Random processes with Discrete Scale Invariance: Methods of synthesis

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Abstract

The property of *discrete scale invariance*, proposed in a recent past as relevant for some deterministic signals issued from physical systems, is considered here in a stochastic context. By means of the Lamperti transformation we connect this property, based on dilations, to the property of cyclostationarity, based on shifts in time. Most of the present article is devoted to one aspect of the study of processes with DSI, namely the problem of synthesis. Some efficient approaches for synthesis of discrete-time random sequences are described and illustrated: methods based on stationary increments built as generalizations of the Weierstrass function; models on difference equations built, on the one hand, on the exact discretization of continuous-time models (the so-called Euler-Cauchy systems), on the other hand by means of fractional differencing. The last method detailed is a revisiting of tree-based techniques: it is in particular shown how random Iterated Function Systems give rise to DSI.

1 Beyond Scale Invariance

A particular symmetry breaking of the scale invariance was put forward by Sornette and co-authors ([1] and references therein): the property of Discrete Scale Invariance (DSI). This weakened version of self-similarity seems to occur spontaneously in many physical systems (geophysics, DLA growth, critical phenomena, maybe turbulence) and is of special interest when one wants to seek behaviours of a physical system beyond simple scale invariance [2].

A deterministic signal or figure is said to have discrete scale invariance if it is invariant only by dilations of a certain preferred scale factor. The simplest example to understand this is the Cantor middle-third set. This set is fractal but in fact stays the same only if dilated by 3^n, where $n \in \mathbb{Z}$. A dilation by a different factor gives a figure which can not be superposed to the original Cantor set.

An inner signature of DSI is the existence of power-laws with a complex exponent, whereas proper scale invariance gives rise to real exponents. They are at the origin of the log-periodic oscillations (periodicity for the variable $\log t$) observed in some systems, for example in the estimation of the fractal dimension of the Cantor set. Classical tools were adapted by Sornette and co-authors to describe, study and estimate the log-periodic corrections due to DSI, explicitly in a deterministic framework (e.g., [3], [4]).

Another possibility to study DSI is to build a properly stochastic theory for DSI [5]. A new formalism, different from that of [1], has to be proposed. In order to settle the basis of DSI for random processes, three domains have to be explored. First are the definitions, the description and the properties of random processes with DSI, which can be properly settled with the aid of the Lamperti transformation as advocated in [6]. Section 2 will summarize this theoretical description. The second question is the analysis of DSI in a stochastic context for experimental signals and the all-important estimation of the preferred scale factor (or factors). This question, by far the most subtle, is left for the future. Some results were proposed in [7, 8] along with a first method of analysis. However, the necessity to re-address this issue, in particular to compare the quality of the synthesis we will propose in the following, has not to be forgotten.

The third problem, which is the major scope of this article, is to find the many ways to generate random processes with DSI. This question is a question on its own, because the general theory characterizing stochastic DSI is established for continuous-time processes whereas one generates or studies in experiments discrete-time sequences. As the properties based on dilations are not easily imported when a discrete sampling in time is supposed, the synthesis of DSI processes is more complicated that the continuous-time theory might suggest. We will argue in the following that several efficient ways exist to obtain random sequences in discrete-time that have DSI.

As announced, we will first recall in section 2 the theory based on the Lamperti transformation for DSI processes. Section 3 will be devoted to how one can reinject some kind of stationarity in DSI to use spectral methods of synthesis; this approach is based on stationary increments. Section 4 goes a different way: it proposes to build models for DSI on fractional differences, as discrete-time equivalent to ARMA (or for DSI : Euler-Cauchy) modelling. Section 5 will finally comment on tree-based synthesis methods which give DSI.

2 Stochastic Discrete Scale Invariance

2.1 Random processes with DSI

We first recall the following formalism for stochastic processes [9]. First, we define self-similar processes, then processes with DSI. Let $(\mathcal{D}_{H,k}X)(t) \hat{=} k^{-H}X(kt)$ be the dilation operator of factor k with rescaling.

