the consistency and the speed of convergence. The rigorous proof is tedious and we just give in Appendix D the main points of this proof, leaving to the reader some uninteresting and long (though straightforward) computations. ∎

V. APPLICATION TO FINANCIAL TIME-SERIES

One of the most important problem in finance is the modelling of price fluctuations of a risky asset. Since Mandelbrot famous work on the fluctuations of cotton price in early sixties [42], it is well known that speculative price variations are poorly described by the standard Geometric Brownian motion (see e.g., [43]–[47]) that does not permit to explain the well known intermittent and correlated nature of volatility variations [2], [48], [49]. During the last decade, the availability of huge data sets of high frequency time series has permitted intensive statistical studies that lead to uncover a very rich and non trivial statistical structure, that is to some degree universal across different assets. These empirical studies have suggested that financial data share many statistical properties with turbulent velocity "intermittent" fluctuations and notably display *multiscaling* properties [2], [11], [50], [51]. In that respect, the phenomenology of multifractal models [26], [34], [50], [52] has provided new concepts and tools to analyze market fluctuations and in particular the log-normal MRW disussed in this paper has been shown to account very well the return fluctuations and volatility correlations over a wide range of time horizons. In this section we use the previous GMM method to calibrate the MRW model from daily return time series. We also show that this model provide a simple way to forecast Value at Risk with better performances than classical GARCH models.

A. The financial time-series

The financial data we have used in this section are daily ($\tau = 1$) close prices of some french stocks that are part of the CAC40 french index. This index is computed using 40 of the largest french stocks of the euronext market. We only kept those with the longest historic. Thus the data consist in the close prices $P(t)$ of 29 stocks between the years 1990 and 2005[4]. Thus the time-series associated with each stock has $L = 3770$ samples. We use here the MRW process as a model for the log of the price $\ln P(t)$. Since the GMM estimation described in section IV-A is partly based on the computation of the logarithms of the increments of the MRW, in the case the close price does not change from one day to the next, one cannot compute this increment. We chose, in that case, to change the value of the second price randomly by one tick up or down (the tick is the smallest effective change of the price of a stock).

---

[4]We have adjusted these prices taking into account the dividends and the eventual splits.

	GMM					
Stock name	$\ln(\sigma)$	λ^2	$\ln(T)$	σ	T (days)
Accor	0.711	0.0327	6.792	2.0363	891
Air Liquide	0.533	0.0157	8.183	1.7039	3580
Alcatel	1.122	0.0157	19.817	3.0718	40414
Axa	0.812	0.0303	7.796	2.2529	2430
Bouygues	0.823	0.0228	8.341	2.2774	4193
Capgemini	1.093	0.0283	8.936	2.9834	7605
Carrefour	0.624	0.0183	8.220	1.8665	3716
Casino Guichard	0.626	0.0338	5.340	1.8707	209
Danone	0.420	0.0126	8.808	1.5218	6687
Essilor International	0.684	0.0266	6.384	1.9822	592
L'Oréal	0.675	0.0108	9.265	1.9641	10561
Lafarge	0.692	0.0156	7.343	1.9968	1546
Lagardere	0.908	0.0613	6.186	2.4800	486
LVMH	0.715	0.0275	7.602	2.0440	2003
Michelin	0.739	0.0174	6.579	2.0934	720
Pernod Ricard	0.639	0.0169	6.546	1.8945	697
PSA Peugeot Citroën	0.643	0.0251	5.737	1.9015	310
Pinault Printemps	0.761	0.0555	6.380	2.1411	590
Publicis	0.895	0.0473	8.317	2.4484	4092
Saint Gobain	0.720	0.0231	7.153	2.0554	1278
Sanofi-Aventis	0.709	0.0219	6.983	2.0326	1078
Schneider Electric	0.821	0.0216	6.874	2.2719	967
Société Générale	0.757	0.0219	7.993	2.1324	2959
Suez	0.719	0.0290	6.774	2.0518	875
TF1	0.920	0.0294	8.865	2.5094	7079
Thales	0.875	0.0272	6.250	2.3995	518
Total	0.594	0.0181	8.930	1.8116	7553
Vinci	0.727	0.0197	7.103	2.0688	1215
Vivendi Universal	0.888	0.0233	10.491	2.4294	35983

TABLE III

THE FIRST THREE COLUMNS CORRESPOND TO THE ESTIMATIONS OF $\ln \sigma$, λ^2 AND $\ln T$ GIVEN BY THE GMM. THE NEXT TWO COLUMNS SHOW THE CORRESPONDING VALUES OF σ AND T.

The problem of risk estimation or forecasting is essential in quantitative finance. However, several risk measures can be used. In this paper, we address two of them, which are widely used : the (historical) volatility and the Value at Risk (VaR). In any case, risk forecasting will make extensive use of the aggregation properties of section III. Before explaining how the forecasting is perfomed, one should estimate the MRW parameters.

B. Parameter estimation using GMM

In this section, we present the results of the GMM estimation as described in section IV-A. They are sum up in Table III. The first three columns show the estimations of $\ln \sigma$, λ^2 and $\ln T$ given by the GMM for each stock. And the next two columns show the corresponding values of σ and T.

One can see that, in most cases, the estimation of T lead to values greater or of the order of $L = 3770$. For those, the low frequency regime is clearly not reached. The observation scale L is not large enough compared to the integral scale. However, one can consider that the high frequency regime is reached $T \gg 1$. Thus, as shown in section IV-B, (i) the estimator of σ does not converge, (ii) the estimation of T is not reliable and depends essentially on L however, (iii) the estimation of λ^2 is reliable. The only way to get confidence intervals is to use Monte-Carlo. We computed 5% confidence interval using 15000 realizations of the MRW process with $\sigma = 1$ (we normalized the logarithm of the stock prices), $\lambda^2 = 0.02$ and $T = 3770$. We got $\widehat{\lambda^2} \in [0.013, 0.027]$, and $\widehat{T} \in [200, 250000]$. This shows that all the results in Table III are compatible (at a 5% level) with a single set of parameters : $\lambda^2 = 0.02$ and $T = 3770$.

C. Volatility forecasting

Volatility is a model dependent notion. For instance, for GARCH models [53], at a given time the conditional volatility (to all the observed past) is a deterministic number whereas for stochastic volatility models as well as for the MRW model it is a random variable. In order to compare different models on what is generally referred to as "volatility forecasting", one needs to define a common problematic. The problematic we consider here is the forecasting of absolute returns : forecasting $|\delta_s X(t_0 + h)|$ knowing all the past data $\{\delta_\tau X(t)\}_{t \leq t_0}$ (with $s \geq \tau = 1$ day). The parameter s will be referred to as the prediction scale and h as the horizon.

s	h	MRWLin	MRWSq	MRW Log
			MAE	
1 day	1 day	28	26	29
5 days	5 days	28	24	26
10 days	10 days	27	24	19
20 days	20 days	26	23	17
10 days	20 days	28	24	25
20 days	40 days	28	20	23
			MSE	
1 day	1 day	13	22	2
5 days	5 days	17	19	2
10 days	10 days	16	19	4
20 days	20 days	21	19	8
10 days	20 days	22	22	8
20 days	40 days	17	16	8

TABLE IV

FORECASTING PERFORMANCE OF THE THREE MRW-BASED METHODS (MRWLIN, MRWSQ, MRWLOG) DEPENDING ON THE ERROR CRITERION (EITHER L^2 : MSE OR L^1 : MAE) AND ON THE SCALE s AND HORIZON h FORECASTING. EACH ENTRY CORRESPONDS TO THE NUMBER OF STOCKS (OUT OF 29) FOR WHICH THE CORRESPONDING MRW-BASED FORECASTING BEAT BOTH GARCH(1,1) AND TGARCH(1,1)-BASED FORECASTING.

Three methods will be used for volatility forecasting using MRW model. The first one, referred to as MRWLin simply corresponds to solving the linear prediction problem of estimating $\delta_s X(t_0 + h)$ as a linear combination of $\{\delta_\tau X(t)\}_{t \leq t_0}$. For this purpose we need analytical expression of the mean and the auto-covariance of the increments process. This is given by Eq. (57) of Theorem 6 along with Eqs (40), (41) and (42) of Proposition 10. The second one, referred to as MRWSq, is based on the exact same equations. It simply corresponds to solving the linear prediction problem of estimating $|\delta_s X(t_0 + h)|^2$ as a linear combination of $\{|\delta_\tau X(t)|^2\}_{t \leq t_0}$. The last one, referred to as MRWLog simply corresponds to solving the linear prediction problem of estimating $\ln |\delta_s X(t_0 + h)|$ as a linear combination of $\{\ln |\delta_\tau X(t)|\}_{t \leq t_0}$. For this purpose we need an analytical expression of the mean and the auto-covariance of the logarithm of the increments process. This is given by Eqs (48) and (49) of Theorem 4 along with Eqs (40), (41) and (42) of Proposition 10.

