Stochastic discrete scale invariance: renormalization group operators and iterated function systems

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Abstract

We revisit here the notion of discrete scale invariance. Initially defined for signal indexed by the positive reals, we present a generalized version of discrete scale invariant signals relying on a renormalization group approach. In this view, the signals are seen as fixed point of a renormalization operator acting on a space of signal. We recall how to show that these fixed point present discrete scale invariance. As an illustration we use the random iterated function system as generators of random processes of the interval that are dicretely scale invariant.

1. Introduction

Of central importance when studying the concept of fractal, is the close notion of scale invariance. For stochastic processes, it leads to fractal and multifractal signals, which are fundamental in several domains of physics, from fully developped turbulence to fracture of materials. We propose in this communication some insights on how one can study processes that are both multifractal and with a specific breaking of the symmetry in scale: **Discrete Scale Invariance**, or DSI.

Let us recall a first approach to define this. DSI corresponds to a symmetry breaking of scale invariance where there exists a preferred scale *ratio* λ_0 (that is some periodicity in scale), see Sornette (1998). A first natural definition (adapted from the definition of proper self-similarity) given in Borgnat *et.al.* (2002) is to consider that a random process has DSI if its finite probability distributions are globally invariant under the action of scaling operators of a fixed *ratio* λ_0 – and of course its powers by iteration of the equality $(\mathcal{D}_{\lambda_0,H}X)(t) = \lambda_0^{-H}X(\lambda_0 t) \overset{d}{=} X(t)$ The l.h.s. in this equation is the definition of a scaling operator $\mathcal{D}$, and the r.h.s. is the probabilistic equality. The index H imposes a monofractality of the signal, and also their nonstationarity. Previous studies of methods of synthesis and analysis for DSI relied on the equivalence of DSI with cyclostationarity by means of the so-called Lamperti transform (Borgnat *et.al.* (2002), Flandrin *et.al.* (2003)).

We propose here to delve into another approach to define processes with DSI that could be multifractal. Scale invariant signals can be obtained as the solutions of some **Renormalization Group equation**, see Sornette (1998). The renormalization group approach provides a framework to analyse DSI in a more general setting. Furthermore, the notion of Iterated Function Systems (Barnsley (1988), Massopust (1994)) can be seen as a particular instance of the renormalisation group and is a method of synthesis. This is quite well known in the deterministic case, and we show here how IFS can be used to synthetize random signals. The following section presents the ideas of the renormalizationgroup approach to DSI, and section 3 deals with random IFS for the modelling of random signal with DSI.

2. DSI and the renormalization group approach

Let f be a field indexed by some set whose variables are generically denoted as t. A renormalization of the field amounts to assume the existence of a renormalization operator $\mathcal{T}$, such that the field satisfies $f(t) = \mathcal{T}\{f(t)\} + g(t)$. where g is some smooth field. Assuming linearity of the operator, n iteration of this defining equation leads to a solution, expressed by means of $f^\star(t)$, a fixed point of the renormalization operator, as

$$f(t) = \sum_{n=0}^{\infty} \mathcal{T}^n\{g(t)\} + f^\star(t) \tag{2.1}$$

† P. B. was supported by an INRIA post-doctoral fellowship for international mobility during this work

Generically, the solution f of this equation has DSI. This can be shown by means of the Mellin transform. Recall that the Mellin transforms reads: $(\mathbf{M}f)(s) = \int_0^{+\infty} f(t) t^{-s-1} \mathrm{d}t$. In this expression, variable s as the meaning of a scale, in the sense that it is the ratio between two variable t. Indeed, the Mellin transform can be seen as the Fourier transform over the multiplicative group. Hence, usual frequencies (difference of times) used by the usual Fourier transform (over the additive group) are transposed by scales (ratios of times) for the Mellin transform. A scale periodicity of f *i.e.*, DSI, may be identified by a peak in its Mellin transform. To proove this, we expand the renormalization operator as its linear equivalent close to a fixed point t_0 in the renormalization operator. $(\mathcal{T}f)(t) \simeq \mu f(t_0 + \lambda_0(t - t_0))$, and study the Mellin transform of the solution given in (2.1). Assuming that $t_0 = 0$ for the ease of notation, the Mellin transform of f reads

$$(\mathbf{M}f)(s) = \frac{1}{1 - \mu\lambda_0^s} (\mathbf{M}g)(s) + (\mathbf{M}f^\star)(s). \tag{2.2}$$