Definition 1 *A random process $\{X(t), t > 0\}$ is said to be self-similar of index H (or scale invariant, noted "H-ss") if for any $k > 0$,*

$$(\mathcal{D}_{H,k}X)(t) \stackrel{d}{=} X(t), \ t > 0. \tag{1}$$

$\stackrel{d}{=}$ is the notation for the probabilistic equality, that is equality of all finite-dimensional distributions. This equation, similar to a renormalization group one, has simple solutions in a deterministic context: those are the power-laws $X(t) \propto t^H$, ubiquitous when one seeks exact scale invariance.

As we will consider here mainly zero-mean Gaussian processes, we focus on second-order statistics and introduce the covariance function $R_X(t,s)$. A random process $\{X(t), t > 0\}$ is wide-sense H-ss if its covariance function, $R_X(t,s) \hat{=} \mathbb{E}\left\{X(t)\overline{X(s)}\right\}$, verifies for any $k > 0$:

$$R_X(kt, ks) = k^{2H} R_X(t,s). \tag{2}$$

Definition 2 *A random process $\{X(t), t > 0\}$ has discrete scale invariance of index H and of scaling factor λ (noted "(H,λ)-DSI") if*

$$(\mathcal{D}_{H,\lambda}X)(t) \stackrel{d}{=} X(t), \ t > 0. \tag{3}$$

The invariance of the process is only required for dilation by λ, and consequently for any λ^n, $n \in \mathbb{Z}$. The group of symmetry is thus an infinite but countable group $\{\mathcal{D}_{H,\lambda^n}, n \in \mathbb{Z}\}$, isomorphic to a multiplicative subgroup of $\mathbb{R}^+$. Second-order definition of DSI is immediate. It requires that the covariance verifies (2) for $k = \lambda$, the preferred scale factor.

The deterministic DSI property introduced by Sornette *et al.* has the solutions t^γ for (3), with $\lambda^\gamma = \lambda^H$. Looking for a general solution for $\gamma \in \mathbb{C}$, one finds that $\gamma = H + i2\pi n/\log\lambda$ with $n \in \mathbb{Z}$. Those functions t^γ have the log-periodic oscillations mentioned as a signature of DSI. Furthermore, the functions $t^{H+i2\pi n/\log\lambda}$ constitute the basis of the Mellin functions [10] and they are also central to the study of stochastic DSI.

2.2 A main theoretical tool: the Lamperti transformation

A general way of thinking for self-similar processes was introduced by Lamperti in 1962 [11]. The idea of this approach is to relate stationary processes with self-similar processes by means of a proper transformation. Some results have been established for self-similar processes with this transformation in the past but its scope can be enlarged to study problems with incomplete or broken scale invariance. This result has then recently received more attention [6]. Let us first recall the theorem.

Theorem 1 ([11]) *If $\{X(t), t > 0\}$ is H-ss, then*

$$\left(\mathcal{L}_H{}^{-1}X\right)(t) \hat{=} e^{-Ht}X(e^t), \quad t \in \mathbb{R}, \tag{4}$$

is stationary. Conversely, if $\{Y(t),\ t \in \mathbb{R}\}$ is stationary, then

$$\left(\mathcal{L}_H Y\right)(t) \hat{=} t^H Y(\log t), \quad t > 0 \tag{5}$$

is H-ss.

Equation (5) defines the direct Lamperti transformation which is invertible; its inverse is given by (4). One can extend the result to variations around stationarity and self-similarity. The correspondence is in fact between dilation operators $\mathcal{D}_{H,\lambda}$ and shift operators, defined as $\left(\mathcal{S}_\tau Y\right)(t) \hat{=} Y(t+\tau)$.

Proposition 1 ([5]) *The Lamperti transform guarantees, for any $\lambda > 0$, the equivalence*

$$\mathcal{L}_H{}^{-1}\, \mathcal{D}_{H,\lambda}\, \mathcal{L}_H \;=\; \mathcal{S}_{\log\lambda}. \tag{6}$$

A first class of consequences of this connection allows to reformulate properties of a self-similar process with the corresponding stationary process. We refer to [6] for many restatements and new results of this kind. The following one will be useful.