We compare the results of the MRW-based forecasting with econometric models that are standard for volatility forecasting. We use the standard GARCH(1,1) model (with normal innovations) and the t-student GARCH(1,1) model (with t-student innovations) referred to as the tGARCH(1,1) model [54].

For all the MRW-based forecasting, we use the same two parameters $\lambda^2 = 0.02$ and $T = L = 3770$ (in the previous section, we have seen that these values were compatible with the estimations performed on most of the stocks). In order to make the GARCH-based forecasting harder to beat, we choose to estimate the GARCH parameters using maximum likelihood estimators [55] *separately* on each stock time-series, using the *entire* time-series. Thus, we perform in-sample GARCH-based forecasting and out-of-sample MRW-based forecasting. The forecasting errors are computed using both an L^2 norm (MSE) and a L^1 norm (MAE). Table IV displays the number of stocks (out of the 29) for which the corresponding MRW-based forecasting beat both the GARCH(1,1) and the tGARCH(1,1) forecasting. This is done independently for each error (MSE or MAE) and for different horizons and scales. We see that for MAE error all MRW-based forecasting clearly outperform both GARCH(1,1) forecasting at any horizon and any scale. MRWLin performs even better than the two other MRW-based methods. For MSE error, both MRWLin and MRWSq forecasting outperform both GARCH forecasting with a preference toward MRWSq (which is not surprising since the corresponding linear prediction, by definition, minimizes the mean square root error).

D. Value at Risk forecasting

Given the log return process $X(n\tau)$ (remember that $\tau = 1$ day in our case) and the present time $t_0 = n_0\tau$, the conditionnal Value at Risk VaR_p at confidence level p, at scale $s \geq \tau$ and at horizon $h \geq 0$ is defined by he relation

$$\mathbb{P}\big[\delta_s X(t_0 + h) > -\mathrm{VaR}_p(t_0)\big|X(n\tau), n \leq n_0\big] = p. \tag{94}$$

It thus corresponds to the maximum loss on a given scale and horizon and at a given confidence level. The highest it is the riskiest the asset is.

The estimation of VaR_p in the case X is an MRW process is based on Eq. (58) of Theorem 7. This equation means that the process $\ln |\delta_\tau X(n\tau)|$ can be seen (in the first order in λ^2 as the sum of the logarithm of a white gaussian noise $\epsilon[n]$ and of the renormalized magnitude which is gaussian and independant of $\epsilon[n]$. Thus, at time t_0, the conditionnal law of $|\delta_\tau X(n\tau)|$ and consequently the associated value at risk can be estimated using an estimation of the conditionnal law (i.e., the conditionnal mean and variance) of the renormalized magnitude. Conditionnal mean and variance estimations can be performed solving, as in the previous section, the linear prediction problem of estimating $\ln |\delta_s X(t_0 + h)|$ as a linear combination of

	nGARCH	tGARCH	MRW
p	$s = 1$ day,	$h = 1$ day	
0.5%	11, 15	2, 4	**24, 23**
1%	21, 21	0, 2	**21**, 20
5%	16, 13	0, 0	**22, 13**
10%	3, 5	0, 0	**24, 20**
20%	2, 2	0, 0	**25, 24**
p	$s = 1$ day,	$h = 6$ days	
0.5%	7, 10	2, 7	**26, 20**
1%	22, 15	1, 1	**22, 16**
5%	11, 7	0, 0	**20, 10**
10%	2, 3	0, 0	**21, 12**
20%	0, 1	0, 0	**25, 18**
p	$s = 1$ day,	$h = 11$ days	
0.5%	9, 10	5, 9	**26, 19**
1%	22, 15	1, 1	**24**, 14
5%	11, 6	0, 0	**20, 8**
10%	2, 3	0, 0	**22, 9**
20%	0, 0	0, 0	**26, 19**
p	$s = 5$ days,	$h = 5$ days	
0.5%	22, 21	22, 22	**23**, 21
1%	23, 24	14, 19	**26, 24**
5%	23, 24	4, 4	21, **24**
10%	19, 18	4, 4	**23, 24**
20%	14, 17	2, 7	**21, 22**
p	$s = 5$ days,	$h = 10$ days	
0.5%	25, 20	23, 18	**26**, 18
1%	26, 25	14, 17	24, 23
5%	22, 19	3, 4	**23, 24**
10%	15, 18	1, 4	**24, 24**
20%	10, 14	0, 4	**21, 22**

TABLE V

NUMBER OF STOCKS (OUT OF 29) THAT ARE ACCEPTED BY THE KUPIEC TEST (LEFT NUMBER) AND THE CHRISTOFFERSEN TEST (RIGHT NUMBER) FOR A CONFIDENCE LEVEL OF 95%. BOLD FACE NUMBERS CORRESPOND TO THE CASE WHERE THE MRW-BASED ESTIMATION PASSES THE TEST MORE TIMES THAN BOTH GARCH AND tGARCH BASED ESTIMATION.

$\{\ln |\delta_\tau X(t)|\}_{t \leq t_0}$. The prediction obtained corresponds to the conditionnal mean estimation and the variance of the prediction corresponds to the conditionnal variance estimation.

We use two different tests for testing the estimated conditonnal value at risk. They are both based on the series

$$I_p[n_0] = \begin{cases} 1, & \text{if } \delta_\tau X(n_0\tau + h) < -\text{VaR}_p, \\ 0, & \text{if } \delta_\tau X(n_0\tau + h) \geq -\text{VaR}_p, \end{cases} \tag{95}$$

One can easily show that the process $\{I_p[n]\}$ is a Bernoulli process with parameter p. Thus, the first test, generally called the Kupiec test [56], is based on the equation

$$\mathbb{P}[I_p[n] = 1] = p. \tag{96}$$

Thus, this test does not take into account the dynamic of the $\{I_p[n]\}$. The Christoffersen test [57] does. It is based on the fact that

$$\mathbb{P}[I_p[n] = 1 | I_p[n-1] = 0] = \mathbb{P}[I_p[n] = 1 | I_p[n-1] = 1] = p. \tag{97}$$

The results of these tests are illustrated in Table V for both MRW-based estimation (using the estimation of the renormalized magnitude conditionnal law as described above) and GARCH-based estimation []. As in the previous section, the MRW parameters λ^2 and T are fixed once for all ($\lambda^2 = 0.02$, $T = 3770$) whereas, the GARCH (and tGARCH) parameters are estimated using a maximum likelihood estimation independantly for *each* stock on the *entire* time-series. Each entry of Table V is composed of two integers separated by a comma. The left number is the number of stocks (out of 29) which passed the Kupiec test, the right number to the number of stocks that passed the Christoffersen test (both tests are performed using a confidence level of 95%). Bold face numbers correspond to the case where the MRW-based estimation passes the test more times than both GARCH and tGARCH based estimation.

The superiority of the MRW-based estimation over the GARCH or tGARCH-based estimation appears very clearly at all scales and horizons and at all level p.

VI. CONCLUSIONS AND PROSPECTS

In this paper, we have reviewed the main properties of log-normal continuous cascades and developed an approximation framework in the limit of small intermittency. We have shown that, within this approximation, the law of the the process increments, at each time scale, can be expressed under simple analytic forms so that the process aggregation properties are easy to control and deal with. As far as parameter estimation problems are concerned, we have pointed out that one has to distinguish "low" and "high" frequency asymptotic regimes according to which the properties of the samples are somehow different. In the "low" frequency regime, one considers samples of abritrary increasing sizes at fixed sampling frequency. In that case, we have shown that the model parameters can be estimated with a GMM method mainly relying upon empirical covariance function of log-increments of the processes. The high frequency regime corresponds to a situation when the process is observed over a finite lenght and sampled at increasing rate. This case is not equivalent to the former one and only the intermittency p aramter λ^2 can be faithfully estimated. Indeed, because of the self-similarity of the process, the integral scale is no longer a parameter and can arbitrarily chosen to be the overall sample lenght while the estimator of σ^2 converges towards a random value.