The first part of the solution has poles in Mellin scales $s_n = (-\log\mu + i2\pi n)/\log\lambda_0$, and this is precisely the signature of a preferred scale *ratio* λ_0. The second part $(\mathbf{M}f^\star)$ in the approximation is the Mellin transform of $f^\star(t) = \mu f^\star(\lambda_0 t)$ whose solution is $\sum_n t^{s_n}$. The Mellin transform reads $\sum_n \delta(s - s_n)$. It is therefore a singular spectrum in the same way as the Fourier transform of periodic signals is singular. Therefore, both terms have the property of DSI. In some case the symmetry can be stronger (*e.g.*, full scale invariance) but we hereafter study the property of DSI mainly.

The analysis presented in this section relies on the renormalization of some deterministic field. Since we are interested in random signals, we have to transpose this to a stochastic setting. Some work have been done on renormalization of the statistical functions that describe stochastic processes. However, these analyses are not always complete since they do not work directly on the multidimensional probability measures that define the process. However, an elegant theory that works on probability distributions and relies on the renormalization group approach exists: the random Iterated Function Systems.

3. Random Iterated Function Systems for signals

We are mainly concerned here by the construction and the properties of **random** signals by means of renormalization equations. Such signals are obtained as fixed point of **random Iterated Function System** (IFS). An IFS for signals is a renormalization operator and is constructed as follows (Barnsley (1988), Massopust (1994), Forte & Vrscay (1995)). Consider a compact interval $\mathbb{X}$ of the real line (everything here can be easily transposed to higher dimensions) and N contractive applications $w_i : \mathbb{X} \to \mathbb{X}$ that provide a partition of $\mathbb{X}$ ($\bigcup w_i(\mathbb{X}) = \mathbb{X}, w_i(\mathbb{X}) \bigcap w_j(\mathbb{X}) = \emptyset$ if $i \neq j$). Then, a contractive operator on a function space $\mathbb{Y}$ is defined as

$$(Tf)(t) = \sum_{i=1}^{N} (T_i f)(t), \qquad (T_i f)(t) = \varphi_i \left(f(w_i^{-1}(t)), w_i^{-1}(t) \right) \mathbf{1}_{w_i(\mathbb{X})}(t) \tag{3.1}$$

where $\mathbf{1}_I(t) = 1$ if $t \in I$ and 0 if not. If the functions $\varphi_i(t, y)$ are supposed to be lipschitz in variable t, then it can be shown that T is contractive, and by the fixed point theorem in Banach spaces, the series $T^n f_0$ admits a unique limit $f^\star$ whose graph is usually a fractal set. The system $(\mathbb{X}, \mathbb{Y}, \{w_i\}, \{\varphi_i\})$ defines an IFS. The limit $f^\star$ obviously satisfies the fixed point equation $f^\star = Tf^\star$. In a random setting, this equality is to be understood in terms of probability distributions. In that case the function $f^\star$ is a scale invariant random function, in the sense $f^\star = Tf^\star$. To be usable, the operator T itself has to be random, otherwise, whatever the initial condition f_0 (random function or not), the sequence $T^n f_0$ converges almost surely to the fixed point of the deterministic IFS.

In the random case (Falconer (1986), Mauldin & Williams (1986), Hutchinson & Ruschendorff (2000)), the application of the operator T at the nth iteration writes $T^n f_0 = \sum_{i_1, i_2, \ldots, i_n} T_{i_1} \circ T_{i_2}^{i_1} \circ \ldots \circ T_{i_n}^{i_1, i_2, \ldots, i_{n-1}} f_0$ where the $T_{i_n}^{i_1, i_2, \ldots, i_{n-1}}$ are drawn from the random operators $T_{i=1\ldots N}$, independently of the previous iterations. Furthermore, $T_{i_n}^{i_1, i_2, \ldots, i_{n-1}}$ and $T_{i_n}^{j_1, j_2, \ldots, j_{n-1}}$ are independent if $i \neq j$. This is illustrated by figure 1 where for the particular case $N = 2$, $n = 3$ levels of the construction are represented. The $N^n = 8$ functions f_0 are independent and identically distributed copies of a random initial function. The operators on the edges going from one node to $N = 2$ leaves can be correlated but are independent of the other operators anywhere on the tree. Therefore, the functions at two nodes labelled i_3 are still independent. This is the fundamental property leading to the existence of a fixed point. The main result of Hutchinson & Ruschendorff (2000), proved for single valued φ_is ($\varphi_i(t, y) = \varphi_i(t)$) is sumed up in