Proposition 2 *Given $\{X(t), t > 0\}$ some process and $\{Y(t) = (\mathcal{L}_H{}^{-1}X)(t), t \in \mathbb{R}\}$ its inverse Lamperti transform, we have*

$$R_X(t,s) = (st)^H R_Y(\log t, \log s),\ t > 0,\ s > 0. \tag{7}$$

The second kind of applications is to connect properties which are not strict scale invariance with properties which are not strict stationarity. See [5] and [6] for some insights in this way. The point here is that the periodicity in scale of the statistics in which DSI is rooted, is transformed in a periodicity in time of statistics of the inverse Lamperti process: this is known as cyclostationarity.

2.3 DSI and cyclostationarity

Definition 3 *A process $\{Y(t), t \in \mathbb{R}\}$ is said to be periodically correlated [12, 13] or cyclostationary [14] of period T ("T-cyclostationary") if*

$$\{Y(t+T), t \in \mathbb{R}\} \stackrel{d}{=} \{Y(t), t \in \mathbb{R}\}. \tag{8}$$

The invariance is required for discrete shifts only, the group $\{S_{nT}, n \in \mathbb{Z}\}$. The corresponding second-order property for zero-mean processes is: $R_Y(t+T, s+T) = R_Y(t,s)$. It follows that $R_Y(t, t+\tau)$ is a periodic function of t and can be written as a Fourier series:

$$R_Y(t, t+\tau) \;=\; \sum_{n=-\infty}^{+\infty} C_n(\tau) e^{i2\pi nt/T}. \tag{9}$$

Theorem 2 ([5]) *If $\{X(t), t > 0\}$ has (H,λ)-DSI, then $\{(\mathcal{L}_H{}^{-1}X)(t), t \in \mathbb{R}\}$ is cyclostationary of period $\log\lambda$. Conversely, if $\{Y(t), t \in \mathbb{R}\}$ is T-cyclostationary, $\{(\mathcal{L}_H Y)(t), t > 0\}$ has (H, e^T)-DSI.*

The proof is straightforward given the definitions and Proposition 1. This theorem is a characterization of DSI processes and Proposition 2 combined with (9) allows to write the general form of the covariance of processes with DSI:

Proposition 3 *Let $\{X(t), t > 0\}$ be a (H,λ)-DSI process. Its covariance reads*

$$R_X(t, kt) \;=\; k^H t^{2H} \sum_{n=-\infty}^{+\infty} C_n(\log k) t^{i2\pi n/\log\lambda}. \tag{10}$$

Once again we encounter Mellin functions which are central to the study of DSI. The mathematical formalism is natural on the Mellin basis because each function has deterministic DSI. It is analog to the property that the Fourier basis is adapted to stationary processes. Mellin functions appear for processes whose properties are constructed on dilations. The reason is shown in [5] and [6] along with the spectral analysis on Mellin functions: the Mellin basis is the image by $\mathcal{L}_H$ of the Fourier basis.

2.4 DSI as an image of cyclostationarity: the sampling problem

The proposed theory is for continuous-time processes and gives a direct way to generate random sequences with DSI. One can simulate a cyclostationary process, then takes its Lamperti transform. A suitable discretization scheme has to be used to obtain a discrete-time signal. There are two possibilities.

The first way is to adopt a regular sampling for the cyclostationary sequence. The DSI sequence obtained by $\mathcal{L}_H$ has a geometrical sampling of the form q^n. This geometrical sampling is natural to deal with DSI properties and Mellin representation in discrete time [10]. It was also proved to be well-suited for the analysis of self-similar processes [15].

But to confront synthetic sequences with DSI to experimental data, the other choice is to work with the usual regular sampling $n \in \mathbb{N}$ for the DSI sequences. This requires a sampling in $\log n$ (where $n \in \mathbb{N}$) of the discrete-time cyclostationary processes - an unusual sampling. Then from a cyclostationary $Y_{\log n}$, we obtain with $\mathcal{L}_H$ a sequence X_n with DSI. In the following, we propose alternative ways to obtain DSI sequences directly expressed with regular sampling in discrete-time, where the problem of discretization is coped with in different manners.

3 Spectral synthesis / Correlation synthesis

3.1 DSI and stationary increments

A known direct way to synthesize a random sequence with a required covariance matrix (e.g., (10) to guarantee a process with DSI) is to use a Choleski decomposition of this covariance matrix. As a DSI process is not stationary, R_X has not a Toeplitz structure and the decomposition is a costly algorithm in $\mathcal{O}(N^3)$ to generate N points. As no proper spectrum exists, spectral methods based on FFT seem also pointless. The problem is the same as for self-similar processes which can not be stationary (when $H > 0$).