Our approach has been applied to financial time series for which it is well known that log-normal MRW provides a particularly parcimonious model that allows one to reproduce most of well documented stylized facts. At 5% confidence level, our estimates show that all the analyzed stock return series are multifractal but with a small intermittency coefficient $\lambda^2 \simeq 0.02$. The low intermittency $\lambda^2 \ll 1$ approximation is thus likely to be sound. Moreover, our estimates of T values suggest that the integral scale magnitude order is greater than one or several years. The ability of log-normal MRW to model volatility dynamics has been illustrated by its perfomances in conditional Value at Risk forecasting. From a practical point of view, the main interest of MRW-like models is that they capture the "heteroskedastic" nature of return fluctuations, by preserving, in some sense, the nice stability properties accross time scales of the Brownian motion.

Time series analysis involving multifractal processes is still in its infancy. In forthcoming studies, we will extend the approach presented here to continuous cascades with arbitrary log-infinitely divisible laws. In particular the prospect to define a semi-parametric test for the multifractal nature of a time series is very appealing. The two asymptotic regimes discussed in this paper, also challenge many interesting issues: in some recent work, we have shown that they can be described within the general framework of "mixed asymptotics". In this regime, the overall sample lenght L increases while the sampling scale $\tau \to 0$ [33]–[36]. A priori many statistics related to multifractal processes and notably the extreme value statisitics explicitly depend which "mixed asymptotics" we are, i.e., on the relative velocity according to which $L \to \infty$ and $\tau \to 0$ [33].

APPENDIX A
EXISTENCE OF THE RENORMALIZED MAGNITUDE $\Omega(t)$

Lemma 1 (Convergence of the finite dimensional laws of $\Omega_l(t)$): Let $t_1, \ldots, t_n$, n be real numbers, then the Gaussian vector $\big(\Omega_l(t_1), \ldots, \Omega_l(t_n)\big)$ converges, when l goes to zero, toward the centered Gaussian vector $\big(\Omega(t_1), \ldots, \Omega(t_n)\big)$, which does not depend on λ and which covariance matrix reads:

$$(\Sigma)_{jk} = \mathbb{C}ov\big[\Omega(t_j), \Omega(t_k)\big] = \frac{1}{\lambda^2} \int\limits_0^{t_j} du \int\limits_0^{t_k} dv \rho(u - v), \tag{98}$$

where the function $\rho(t)$ is defined by

$$\rho(t) = \begin{cases} \lambda^2 \ln\left(\frac{T}{|t|}\right), & \text{if } |t| < T, \\ 0, & \text{if } T \le |t|. \end{cases} \tag{99}$$

Proof: The function $\rho_l(t)$ being defined (cf. section II) as the correlation function of the process $\omega_{l,T}(t)$, the vector $\big(\Omega_l(t_1), \ldots, \Omega_l(t_n)\big)$ is a centered Gaussian vector which covariance matrix Σ_l is

$$(\Sigma_l)_{jk} = \mathbb{C}ov\big[\Omega_l(t_j), \Omega_l(t_k)\big] = \frac{1}{\lambda^2} \int\limits_0^{t_j} du \int\limits_0^{t_k} dv \rho_l(|u - v|), \tag{100}$$

This matrix converges toward the matrix Σ (which all coefficients are finite) when $l \to 0$. It thus suffices to show that Σ is semi-defined positive. It can be shown [32] that the function $\rho(t)$ can be written as

$$\rho(t) = \int\limits_{-\infty}^{\infty} \tilde{\rho}(t - s)\tilde{\rho}(s)ds. \tag{101}$$

If one defines the vector $V(s)$ as

$$V(s) = \Big(\int\limits_0^{t_1} \tilde{\rho}(u_1 - s)du_1, \ldots, \int\limits_0^{t_n} \tilde{\rho}(u_n - s)du_n\Big)^T \tag{102}$$

then the matrix Σ can be written as

$$\Sigma = \int\limits_{-\infty}^{\infty} V(s)V^T(s)ds, \tag{103}$$

thanks to the identity $\tilde{\rho}(s) = \tilde{\rho}(-s)$. Consequently, Σ semi-defined positive. ∎

Because the variables $\Omega_l(t)$ and $\Omega(t)$ are Gaussian, in order to show the tightness of the sequence $\{e^{2\Omega_l(t)}\}_t$, it is sufficient to show the following proposition:

Lemma 2 (Tightness): It exists $\epsilon \in [0, 1[$ such that

$$\mathbb{E}\left[\left(e^{2\Omega(t)} - e^{2\Omega(s)}\right)^2\right] = o\left((t-s)^{2-\epsilon}\right), \ \forall t, s. \tag{104}$$

Proof: A direct computation leads to the following equation:

$$\mathbb{E}\left[\left(e^{2\Omega(t)} - e^{2\Omega(s)}\right)^2\right] = e^{8\mathbb{V}ar[\Omega(t)]} + e^{8\mathbb{V}ar[\Omega(s)]} - 2e^{2\mathbb{V}ar[\Omega(t)+\Omega(s)]}$$

$$= \left(e^{4\mathbb{V}ar[\Omega(t)]} - e^{4\mathbb{V}ar[\Omega(s)]}\right)^2 + 2e^{4\mathbb{V}ar[\Omega(t)]+4\mathbb{V}ar[\Omega(t)]}\left(1 - e^{-2\mathbb{V}ar[\Omega(|t-s|)]}\right). \tag{105}$$

The first term of (105) can be estimated as

$$\begin{aligned}\left(e^{4\mathbb{V}ar[\Omega(t)]} - e^{4\mathbb{V}ar[\Omega(s)]}\right)^2 &= 64t^2 e^{8t^2 \ln\left(\frac{Te^{3/2}}{t}\right)}(t-s)^2 + o((t-s)^2)\\ &= o\left((t-s)^{2-\epsilon}\right).\end{aligned}$$

and the second term as

$$\begin{aligned}2e^{4\mathbb{V}ar[\Omega(t)]+4\mathbb{V}ar[\Omega(t)]}\left(1 - e^{-2\mathbb{V}ar[\Omega(|t-s|)]}\right) &= 2e^{8t^2 \ln\left(\frac{Te^{3/2}}{t}\right)} \ln\left(\frac{Te^{3/2}}{|t-s|}\right)(t-s)^2 + o((t-s)^2)\\ &= o\left((t-s)^{2-\epsilon}\right).\end{aligned}$$

∎

Lemma 3: If $I_1, \ldots, I_n$, be n arbitrary intervals then

$$\mathbb{E}\left[\frac{\Omega(I_1)}{|I_1|}, \ldots, \frac{\Omega(I_n)}{|I_n|}\right] = K(I_1, \ldots, I_n), \tag{106}$$

where $K(I_1, \ldots, I_n)$ reads

$$K(I_1, \ldots, I_n) = \begin{cases} \displaystyle\sum_{P(\mathcal{I}_n)} \left[\prod_{k=1}^{n/2} \int_{I_{i_k}} \frac{du_{i_k}}{|I_{i_k}|} \int_{I_{j_k}} \frac{du_{j_k}}{|I_{j_k}|} \rho(u_{i_k} - u_{j_k}), & \text{if } n \text{ is even,} \\ 0, & \text{otherwise,} \end{cases} \tag{107}$$

where $\rho(t)$ is defined by Eq (99) and $P(\mathcal{I}_n)$ is the set of all non ordinated partitions of two elements of $\mathcal{I}_n$. An element of $P(\mathcal{I}_n)$ can thus be written as $\{(i_k, j_k)\}_{k=1,\ldots,n/2}$.

Proof: This result can be obtained from a direct computation relying upon Wick's Theorem [58] and Lemma 1 of Appendix A ∎

APPENDIX B
TAYLOR EXPANSION OF THE MOMENTS OF THE MEASURE

The following proposition links the centered generalized moments of M with those of Ω. It will be used to prove limit theorems of sections III-C and III-D.