Theorem *Let* $\{(\varphi_i(t)\}_{i=1,\ldots,N})$ *N random lipschitz functions, with lipschitz (possibly random) constants* s_i*;*

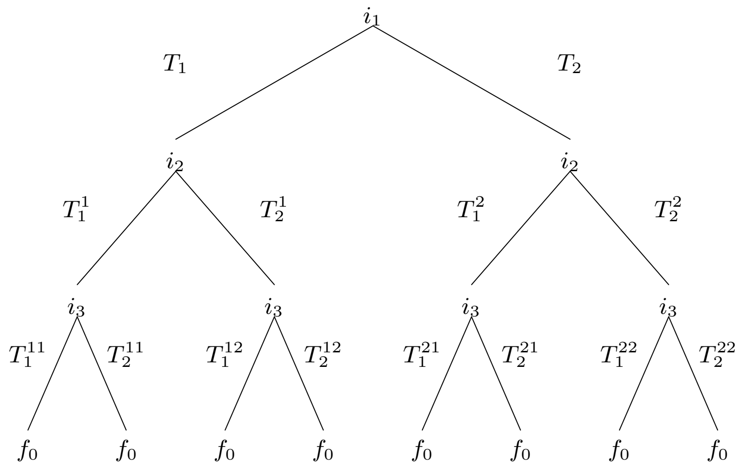

FIGURE 1. Construction tree of a random IFS. Illustration for a binary tree and three iteration of the construction process. Each function f_0 at the leaves are independent copies of a random initial function. Operator linking one node to its leaves may be correlated but are independent from all the other operators in the tree, but are however identically distributed.

let N contractive functions w_i that provide a partition of $\mathbb{X} = [0,1]$, with contractivity factor p_i ; suppose that $\lambda_p = E[\sum_i p_i s_i^p] < 1$ and $E[\sum p_i |\varphi_i(0)|^p] < \infty$ for some $1 < p < \infty$. Then, $\forall f_0 \in L_p(\mathbb{X})$

$$E^{1/p}[d_p^p(T^n f_0, f_\star)] \leq \frac{\lambda_p^{n/p}}{1-\lambda_p^{1/p}} E^{1/p}[d_p^p(f_0, Tf_0)] \longrightarrow 0 \tag{3.2}$$

when $n \to \infty$. Furthermore, in the distribution sense $f_\star$ is a fixed point of T.

To prove the result, one uses the fact that the space $L_p = \left\{ f(\omega,t), \omega \in \Omega / E\left[\int |f(\omega,t)|^p \, dt\right] < +\infty \right\}$ with the distance $E^{1/p}[d_p^p(f,g)]$ is a complete metric space, and that the random operator T as defined above is contractive in that space with this distance. Note that the theorem remains valid for general $\varphi_i(t,y)$ lipschitz in t.

As an illustration, we consider the simple case where the contractive applications w_i are affine functions, $w_i(t) = p_i t + q_i$, and provide a partition of $[0,1]$, and where $(T_i f)(t) = \left(s_i f(w_i^{-1}(t)) + \varepsilon \lambda_i(w_i^{-1}(t))\right) \mathbf{1}_{w_i(\mathbb{X})}(t), \forall i = 1,\ldots,N$.Here, ε is a random variable drawn once for all $i = 1,\ldots,N$ and the λ_i are deterministic functions. Once again, the equality in the definition of T_i stands in distribution, meaning practically that the T_i's are applied to N independent and identically distributed copies of a random function f. The fixed point of the random IFS is the extension to the random case of the so-called affine functions (Barnsley (1988), Massopust (1994), Jaffard (1997)). Since they are obtained as fixed point of renormalization group like equation, they are discretely scale invariant. Figure (2) illustrates these functions for the case $N = 2$.

The fixed point is know through the equality $f^\star = Tf^\star$ which holds in distribution. If we are interested in the statistics of the signal, we can evaluate the equation that follows the cumulants of the signal thanks to the model. For example, we can show that the nth-order cumulant $C_{n,f}(t)$ of $f^\star$ at time t is the fixed point of a deterministic IFS defined by

$$C_{n,Tf}(t) = \sum_i \left(s^i C_{n,f}(w_i^{-1}(t)) + C_{n,\varepsilon}\lambda^n(w_i^{-1}(t))\right) \mathbf{1}_{w_i(\mathbb{X})}(t) \tag{3.3}$$