A classical assumption for re-injecting some kind of stationarity in self-similar signals, in order to use spectral methods and bypass the complexity of H-ss processes, is to suppose that their increments are stationary. For instance, fractional Brownian motion (fBm) is known to be the only Gaussian, H-ss process with stationary increments (s.i.). Its covariance and the covariance of its increment process $G_{H,\tau}(t) = B_H(t+\tau) - B_H(t)$ (fractional Gaussian noise) are:

$$R_{B_H}(t,s) = \frac{\sigma^2}{2}(|t|^{2H} + |s|^{2H} - |s-t|^{2H}), \tag{11}$$

$$R_{G_{H,\tau}}(t,t+k) = \frac{\sigma^2}{2}(|k+\tau|^{2H} + |k-\tau|^{2H} - 2|k|^{2H}). \tag{12}$$

The general approach [16] is to find the second order structure function $\mathbb{E}|X(t) - X(s)|^2$ (sufficient to write the covariance of a real-valued s.i. process). The assumption putting back some stationarity may also be generalized to any nth-order increments, following [17]. The explicit construction of fBm with nth-order stationary increments was applied in [18].

Following step by step the usual derivation, one can establish the general form of the covariance of a (H,λ)-DSI process with stationary increments of order 1. We are complied to specify a fixed value for $X(0) = m$ (we will then assume $m = 0$ with no loss of generality) to account for the specific property of the origin with respect to dilations. The covariances of a s.i. process with DSI, $X(t)$, and of its increments $Z_\tau(t) = X(t+\tau) - X(t)$ read

$$R_X(t,s) = \sum_{n\in\mathbb{Z}} \frac{c_n}{2}(|t|^{2H+in2\pi/\log\lambda} + |s|^{2H+in2\pi/\log\lambda} - |s-t|^{2H+in2\pi/\log\lambda}), \tag{13}$$

$$R_{Z_\tau}(t,t+k) = \sum_{n\in\mathbb{Z}} \frac{c_n}{2}(|k+\tau|^{2H+in2\pi/\log\lambda} + |k-\tau|^{2H+in2\pi/\log\lambda} - 2|k|^{2H+in2\pi/\log\lambda}). \tag{14}$$

The difficulty here is to find useful conditions on the c_n to ensure that (13) and (14) are true correlation functions, i.e., non-negative definite. For example, the simplified model which keeps only c_0 and $c_{\pm 1}$ is found generally insufficient and gives a function which is not a correlation function of any existing process. We propose to study a more specific model which is then easier to deal with.

3.2 The Weierstrass-Mandelbrot function as a model for DSI

The Weierstrass function, known as a fractal function [19] [20], will serve us as a guide for a slightly modified model with s.i. and DSI. We will use its Weierstrass-Mandelbrot version which turns out to be exactly (H,λ)-DSI in the sense of definition 3. A detailed analysis of this function is given in [21]. It reads

$$W_{H,\lambda}(t) = \sum_{n=-\infty}^{+\infty} \lambda^{-nH}(1-e^{i\lambda^n t})e^{i\phi_n}. \tag{15}$$

The Weierstrass-Mandelbrot function is a superposition of the same pattern at different scales, centered in the Fourier domain at frequencies $\lambda^n/2\pi$, $n \in \mathbb{Z}$. The phases ϕ_n might be deterministic ($\phi_n = \mu n$ for example) but stochastic versions of this function are obtained if they are random, i.i.d. and uniformly distributed in $[0\ ;\ 2\pi[$. Then $W_{H,\lambda}(t)$ has (H,λ)-DSI in a stochastic way and its covariance is

$$R_W(t,s) = \sum_n \lambda^{-2nH}(1+e^{i\lambda^n(t-s)} - e^{i\lambda^n t} - e^{-i\lambda^n s}). \tag{16}$$

This function is a first example of a s.i. process with DSI: its increments $V_{H,\lambda,\tau}(t) = W_{H,\lambda}(t+\tau) - W_{H,\lambda}(t)$ are stationary with respect to t. This follows from [21]

$$R_{V_{H,\lambda,\tau}}(t,s) = 2\sum_n \lambda^{-2nH}(1-\cos\lambda^n\tau)e^{i\lambda^n(t-s)}. \tag{17}$$