Proposition 12: Let n some positive integer. The generalized centered moment of the log-normal MRM measure of the intervals $I_1, \ldots, I_n$ admits the following Taylor series expansion when $\lambda^2 \to 0$:

$$\mathbb{E}\left[\left(\frac{M(I_1)}{|I_1|} - 1\right) \cdots \left(\frac{M(I_n)}{|I_n|} - 1\right)\right] = 2^n \lambda^n \mathbb{E}\left[\frac{\Omega(I_1)}{|I_1|}, \ldots, \frac{\Omega(I_n)}{|I_n|}\right] + o(\lambda^n), \tag{108}$$

where Ω is the renormalized magnitude defined in Section III-B.

Proof: Let us note that the right handside of Eq. (108) is given by Lemma 3 of Appendix A. We are going to prove that the left handside of Eq. (108) is equal to the same expression.

Let us first indroduce the following random variables:

$$M_j = \frac{M(I_j)}{|I_j|}, \ \text{pour } j = 1, \ldots, n. \tag{109}$$

The centered generalized moment corresponding to intervals $I_1, \ldots, I_n$ can be written as a linear combination of generalized moments:

$$\mathbb{E}\left[(M_1 - 1) \cdots (M_n - 1)\right] = \sum_{m=0}^{n} (-1)^{n-m} \sum_{\mathcal{I}_m \subset \mathcal{I}_n} \mathbb{E}\left[M_{i_1} \cdots M_{i_m}\right]. \tag{110}$$

From integral representation (121), we have:

$$\mathbb{E}\left[(M_1 - 1) \cdots (M_n - 1)\right] = \sum_{m=0}^{n} (-1)^{n-m} \sum_{\mathcal{I}_m \subset \mathcal{I}_n} \int_{I_{i_1}} \frac{du_{i_1}}{|I_{i_1}|} \cdots \int_{I_{i_m}} \frac{du_{i_m}}{|I_{i_m}|} e^{\lambda^2 \mathcal{S}(\mathcal{I}_m)}, \tag{111}$$

where $\mathcal{S}(\mathcal{I}_m)$ is the following symmetric sum;

$$\mathcal{S}(\mathcal{I}_m) = \sum_{\substack{i_j, i_k \in \mathcal{I}_m \\ i_j < i_k}} X_{i_j i_k}, \tag{112}$$

with

$$X_{i_j i_k} = 4 \ln\left(\frac{T}{|u_{i_j} - u_{i_k}|}\right) \mathbf{1}_{\{|u_{i_j} - u_{i_k}| \leq T\}}. \tag{113}$$

It is possible to integrate previous expression as respect $u_1, \ldots, u_n$:

$$\int_{I_{i_1}} \frac{du_{i_1}}{|I_{i_1}|} \cdots \int_{I_{i_m}} \frac{du_{i_m}}{|I_{i_m}|} e^{\lambda^2 \mathcal{S}(\mathcal{I}_m)} = \int_{I_1} \frac{du_1}{|I_1|} \cdots \int_{I_n} \frac{du_n}{|I_n|} e^{\lambda^2 \mathcal{S}(\mathcal{I}_m)}. \tag{114}$$

and changing the order of integration leads to:

$$\mathbb{E}\left[(M_1 - 1) \cdots (M_n - 1)\right] = \int_{I_1} \frac{du_1}{|I_1|} \cdots \int_{I_n} \frac{du_n}{|I_n|} \sum_{m=0}^{n} (-1)^{n-m} \sum_{\mathcal{I}_m \subset \mathcal{I}_n} e^{\lambda^2 \mathcal{S}(\mathcal{I}_m)}. \tag{115}$$

In [32] it is shown that the generalized moments as functions of λ^2 belong to $C^{\frac{n}{2}}(\mathbf{R})$ the class of $\frac{n}{2}$ times continuously differentiable functions. The k-th derivative of the generalized centered moment as respect to λ^2 in $\lambda^2 = 0$ reads:

$$\left(\frac{\partial}{\partial \lambda^2}\right)^k \mathbb{E}\left[(M_1 - 1) \cdots (M_n - 1)\right]\Big|_{\lambda^2 = 0} = \int_{I_1} \frac{du_1}{|I_1|} \cdots \int_{I_n} \frac{du_n}{|I_n|} \sum_{m=0}^{n} (-1)^{n-m} \sum_{\mathcal{I}_m \subset \mathcal{I}_n} \left(\mathcal{S}(\mathcal{I}_m)\right)^k \tag{116}$$

Let us consider some arbitrary integer j. One can regroup the terms under the integral in (116)

$$\sum_{m=0}^{n} (-1)^{n-m} \sum_{\mathcal{I}_m \subset \mathcal{I}_n} \left(\mathcal{S}(\mathcal{I}_m)\right)^k = \sum_{m=0}^{n-1} (-1)^{n-m} \sum_{\mathcal{I}_m \subset \mathcal{I}_n \setminus \{j\}} \left(\mathcal{S}(\mathcal{I}_m)\right)^k - \left(\mathcal{S}(\mathcal{I}_m \cup \{j\})\right)^k \tag{117}$$

By noticing that the symmetric sum $\mathcal{S}(\mathcal{I}_m \cup \{j\})$ can be rewritten as

$$\mathcal{S}(\mathcal{I}_m \cup \{j\}) = \mathcal{S}(\mathcal{I}_m) + \mathcal{C}(j, \mathcal{I}_m), \tag{118}$$

where $\mathcal{C}(j, \mathcal{I}_m)$ is defined as

$$\mathcal{C}(j, \mathcal{I}_m) = \sum_{i_k \in I_m} X_{j i_k}, \tag{119}$$

we can see that

$$\left(\mathcal{S}(\mathcal{I}_m)\right)^k - \left(\mathcal{S}(\mathcal{I}_m \cup \{j\})\right)^k = -\mathcal{C}(j, \mathcal{I}_m) \sum_{i=1}^{k} \frac{k!}{i!(k-i)!} \left(\mathcal{C}(j, \mathcal{I}_m)\right)^{i-1} \left(\mathcal{S}(\mathcal{I}_m)\right)^{k-i}. \tag{120}$$

This last relationship means that each term of the sum (117) contains at least one factor like $X_{j i_m}$. Because j is arbitrarily fixed, the sets of factor indices of each term of the sum (117) must contain all indices $1, \ldots, n$. Therefore, if the derivative order k is smaller than $n/2$, the sum (117) must vanish. It results that the first non trivial order in the Taylor series in power of λ^2 of the centered generalized moment is at least $\frac{n+1}{2}$ if n est odd, and $\frac{n}{2}$ if n is even. In this latter case, the first order terms in the Taylor series are proportionnal to $\lambda^n X_{i_1 j_1} \cdots X_{i_{n/2} j_{n/2}}$, where the set of indices $\{i_1, j_1, \ldots, i_{n/2}, j_{n/2}\}$ contains all the values $1, \ldots, n$. Such terms can only come from the expansion of $e^{\lambda^2 \mathcal{S}(\mathcal{I}_n)}$, which leads to the expected result. ∎

The following Lemma will be used to prove the limit theorem of Section III-E.

Lemma 4: For some arbitrary intervals $I_1 \ldots, I_n$, the generalized moments of a log-normal MRM measure has the following integral representation:

$$\mathbb{E}\left[\frac{M(I_1)}{|I_1|} \cdots \frac{M(I_n)}{|I_n|}\right] = \int\limits_{I_1} \frac{du_1}{|I_1|} \cdots \int\limits_{I_n} \frac{du_n}{|I_n|} f_{\lambda^2}(u_1, \ldots, u_n), \tag{121}$$

where the function $f_{\lambda^2}(u_1, \ldots, u_n)$ is

$$f_{\lambda^2}(u_1, \ldots, u_n) = \exp\left(4\lambda^2 \sum_{1 \leq i < j \leq n} \rho(u_i u_j)\right), \tag{122}$$

where $\rho(t)$ is defined by Eq (99).

Proof: This directly results from the fact that $\lim_{l \to 0+} \mathbb{E}[M_{l,T}[0,t]] = \mathbb{E}[M_T[0,t]] = t$ [14] and some simple algebra (see [32]). ∎

For the sake of simplicity let us introduce the following set of indices:

$$\mathcal{I}_m = \{i_1, \ldots, i_m\} \subset \mathcal{I}_n = \{1, \ldots, n\}. \tag{123}$$

APPENDIX C
TAYLOR EXPANSION OF THE MOMENTS OF THE LOGARITHM OF THE MEASURE

In this section we establish some results useful to prove limit theorems of Sections III-D and III-C. According to Proposition 5, all the moments of the MRM of negative orders are finite, consequently the moments of the logarithm of the MRM are also finite.