Furthermore, we can also show that the nth-order multicorrelation of the fixed point is the fixed point of an n-dimensional deterministic IFS. For example, the correlation $C_{f^\star}(t_1,t_2) = \mathrm{Cum}[f^\star(t_1), f^\star(t_2)]$ is the fixed point of the following 2-dimensional IFS

$$C_{Tf}(t_1,t_2) = \sum_{i,j} \left(s^2 \delta_{i,j} C_f(w_i^{-1}(t_1), w_j^{-1}(t_2)) + C_{2,\varepsilon}\lambda_i(w_i^{-1}(t_1))\lambda_j(w_j^{-1}(t_2))\right) \mathbf{1}_{w_i(\mathbb{X})\times w_j(\mathbb{X})}(t_1,t_2) \tag{3.4}$$

These statistics are illustrated in figure (2), where they are plotted using the deterministic fixed point of the IFS, or by estimation using average over 4000 snapshots of the random fixed point. We especially plot the Fourier transform of the covariance to reveal the presence of DSI (log-periodicity). Indeed, in the case of DSI, plotting in log scale the Fourier transform allows to display dirac function at the multiple of the log preferred ratio λ_0. We can see that this method to assess DSI is highly dependent on the value of this log-frequency with respect to

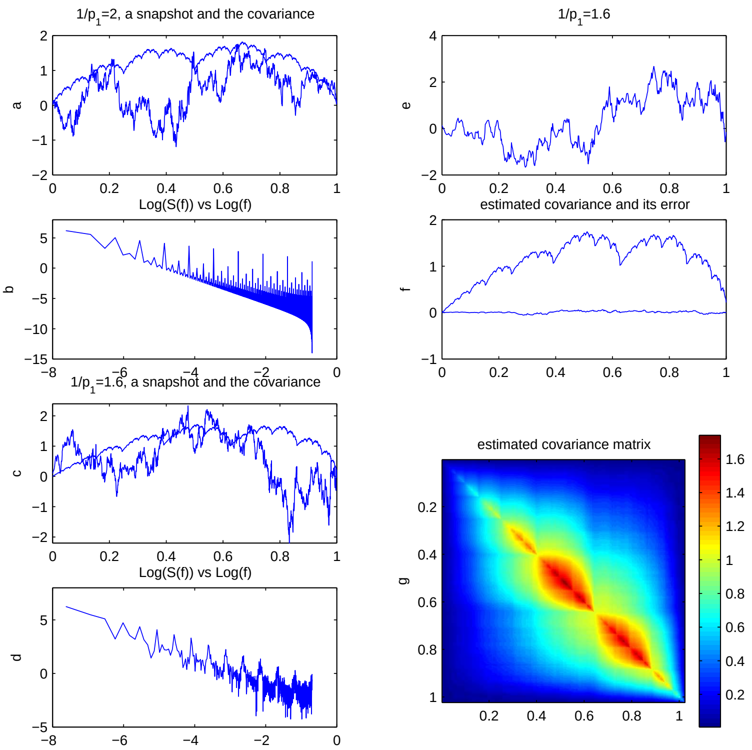

FIGURE 2. Random IFS for $N = 2$. (a) $p_1 = 1/2 = p_2$, a snapshot of the random fixed point and its covariance given by eq. (3.3). (b) $p_1 = 1/2 = p_2$ Fourier transform of the covariance represented in log scales. (c) $p_1 = 5/8 = 5p_2/3$ a snapshot of the random fixed point and its covariance. (d) $p_1 = 5/8 = 5p_2/3$ Fourier transform of the covariance represented in log scales. (e) : Another snapshot for $p_1 = 5/8 = 5p_2/3$, the estimated covariance and the difference with the true covariance (f). (g) the estimated covariance matrix. Estimation are done by averaging over 4000 snapshots.

the sampling grid. In figure (2b), this frequency is on the grid and we have a perfect sampling, whereas in figure (2d), the log-frequency is off the grid, and the effect of aliasing is terrible. It may be even more desastrous for certain values, completly destroying the effect of DSI. This shows the difficulty of assessing DSI on real data, especially when the effect is only a correction to pure scaling.

As a conclusion, let us mention the fact that the fixed point of deterministic IFS with affine contraction mappings (the class we consider to illustrate the ideas) have be shown to be multifractal functions. This remains to be studied for the random signals we have considered here. If it is clear that the one point statistics are multifractal function (as fixed point of affine IFS), it is not an evidence that the random fixed point will itself be a random multifractal process. This has to be investigated further.

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