$W_{H,\lambda}(t)$ admits a generalization with another pattern than $e^{i(t+\Phi)}$. It has the form: $W_{g_{H,\lambda}}(t) = \sum_n \lambda^{-nH}(g_n(0;\omega) - g_n(\lambda^n t;\omega))$ where $g_n(\cdot;\omega)$ is a random function. If we impose that the increments of W_g are stationary, we obtain a functional equation on g that is not easily solved. We prefer to suppose directly that there exists a function G such that $\mathbb{E}\left\{g_n(t;\omega)\overline{g_m(s;\omega)}\right\} = G(t-s)\delta_{n,m}$. The knowledge of G is sufficient to synthetize X.

We thus propose the following construction for model of DSI with s.i.: let $\{X(t), t>0\}$ be a wanted DSI process and $\{Z_\tau(t) = X(t+\tau) - X(t), t > 0\}$ its increments. We suppose that $X(0) = m$ (to particularize the origin) and that the increments have the stationary covariance

$$R_{Z_\tau}(t,t+k) = \sum_n \lambda^{-2nH}\left\{G(\lambda^n(k+\tau)) + G(\lambda^n(k-\tau)) - 2G(\lambda^n k)\right\}. \tag{18}$$

Z_τ is generated by a correlation synthesis method from (18) then $X(t)$ is calculated at times $t = n\tau, n \in \mathbb{N}$, from the origin by summation of Z_τ. An example of this model is shown on figure 1, along with the original deterministic Weierstrass function.

As Z_τ is stationary, the construction imposes that X has s.i.; we show then that it has DSI by studying the function $k \longrightarrow \sum_n \lambda^{-2nH}G(\lambda^n k)$. This function has obviously $(2H,\lambda)$-DSI and its exact decomposition on the Mellin basis of $(\log\lambda)$-log periodic functions is obtained by means of the Poisson formula, to transform the sum over n:

$$\sum_{n\in\mathbb{Z}} \lambda^{-2nH}G(\lambda^n k) = \sum_{m\in\mathbb{Z}} \left(\int_{-\infty}^{+\infty} \lambda^{-2Hz}G(\lambda^z)e^{-i2\pi mz}\mathrm{d}z\right) k^{2H+im\frac{2\pi}{\log\lambda}}. \tag{19}$$

Consequently (18) admits a form as (14) with $c_m/2$ equals the integral in (19), and the constructed process X has proper (H,λ)-DSI.

3.3 Algorithmic issues

We do not want to survey all possible algorithms that, given a stationary covariance as (18), allow for a synthesis of the process. A bibliographical study for the synthesis of H-ss with s.i. can be found for example in [22], appendix A.

Methods using the Toeplitz structure of $(R)_{nl} = R_{Z_\tau}(n-l)$ of stationary process Z_τ (e.g, the Levinson algorithm), have a complexity reduced compared to that of the Choleski factorization. We give some details about a recent method, which is really fast, following from the works of Wood and Chan [23]