Along the same line as in previous appendix, one can write an expansion for the generalized moment of the logarithm of a log-normal MRM measure.

Proposition 13: Let n be a positive integer. The generalized centered moment of the logarithm of the MRM measure of intervals $I_1, \ldots, I_n$ admits the following Taylor series expansion around $\lambda^2 = 0$:

$$\mathbb{E}\left[\ln\left(\frac{M(I_1)}{|I_1|}\right) \cdots \ln\left(\frac{M(I_n)}{|I_n|}\right)\right] = \mathbb{E}\left[\left(\frac{M(I_1)}{|I_1|} - 1\right) \cdots \left(\frac{M(I_n)}{|I_n|} - 1\right)\right] + o(\lambda^n), \tag{124}$$

where $K(I_1, \ldots, I_n)$ are defined Eq. (107). Let us recall that $K(I_1, \ldots, I_n) = 0$ if n is odd.

The proof of this Proposition is postponed to the end of this section, it is based on the following Lemma 5, 6 and 7 :

Lemma 5: Let $0 < \epsilon < 1$ and m be a positive integer. One has the following inequality:

$$\mathbb{P}[M \notin B_\epsilon] \leq o(\lambda^{2m}), \tag{125}$$

where the compact subset of $\mathbb{R}^n$, B_ϵ is defined by:

$$B_\epsilon = \left\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n; \max_{1 \leq k \leq n}(|x_k - 1|) \leq \epsilon\right\}. \tag{126}$$

and $o(\lambda^{2m})$ depends on de ϵ.

Proof: Since $\lambda^2 \to 0$, then, without loss of generality, one can assume that the order $2m + 2$ centered moment of the log-normal MRM measure exists (see Theorem 3). Thanks to Bienaymé-Tchebychev inequality, one has:

$$\mathbb{P}[M \notin B_\epsilon] \leq \sum_{j=1}^n \mathbb{P}[|M_j - 1| > \epsilon] \leq \sum_{j=1}^n \frac{1}{\epsilon^{2m+2}} \mathbb{E}[(M_j - 1)^{2m+2}] = o(\lambda^{2m}). \tag{127}$$

∎

Lemma 6: Let $0 < \epsilon < 1$. For all continuous function $f(M)$ over the compact set B_ϵ defined in (126), the following inequality holds:

$$\mathbb{E}[f(M)] - \mathbb{E}[f(M)|M \in B_\epsilon] \leq \sqrt{\mathbb{E}[f(M)^2]}\sqrt{\mathbb{P}[M \notin B_\epsilon]} + \left(\sup_{M \in B_\epsilon} |f(M)|\right)\mathbb{P}[M \notin B_\epsilon]. \tag{128}$$

Proof: From the law of total probabilities, it follows that

$$\begin{aligned}
\mathbb{E}[f(M)] - \mathbb{E}[f(M)|M \in B_\epsilon] &= \mathbb{E}[f(M)|M \notin B_\epsilon]\mathbb{P}[M \notin B_\epsilon] + \mathbb{E}[f(M)|M \in B_\epsilon](\mathbb{P}[M \in B_\epsilon] - 1) \\
&= \mathbb{E}[f(M)|M \notin B_\epsilon]\mathbb{P}[M \notin B_\epsilon] - \mathbb{E}[f(M)|M \in B_\epsilon]\mathbb{P}[M \notin B_\epsilon],
\end{aligned}$$

where the first term can be bounded using Cauchy-Schwartz inequality

$$\mathbb{E}[f(M)|M \notin B_\epsilon]\mathbb{P}[M \notin B_\epsilon] = \mathbb{E}[f(M)\mathbf{1}_{\{M \notin B_\epsilon\}}] \leq \sqrt{\mathbb{E}[f(M)^2]}\sqrt{\mathbb{P}[M \notin B_\epsilon]}, \tag{129}$$

whereas the second term is bounded by the supremum over the compact set B_ϵ

$$\mathbb{E}\left[f(M)\big|M \in B_\epsilon\right]\mathbb{P}\left[M \notin B_\epsilon\right] \leq \left(\sup_{M \in B_\epsilon}|f(M)|\right)\mathbb{P}\left[M \notin B_\epsilon\right]. \tag{130}$$

∎

Lemma 7: Let n be a positive integer. In the compact set B_ϵ, with $\epsilon < 1$, one has the following identity of Taylor series expansions up to order n around $\lambda^2 = 0$:

$$\mathbb{E}\left[\prod_{j=1}^{n}\ln(M_j)\Big|M \in B_\epsilon\right] = \mathbb{E}\left[\prod_{j=1}^{n}(M_j - 1)\Big|M \in B_\epsilon\right] + o(\lambda^n) \tag{131}$$

Proof: Thanks to the identity $\ln(x) = x - 1 - (x - 1 - \ln(x))$, one has

$$\prod_{j=1}^{n}\ln(M_j) - \prod_{j=1}^{n}(M_j - 1) = \prod_{j=1}^{n}\left(M_j - 1 - (M_j - 1 - \ln(M_j))\right) - \prod_{j=1}^{n}(M_j - 1). \tag{132}$$

The expansion of the first product of r.h.s. leads to a linear combination of terms such as $\prod_{j=1}^{k}(M_{i_j} - 1 - \ln(M_{i_j}))\prod_{j=k+1}^{n}(M_{i_j} - 1)$ with $1 \leq k \leq n$ which conditional expectation can be bounded using Cauchy-Schwartz inequality and the following inequality established in [32]:

$$\frac{\epsilon - \ln(1 + \epsilon)}{\epsilon^2}x^2 \leq x - \ln(1 + x) \leq -\frac{\epsilon + \ln(1 - \epsilon)}{\epsilon^2}x^2, \text{ for all } 0 < \epsilon < 1 \text{ and } x \in [-\epsilon, \epsilon]. \tag{133}$$

. It results that

$$\mathbb{E}\left[\prod_{j=1}^{k}(M_{i_j} - 1 - \ln(M_{i_j}))\prod_{j=k+1}^{n}(M_{i_j} - 1)\Big|M \in B_\epsilon\right]$$

$$\leq \mathbb{E}\left[\prod_{j=1}^{k}(M_{i_j} - 1 - \ln(M_{i_j}))^2\Big|M \in B_\epsilon\right]^{1/2}\mathbb{E}\left[\prod_{j=k+1}^{n}(M_{i_j} - 1)^2\Big|M \in B_\epsilon\right]^{1/2}$$

$$\leq C_\epsilon\mathbb{E}\left[\prod_{j=1}^{k}(M_{i_j} - 1)^4\Big|M \in B_\epsilon\right]^{1/2}\mathbb{E}\left[\prod_{j=k+1}^{n}(M_{i_j} - 1)^2\Big|M \in B_\epsilon\right]^{1/2}. \tag{134}$$

Using (142), with $n = 2$ and $n = 4$, it is possible to remove the condition $M \in B_\epsilon$

$$\mathbb{E}\left[\prod_{j=1}^{k}(M_{i_j} - 1 - \ln(M_{i_j}))\prod_{j=k+1}^{n}(M_{i_j} - 1)\Big|M \in B_\epsilon\right]$$

$$= \left(\mathcal{O}(\lambda^{4k}) + o(\lambda^{4n})\right)^{1/2}\left(\mathcal{O}(\lambda^{2n-2k}) + o(\lambda^{2n})\right)^{1/2} = \mathcal{O}(\lambda^{n+k}) = o(\lambda^n). \tag{135}$$

∎

We are now ready to give the proof of Proposition 13:

Proof of Proposition 13: For simplicity purpose let us introduce the random variables:

$$M_j = \frac{M(I_j)}{|I_j|}, \text{ for } j = 1, \ldots, n. \tag{136}$$

The major difficulty of the proof relies in the fact that the Taylor series of $\ln(1 + x)$ around $x = 0$ converges in the interval $(-1, 1)$. Let $0 < \epsilon < 1$ and let us consider thevector $M = (M_1, \ldots, M_n)$ and the compact set B_ϵ defined in (126). Let us also define the two remaining parts $R^{\ln}(\lambda)$ and $R^c(\lambda)$ as:

$$R^{\ln}(\lambda) = \mathbb{E}\left[\ln(M_1)\cdots\ln(M_n)\right] - \mathbb{E}\left[\ln(M_1)\cdots\ln(M_n)\big|M \in B_\epsilon\right], \tag{137}$$

$$R^c(\lambda) = \mathbb{E}\left[(M_1 - 1)\cdots(M_n - 1)\right] - \mathbb{E}\left[(M_1 - 1)\cdots(M_n - 1)\big|M \in B_\epsilon\right]. \tag{138}$$

