FROM A VORTEX GAS MODEL OF TURBULENCE TO MELLIN FUNCTIONS

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Finding a way to describe essential properties of fluid flows and to measure the parameters involved in modelling is a road towards comprehension of turbulence. One model (see a review in [1]) was developped by Lundgren, following an idea of Townsend, in which small-scale turbulence is described as a collection of vorticity structures, aging as time goes on. Coherent vorticity structures have been reported in numerous experimental, theoretical or numerical studies and are presumably related to intermittency. We focus here on estimates for models describing turbulence as a dilute "gas of objects".

Lundgren Model. Lundgren established a long-time solution of Navier-Stokes equations for an unsteady, stretched, non-axisymmetric spiral vortex. The strain field is supposed to be axial and decoupled from the vorticity $\vec{\omega}(\vec{r},t)$. The field $\vec{\omega}$ of a vortex in this class of solution is expressed in cylidrical coordinates (r,θ,z) as :

$$\vec{\omega}(r,\theta,t) = S(t)[\omega_0(\xi) + \textstyle\sum_{n\neq 0} f_n(\xi) e^{in(\theta-\Omega(\xi)T)} e^{-\frac{1}{3}\nu n^2|\Omega'(\xi)|^2T^3}]\hat{z}\ ;$$

where $\Omega(r)$ is the θ-averaged angular velocity and $S(t) = \exp(\int_0^t s(\tau)d\tau)$, $T(t) = \int_0^t S(\tau)d\tau$, $\xi = rS^{1/2}$. It was proved that the time-averaged energy spectrum scales like $k^{-5/3}$ in the inertial range (K41 assumption). This requires to set up the following parameters: first, Ω (related to the azimuthal-averaged vorticity ω_o by $r\omega_o(r) = \frac{d}{dr}\left(r^2\Omega\right)$) and f_n which both set the spiral geometry; second, the strain rate s, the radial length-scale a, and the vortex Reynolds number Γ_0/ν. This rises up the question about the sensitivity of the obtained solution with respect to the initial conditions (Ω, f_n).

Tuning the model with experimental results. The parameters a, Γ_0 and s are deduced (at least approximatively) from the experimental study of the velocity field in fully developped turbulence. In [2], Baudet and al. proposed a spectral, hence, global approach based on acoustic scattering, thus allowing to measure the vorticity field at a given set of scales. This latter approach evidences both the short lifetime of vorticity structures and their periodic structure in the space-Fourier domain. This naturally leads to propose the spiral objects as good candidates for being the "atoms" of the Townsend gas. We compute vorticity, velocity and pressure fields of a Lundgren vortex and show that, under assumption of isotropy and homogeneity of the vortices distribution, coherent fine-scale details of this model are blurred out by core of the spiral, except when measuring averaged values. Again, this implies to find means to detect fine structures, which cannot be made possible unless one resorts to global measures such as acoustic scattering methods.

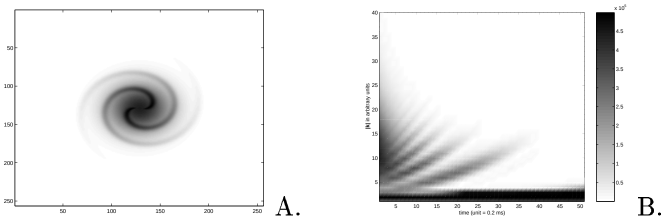

Figure 1. A. Vorticity of a spiral vortex, 6 ms after the forming of the vortex. Parameters are: $a = 3$ mm, $s = 100$ s^{-1}, $\Gamma_0/\nu = 600$. B. Time-wave vector representation of the radial-averaged enstrophy spectrum of this vortex. The arms are bent by the strain, then dissipated by viscocity in less than 10 ms. The influence of the core is predominant for low $|\vec{k}|$ and is responsible of the high enstrophy area appearing at the bottom of the graph.

Geometric analysis. Vassilicos and al. [3] proved that geometric studies are helpful to analyze turbulent flows in terms of structures, especially to find their local properties. The Kolmogorov capacity dimension of a complex singularity (e.g. a spiral) is closely related to its energy spectrum and the former is easier to measure than the latter, in particular if the structure is defined only on a small scale range. Thus we compute $D_K^{'}$, the capacity dimension of a cross-section in various spiral vortices and separately study their form (algebric, logarithmic, etc). There is twofold difficulty for interpreting the results. First, the measures of $D_K^{'}$ are not precise : D'_K changes in course of time because of aging of the vortex. Moreover it is found to depend on the location of the choosen 1D cut, as the spiral is observed to count only a few rolling turns: therefore this measure cannot be used for detecting structures. Second, the interesting spectrum of the spiral vortex in Lundgren model arises with time-average whereas time-average of D'_K is by no simple way related to K41 spectrum. Then it is hazardous to infer turbulence properties of a vortices gas from these measurements.

Towards time-scale representation. Another path is explored, trying to build a time-scale representation of an object from which we could analyze a multi-object signal. A time-Fourier space $(t, |\vec{k}|)$ representation of the vorticity evidences the existence of arms going from large to small wavelength in course of time, bent by the strain (cascade effect). We observe a lack of universality in such representation which imperils our wishes to detect individual objects in a "object gas" signal.

As we hardly notice significant behaviour differences for various spiral models, we can choose an idealized form adapted to analysis and interpretation of our measures. As Lundgren solution relies upon a scale transform (r is replaced by ξ and S is the scale change factor), we propose a simplified approach based on the scale invariant Mellin functions.

References

1. D.I. Pullin, P.G. Saffman (1998), *Ann. Rev. Fluid Mech.*, **30**, pp. 31-51.
2. C. Baudet, O. Michel (1998), *Advance in Turbulence VII*, U. Frisch ed., pp.43-46.
3. J.C. Vassilicos, J.C.R. Hunt (1991), *Proc. R. Soc. Lond. A*, **435**, pp. 505-534.