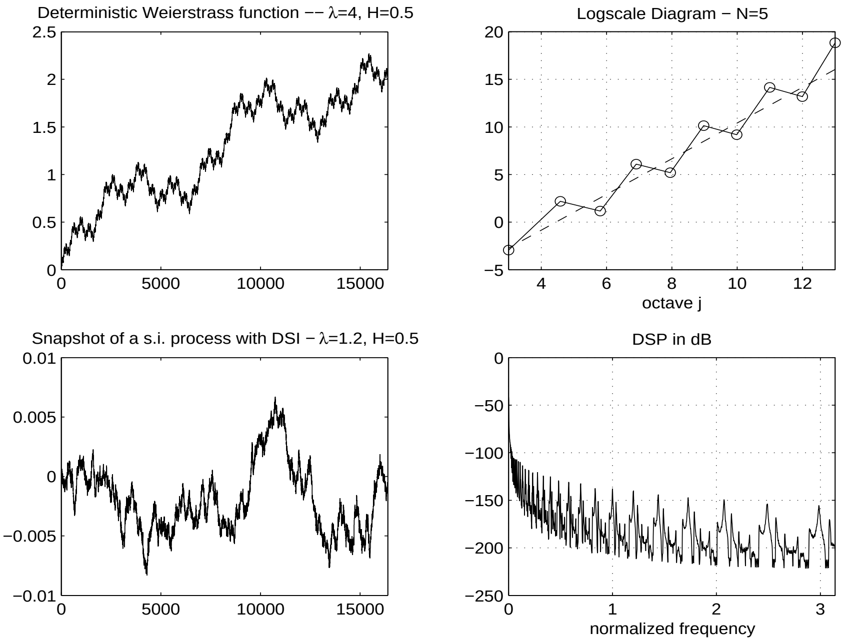

Figure 1: Upper part: the Weierstrass function on the left and its log-scale diagram beside, used for the estimation of H. See the oscillations caused by the DSI on this diagram. Lower part : s.i DSI process defined with (18). On the right, the estimated DSP makes patent the DSI (here we chose a simple model where G has a band-limited spectrum, thus peaks in the DSP are visible, spaced in λ^n).

and refined by Dietrich and Newsam [24]. More details can be found in [25] and [22]. In a nutshell, to generate N points of the process, the matrix R is embedded into a circulant matrix C of size $M \geq 2N$ with $C_m = R(m)$ if $m \leq N$, and $C_m = R(N-m)$ if $M - N \leq m \leq M$. The remaining coefficients C_m for $N < m < M - N$ are chosen at will such that C remains non-negative definite. In general $M = 2N$ is adopted. As C is a circulant matrix, it can be factorized easily as $C = Q\Lambda Q^t$, with Λ a diagonal matrix with real positive eigenvalues and Q a unitary matrix. It follows that $Q\sqrt{\Lambda}U$, where U is a Gaussian random sequence, admits R for covariance matrix of the first N points of the sequence. The algorithmic complexity of the method, implemented by means of FFT, is $\mathcal{O}(N \log N)$.

A practical advantage of the method is that, from a numerical point of view, the correlation function defined with (18) is not always exactly non-negative definite because of finite numerical precision. The Choleski and Levinson methods are sensitive to this numerical approximation (the Choleski algorithm simply does not work) and a proper projection of R to the space of non-negative definite matrices is difficult. The circulant matrix method may have some negative eigenvalues for Λ, but very small, as a signature of the numerical precision. It is then easy to set those values to zero; the process generated is then really close to what we look at.

4 Methods from difference equations

4.1 A general view

The continuous-time theory for parametric models with DSI is straightforward. The first step is to find the equivalent of ARMA models, which are the basis to stationary systems, after a "lampertization". The transformation $\mathcal{L}_H$ ensures that : $(\mathrm{d}X/\mathrm{d}t)(t) = t^{H-1}(HY(\log t) + (\mathrm{d}Y/\mathrm{d}t)(\log t))$. If one employs properly this property to calculate the derivative of X of any order, one will found the following result [6], used in [26] as a model for H-ss processes:

Proposition 4 *The stationary ARMA process*

$$\sum_{n=0}^{p} \alpha_n Y^{(n)}(t) = \sum_{m=0}^{q} \beta_q W^{(m)}(t), \tag{20}$$

where $W(t)$ is white noise, has an H-ss Lamperti counterpart, referred to as an Euler-Cauchy (EC) process, which is solution of an equation of the form

$$\sum_{n=0}^{p} \alpha'_n t^n X^{(n)}(t) = \sum_{m=0}^{q} \beta'_q t^q \tilde{W}^{(m)}(t), \tag{21}$$

with $\tilde{W}(t) = t^{H+1/2} W(t)$, and $t > 0$. Note that $\mathbb{E}\left\{\tilde{W}(t)\overline{\tilde{W}(s)}\right\} = \sigma^2 t^{2H+1} \delta(t-s)$.

Nonstationary ARMA systems are a convenient framework for modelling cyclostationary systems. The precise form is the one given in [27] and studied in detail in [28]. The coefficients in the equation (20) should have a period T, so that: $\alpha_n(t+T) = \alpha_n(t)$ and $\beta_q(t+T) = \beta_q(t)$. This system is then T-cyclostationary.