By applying Lemma 6 to $R^{\ln}(\lambda)$ and $R^c(\lambda)$, it follows

$$\left|R^{\ln}(\lambda)\right| \leq \sqrt{\mathbb{E}\left[\ln(M_1)^2\cdots\ln(M_n)^2\right]}\sqrt{\mathbb{P}\left[M \notin B_\epsilon\right]} + |\ln(1 - \epsilon)|^n\mathbb{P}\left[M \notin B_\epsilon\right], \tag{139}$$

$$\left|R^c(\lambda)\right| \leq \sqrt{\mathbb{E}\left[(M_1 - 1)^2\cdots(M_n - 1)^2\right]}\sqrt{\mathbb{P}\left[M \notin B_\epsilon\right]} + \epsilon^n\mathbb{P}\left[M \notin B_\epsilon\right]. \tag{140}$$

Using the analytical expression of the q-order moment of M (cf Proposition 8), one shows that the expectation term in (139) is uniformely bounded in λ^2, for λ^2 smaller than a (small enough) given λ_0^2. In [32], it is shown that the the expectation term in (140) is also uniformely bounded for $\lambda^2 < \lambda_0^2$. Using Lemma 5 with $m = n$, one gets the following inequalities:

$$\left| R^{\ln}(\lambda) \right| \leq C\lambda^{n+1}, \tag{141}$$

$$\left| R^{c}(\lambda) \right| \leq C\lambda^{n+1}, \tag{142}$$

where C depends on ϵ.

According to 7, from definitions (137) and (138) and the bounds (141) and (142), we have:

$$\mathbb{E}\left[\ln(M_1)\cdots\ln(M_n)\right] - \mathbb{E}\left[(M_1 - 1)\cdots(M_n - 1)\right] = o(\lambda^n). \tag{143}$$

∎

APPENDIX D
PROOF OF THE CONSISTENCY OF THE ESTIMATOR (90) IN THE HIGH FREQUENCY REGIME

In this section we provide of "simplified" proof of the consistency of the estimator defined in Eq. (90). More presisely, we prove the second part of theorem 10 claiming that if n, n' are different integers such that $0 < n\tau < T$ and $0 < n'\tau < T$, then in the high frequency asymptotic regime $\tau \to 0$, the estimator defined by (90) is consistent and its variance decreases as

$$\mathbb{V}ar\left[\widehat{\lambda^2}\right] = \mathcal{O}\left(\frac{\ln(N)}{N}\right), \tag{144}$$

(with $N = L/\tau$).

Proof: In the following we set

$$Z_{l,\tau}[n] = \ln|\delta_\tau \tilde{X}_{l,T}(n\tau)|, \tag{145}$$

where $\tilde{X}_{l,T}$ is the linear-wise process defined in Section II-E. It is defined by Eq. (35) which uses the measure $\tilde{M}_{l,T}$ defined by (34). Let us recall that $\tilde{X}_{l,\tau}(t)$ (resp. $\tilde{M}_{l,T}(dt)$) converges towards $X_T(t)$ (resp. $M_T(dt)$) when $l \to 0$ [14]. With no loss of generality, we choose l such that l/τ is an integer. Notice that according to Eq. (35) one has:

$$\{\delta_\tau \tilde{X}_{l,T}(n\tau)\}_n \underset{Law}{=} \{\epsilon[n]\sqrt{\delta_\tau \tilde{M}_{l,T}(n\tau)}\}_n, \tag{146}$$

where $\epsilon[n]$ is a gaussian white noise independent. It results that

$$Z_{l,\tau}[n] \underset{Law}{=} \frac{1}{2}\ln \delta_\tau \tilde{M}_{l,T}(n\tau) + \ln|\epsilon[n]|, \tag{147}$$

and since we will consider below (empirical) covariances of $Z[n]$ for lags $n > 0$, we will not longer take care of the terms $\ln|\epsilon[n]|$.

The empirical covariance function involving the cut-off scale l will de denoted as $\widehat{R}_{l,\tau}[n]$:

$$\widehat{R}_{l,\tau}[n] = \frac{1}{N}\sum_{k=1}^{N-k} Z_{l,\tau}[k]Z_{l,\tau}[k+n] - \left(\frac{1}{N}\sum_{k=1}^{N} Z_{l,\tau}[k]\right)^2. \tag{148}$$

and naturally

$$\widehat{R}_{l,\tau}[n, n'] = \widehat{R}_{l,\tau}[n] - \widehat{R}_{l,\tau}[n']. \tag{149}$$

A little algebra is sufficient to establish that

$$\mathbb{V}ar\left[\widehat{R}_{l,\tau}[n, n']\right] = N^{-2}\sum_{j,k} \mathbb{C}ov\left[Z_{l,\tau}[j]Z_{l,\tau}[j+n], Z_{l,\tau}[k]Z_{l,\tau}[k+n]\right] + \mathbb{C}ov\left[Z_{l,\tau}[j]Z_{l,\tau}[j+n'], Z_{l,\tau}[k]Z_{l,\tau}[k+n']\right]$$

$$- 2\mathbb{C}ov\left[Z_{l,\tau}[j]Z_{l,\tau}[j+n], Z_{l,\tau}[k]Z_{l,\tau}[k+n']\right] \tag{150}$$

Therefore, in order to control the variance of the estimator one needs to control each term involved in the previous equation. Let us introduce some additional notations. Let $[t_0, t_1]$ some time interval such that t_0/l and t_1/l are integers and such that $|t_1 - t_0| > 2l$. For all $t = t_0 + jl$ with $0 \leq j \leq (t_1 - t_0)/l$, we will decompose $\omega_{l,T}(t) = \mathcal{P}(\mathcal{A}_{l,T}(t))$ as:

$$\omega_l(t) = O_{[t_0,t_1]} + S^l_{[t_0,t_1]}(j) \tag{151}$$

where $O_{[t_0,t_1]}$ and $S^l_{[t_0,t_1]}(j)$ are independant gaussian process defined by

$$O_{[t_0,t_1]} = \frac{1}{2}\mathcal{P}\left(\bigcap_k \mathcal{A}_{l,T}(t_0 + kl)\right)$$

$$S^l_{[t_0,t_1]}(j) = \frac{1}{2}\mathcal{P}\left(\mathcal{A}_{l,T}(t_0 + jl) \setminus \bigcap_k \mathcal{A}_{l,T}(t_0 + kl)\right)$$

Let us now consider two disjoint intervals $[t_0, t_1]$ and $[s_0, s_1]$ and let δ the distance between the middle of each interval: $\delta = (s_0 + s_1 - t_0 - t_1)/2$. Then after some algebra it can be shown that there exists two spherical gaussian vectors[5] $d^l_{[t_0, t_1]}$ (of dimension $3(t_1 - t_0)/l$) and $r^l_{[s_0, s_1]}$ (of dimension $3(s_1 - s_0)/l$) such that, $\forall (j, k) \in [0, (t_1 - t_0)/l] \times [0, (s_1 - s_0)/l]$, $S^l_{[t_0, t_1]}(j)$ is a j-dependent linear combination of the components of $d^l_{[t_0, t_1]}$ while $S^l_{[s_0, s_1]}(k)$ is a k-dependent linear combination of the components of $r^l_{[s_0, s_1]}$. Moreover, when $\delta \gg \sup(|s_1 - s_0|, |t_1 - t_0|)$, one has

$$
\begin{aligned}
\mathbb{C}\text{ov}\left[d^l_{[t_0,t_1]}[k], r^l_{[s_0,s_1]}[i]\right] &= \mathcal{O}\left(\frac{l^2}{\delta^2}\right) \text{ if } 0 \le k \le (t_1 - t_0)/l \text{ and } 0 \le i \le (s_1 - s_0)/l \\
\mathbb{C}\text{ov}\left[d^l_{[t_0,t_1]}[k], r^l_{[s_0,s_1]}[i]\right] &= 0 \text{ otherwise} \\
\mathbb{C}\text{ov}\left[d^l_{[t_0,t_1]}[k], O_{[s_0,s_1]}\right] &= \mathcal{O}\left(\frac{l}{\delta}\right) \text{ if } 0 \le k \le (t_1 - t_0)/l \\
\mathbb{C}\text{ov}\left[d^l_{[t_0,t_1]}[k], O_{[t_0,t_1]}\right] &= 0 \text{ otherwise} \\
\mathbb{C}\text{ov}\left[r^l_{[t_0,t_1]}[k], O_{[t_0,t_1]}\right] &= \mathcal{O}\left(\frac{l}{\delta}\right) \text{ if } 0 \le i \le (s_1 - s_0)/l \\
\mathbb{C}\text{ov}\left[d^l_{[t_0,t_1]}[k], O_{[t_0,t_1]}\right] &= 0 \text{ otherwise} \\
\mathbb{C}\text{ov}\left[O_{[t_0,t_1]}, O_{[s_0,s_1]}\right] &\sim R_\tau(\frac{\delta}{\tau})
\end{aligned}
\tag{152}
$$

Let some integer $n > 0$, according to these notations, $Z_{l,\tau}[k] Z_{l,\tau}[k+n]$ can be rewritten as:

$$
Z_{l,\tau}[k] Z_{l,\tau}[k+n] = O^2_{[(k-1)\tau, (k+n)\tau]} + O_{[(k-1)\tau, (k+n)\tau]} f_1(r^l_{[(k-1)\tau, (k+n)\tau]}) + f_2(r^l_{[(k-1)\tau, (k+n)\tau]})
\tag{153}
$$

where f_1 and f_2 are two non-linear functions of the spherical noise $r^l_{[(k-1)\tau, (k+n)\tau]}$ that can be shown to have a second moment that is bounded uniformely in l. If one uses the decomposition (153) in expression (150), each covariance term in (150) will give 9 terms that can be of 6 different forms: If one denotes I_j the interval $[j\tau, (j+n)\tau]$ or $[j\tau, (j+n')\tau]$:

$$
\begin{aligned}
&(i) \ N^{-2} \mathbb{C}ov\left[O^2_{I_j}, O^2_{I_k}\right] \\
&(ii) \ N^{-2} \mathbb{C}ov\left[O^2_{I_j}, O_{I_k} f_1(r^l_{I_k})\right] \\
&(iii) \ N^{-2} \mathbb{C}ov\left[O_{I_j} f_1(d^l_{I_j}), O_{I_k} f_1(r^l_{I_k})\right] \\
&(iv) \ N^{-2} \mathbb{C}ov\left[f_2(d^l_{I_j}), f_2(r^l_{I_k})\right] \\
&(v) \ N^{-2} \mathbb{C}ov\left[O_{I_j} f_1(d^l_{I_j}), f_2(r^l_{I_k})\right] \\
&(vi) \ N^{-2} \mathbb{C}ov\left[O^2_{I_j}, f_2(r^l_{I_k})\right]
\end{aligned}
$$

In order to prove the consistency of the estimator, we have to prove that the contribution of each term (i)-(vi) vanishes when $N \to \infty$ (after taking the limit $l \to 0$). We will just explain how to take care of the terms of type (iv). The other terms are dealt with in the same way. The terms of type (iv) are of the form

$$
N^{-2} \sum_{j,k} \mathbb{C}ov\left[O^2_{I_j}, f_2(r^l_{I_k})\right]
\tag{154}
$$

We need the following technical Proposition proved in [41] concerning the covariance of non-linear function of gaussian vectors:

Proposition 14: Let $\mathbf{A}$ and $\mathbf{B}$ be two spherical gaussian vectors which dimensions are respectively p and q. Let us denote C_{kl} the cross covariance between components: $C_{kl} = \mathbb{C}ov[A_k, B_l]$ and $\rho = \sum_{i=1}^{p} \sum_{j=1}^{q} |C_{ij}|$. Let $f : \mathbb{R}^p \to \mathbb{R}$ and $g : \mathbb{R}^q \to \mathbb{R}$ be two non-linear functions such that:

$$
\mathbb{E}\left[f(\mathbf{A})^2\right] + \mathbb{E}\left[g(\mathbf{B})^2\right] < \infty
\tag{155}
$$

then, if $\rho < 1/2$ we have:

$$
\mathbb{C}ov\left[f(\mathbf{A}), g(\mathbf{B})\right] \le 2 \left(\mathbb{E}\left[f(\mathbf{A})^2\right] \mathbb{E}\left[g(\mathbf{B})^2\right]\right)^{1/2} \rho
\tag{156}
$$

Taking care of the term (154) is a special case of the previous Lemma with $p = 1$ and $q = |I_k|/l$. Accordingly and using (152) one has (since $|I_k| \sim \tau$)

$$
\rho = \sum_{n=0}^{|I_k|/l} \mathcal{O}(\frac{l}{|j-k|\tau}) = \mathcal{O}(\frac{1}{|j-k|})
\tag{157}
$$

---

[5]A spherical gaussian vector is a vector made of independant standard normal variates

and therefore, there exists $K < \infty$, such that, uniformely in l,

$$\mathbb{C}ov\big[O_{I_j}^2, f_2(r_{I_k}^l)\big] \leq \frac{K}{|j-k|} \tag{158}$$

Let us now remark that in the double sum (150), since $Z_\tau = \ln|\delta_\tau X_T|$ admits a finite four order moment, one has just to consider the case where (j,k) are such that $|j-k| > N^\nu$ with $\nu < 1$ because the contribution of terms $|j-k| < N^\nu$ can be bounded as

$$N^{-2}N^{1+\nu}\mathbb{E}\big[Z_\tau^4\big] = \mathcal{O}(N^{\nu-1}) \tag{159}$$

which converges to zero when $N \to \infty$. Hence, one can choose some $\nu > 0$ in order that $\sup(n,n') \ll |j-k|$. Under that condition, the term (154) can be bounded, uniformely in l by $K'\frac{\ln N}{N}$ which vanishes in the limit $N \to +\infty$.

The same kind of computation can be lead for each of terms like (ii)-(v) and we do not report the details here for the sake of concision. The only problem remains for (i) terms. But in that case, since the random variables O are gaussian wich covariance is nothing but $R_\tau[|j-k|]$, thanks to Wick's theorem the covariance of products of O can be expressed in terms of R_τ and it can be shown, that the main remaining contribution can be bounded like:

$$\frac{2}{N^2}\sum_{|k|<N}(N-|k|)R_\tau[k](R_\tau[k] - R_\tau[k+n-n'] \sim \frac{\ln(N)}{N}. \tag{160}$$

whith some constant the depends only on n and n'. Finally, by merging all contributions together we have proved that there exists a constant C that depends only on n and n' such that:

$$\mathbb{V}ar\big[\widehat{R}_{l,\tau}[n,n']\big] \leq C\frac{\ln N}{N} \tag{161}$$

and by taking the limit $l \to 0$ one obtains (144) and the consistency of the estimator.

∎

References

[1] U. Frisch, *Turbulence*. Cambridge: Cambridge Univ. Press, 1995.

[2] J. P. Bouchaud and M. Potters, *Theory of Financial Risk and Derivative Pricing*. Cambridge: Cambridge University Press, 2003.

[3] R. N. Mantegna and H. E. Stanley, *An introduction to econophysics*. Cambridge: Cambridge University Press, 2000.

[4] Parks and W. Willinger, *Wavelets for the analysis, estimation and synthesis of scaling data*. New-York: Wiley, 1998.

[5] B. West, *Fractal Physiology and Chaos in Medecine*. Singapore: World Scientific, 1990.

[6] B. West and W. Deering, "Fractal physiology for physicists," *Phys. Rep.*, vol. 254:1, 1994.

[7] G. G. W. J. Kanellopoulos and J. Megier, *Fractals in Geoscience and Remote Sensing*. Bruxelles, Luxembourg: Image Understanding Research Series, 1995, vol. 1.

[8] A. N. Kolmogorov, "A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high reynolds number," *Journal of Fluid Mechanics*, vol. 13, pp. 82–85, 1962.

[9] B. B. Mandelbrot, "Intermittent turbulence in self-similar cascades: divirgence of high moments and dimension of the carrier," *Journal of Fluid Mechanics*, vol. 62, pp. 331–358, 1974.

[10] ——, "Multiplications aléatoires itérées et distributions invariantes par moyenne pondérée aléatoire," *C.R. Acad. Sci. Paris*, vol. 278, pp. 289–292, 1974.

[11] J. F. Muzy, J. Delour, and E. Bacry, "Modelling fluctuations of financial time series: from cascade process to stochastic volatility model," *Eur. J. Phys. B*, vol. 17, pp. 537–548, 2000.

[12] J. Barral and B. B. Mandelbrot, "Multifractal products of cylindrical pulses," *Prob. Theory and Relat. Fields*, vol. 124, pp. 409–430, 2002.

[13] J. F. Muzy and E. Bacry, "Multifractal stationary random measures and multifractal random walks with log-innitely divisible scaling laws," *Phys. Rev. E*, vol. 66, pp. 056 121–056 150, 2002.

[14] E. Bacry and J. F. Muzy, "Log-infinitely divisible multifractal process," *Comm. in Math. Phys.*, vol. 236, pp. 449–475, 2003.

[15] J. P. Kahane and J. Peyrière, "Sur certaines martingales de benoit mandelbrot," *Adv. in Mathematics*, vol. 22, pp. 131–145, 1976.

[16] Y. Guivarc'h, "Remarques sur les solutions d'une équation fonctionnelle non linéaire de b. mandelbrot," *C.R. Acad. Sci. Paris*, vol. 305, pp. 139–141, 1987.

[17] G. M. Molchan, "Scaling exponents and multifractal dimensions for independent random cascades," *Comm. in Math. Phys.*, vol. 179, pp. 681–702, 1996.

[18] ——, "Turbulent cascades: Limitations and a statistical test of the lognormal hypothesis," *Phys. Fluids*, vol. 9, pp. 2387–2395, 1997.

[19] Q. Liu, "An extension of a functional equation of poincaré and mandelbrot," *Asian Journal of Mathematics*, vol. 6, pp. 145–168, 2002.

[20] P. Chainais, R. Riedi, and P. Abry, "Scale invariant infinitely divisible cascades," 2003, in Int. Symp. on Physics in Signal and Image Processing, Grenoble, France.

[21] J. Barral, "Non-degeneracy, moments, dimension, and multifractal analysis for random multiplicative measures (random multiplicative multifractal measures, part iii)," vol. 72, AMS, Providence, 2004, pp. 53–90.

[22] J. Barral and B. B. Mandelbrot, "Non-degeneracy, moments, dimension, and multifractal analysis for random multiplicative measures (random multiplicative multifractal measures, part ii)," *Proc. Symp. Pures Math.*, vol. 72, pp. 17–52, 2004.

[23] M. Ossiander and E. C. Waymire, "Statistical estimation for multiplicative cascades," *The Ann. of Stat.*, vol. 28, pp. 1533–1560, 2000.

[24] S. Resnick, G. Samorodnitsky, A. Gilbert, and W. Willinger, "Wavelet analysis of conservative cascades," *Bernoulli*, vol. 9, pp. 97–135, 2003.

[25] T. Lux, "The multi-fractal model of asset returns: Its estimation via gmm estimation and its use for volatility forecasting," 2003, university of Kiel.

[26] ——, "The markov-switching multi-fractal model of asset returns: Gmm estimation and linear forecasting of volatility," 2004, university of Kiel.

[27] L. Calvet and A. Fisher, "How to forecast long-run volatility: Regime-switching and the estimation of multifractal processes," *Journal of Financial Econometrics*, vol. 2, pp. 49–83, 2004.

[28] J. P. Kahane, "Positive martingales and random measures," *Chi. Ann. of Math.*, vol. 8B, pp. 1–12, 1987.

[29] B. B. Mandelbrot and H. M. Taylor, "On the distribution of stock price differences," *Op. Research*, vol. 15, pp. 1057–1062, 1967.

[30] E. Bacry, J. Delour, and J. F. Muzy, "Multifractal random walks," *Phys. Rev. E*, vol. 64, pp. 026 103–026 106, 2001.

[31] K. L. Chung, *A course in probability theory*, 2nd ed. London: Academic Press Inc., 1974.

[32] A. Kozhemyak, "Modélisation de séries financières à l'aide de processus invariants d'échelle. application à la prédiction du risque," Ph.D. dissertation, Ecole Polytechnique, Palaiseau, France, 2006.

[33] J. F. Muzy, E. Bacry, and A. Kozhemyak, "Extreme values and fat tails of multifractal fluctuations," *Physical Review E*, vol. 73, p. 066114, 2006.

[34] E. Bacry, A. Kozhemyak, and J. F. Muzy, "Continuous cascade models for asset returns," *Journal of Economic Dynamics and Control*, vol. 32, pp. 156–199, 2008.

[35] J. F. Muzy, E. Bacry, R. Baile, and P. Poggi, "Uncovering latent singularities from multifractal scaling laws in mixed asymptotic regime. application to turbulence," 2008, submitted to Phys. Rev. Lett.

[36] E. Bacry, A. Gloter, M. Hoffmann, and J. F. Muzy, "Estimation of the multifractal spectrum in a mixed asymptotic framework," 2008, preprint.

[37] L. Calvet and A. Fisher, "Forecasting multifractal volatility," *Journal of Econometrics*, vol. 105, pp. 27–58, 2001.

[38] L. Hansen, "Large sample properties of generalized method of moments estimators," *Econometrica*, vol. 50, pp. 1029–1054, 1982.

[39] A. R. Hall, *Generalized method of moments*, 2005.

[40] G. Shorack and J. Wellner, *Empirical Processes With Applications to Statistics*. John Wiley & Sons Inc., 1986.

[41] P. M. Robinson, "The memory of stochastic volatility models," *Journal of Econometrics*, vol. 101, pp. 195–218, 2001.

[42] B. B. Mandelbrot, "The variation of certain speculative prices," *Journal of Business*, vol. 36, pp. 394–419, 1963.

[43] ——, "Formes nouvelles du hasard dans les sciences," *Economie appliquée*, vol. 26, pp. 307–319, 1973.

[44] ——, "Le syndrome de la variance infinie et ses rapports avec la discontinuité des prix," *Economie appliquée*, vol. 26, pp. 321–348, 1973.

[45] ——, "Le problème de la réalité des cycles lents et le syndrome de joseph," *Economie appliquée*, vol. 26, pp. 349–365, 1973.

[46] A. Orléan, *Le pouvoir de la finance*. Paris: Odile Jacob, 1999.

[47] C. Walter, "1900-2000 : Un siècle de description statistique des fluctuations boursières ou les aléas du modèle de marche au hasard en finances," 2003, communication at colloquium Marché boursier, Collège de France.

[48] Z. Ding and C. W. J. Granger, "Modeling volatility persistence of speculative returns : a new approach," *Journal of Econometrics*, vol. 73, pp. 185–215, 1996.

[49] B. B. Mandelbrot, *Fractals and scaling in finance. Discontinuity, concentration, risk*. New-York: Springer, 1997.

[50] B. B. Mandelbrot, A. Fisher, and L. Calvet, "A multifractal model of asset returns," 1997, cowles Foundation Discussion Paper, 1164.

[51] S. Ghashghaie, W. Breymann, J. Peinke, P. Talkner, and Y. Dodge, "Turbulent cascades in foreign exchange markets," *Nature*, vol. 386, pp. 767–770, 1996.

[52] L. Calvet and A. Fisher, "Multifractality in asset returns : theory and evidence," *Review of Economics and Statistics*, vol. 84, pp. 381–406, 2002.

[53] T. Bollerslev, "A generalized autoregressive conditional heteroskedasticity," *Journal of Econometrics*, vol. 31, pp. 307–327, 1986.

[54] T. G. Andersen, T. Bollerslev, P. Christoffersen, and F. X. Diebold, *Volatility and Correlation Forecasting*. Amsterdam: North-Holland: In G. Elliott, C.W.J. Granger and Allan Timmermann (Eds.), Handbook of Economic Forecasting, 2005.

[55] S. W. Lee and B. E. Hansen, "Asymptotic theory for the garch(1,1) quasi-maximum likelihood estimator," *Econometric Theory*, vol. 10, pp. 29–52, 1994.

[56] P. Kupiec, "Techniques for verifying the accuracy of risk measurement models," *Journal of Derivatives*, vol. 2, pp. 73–84, 1995.

[57] P. Christoffersen, "Evaluating interval forecasts," *International Economic Review*, vol. 39, pp. 841–862, 1998.

[58] G. C. Wick, "The evaluation of the collision matrix," *Physical review*, vol. 80, pp. 268–272, 1950.