The Lamperti transform of cyclostationary ARMA gives a general expression for a parametric, continuous-time model of DSI processes. It has the expected form of the Euler-Cauchy model (21), driven by the multiplicative (nonstationary) white-noise $\tilde{W}(t)$, with time-varying coefficients α'_n and β'_q which must be log-periodic of period $\log \lambda$.

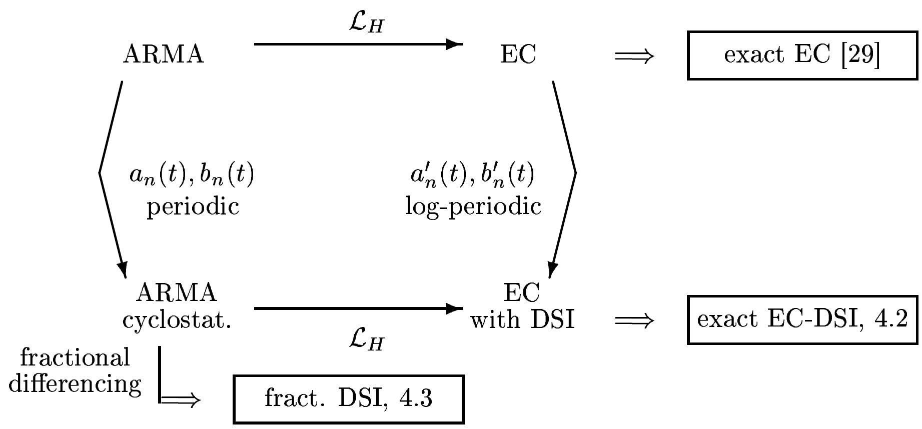

Figure 2: Guideline of the construction of models from difference equations. Discrete-time models are in the frameboxes; continuous-time systems are not.

A general scheme of what we must do to find a discrete-time counterpart of this model is given in figure 2. There are two natural ways to obtain effective discrete-time DSI models. First is to adopt an exact discretization scheme of the differential equation as proposed in [29] for EC systems. The drawback is that the nonstationary model obtained rapidly looses its specific self-similar evolution in time. The second way is to find another argument to construct pseudo H-ss systems by means of a representation with fractional integration. Those representations have a discrete-time corresponding model based on fractional differencing (the so-called FARIMA for instance) and we add log-periodicity to the coefficients to obtain DSI.

4.2 An exact discretization of the Euler-Cauchy system

The scheme is adapted from [29]. It consists in two steps; the first one is to find the solution $X(t_k)$ in term of $X(t_{k-1})$ at a previous instant. A propagator $G(t,u)$ of the equation, which obeys

$$\sum_{n=0}^{p} \alpha'_n(t) t^n \frac{\partial^n G(t,u)}{\partial t^n} = \delta(t-u), \tag{22}$$

with initial condition $G(u,u) = 1$, allows to express the solution in a general form as

$$X(t_k) = G(t_k, t_{k-1}) X(t_{k-1}) + \int_{t_{k-1}}^{t_k} G(t_k, u) \left(\sum_{m=0}^{q} \beta'_q(u) u^q \tilde{W}^{(m)}(u) \right) \mathrm{d}u. \tag{23}$$

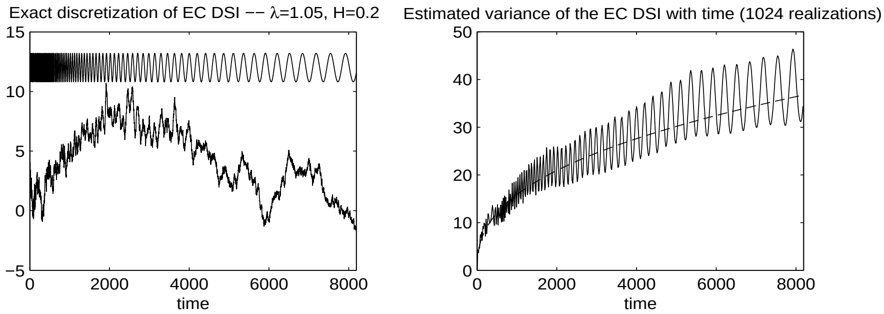

Figure 3: Left: a snapshot of a realization of 4.2; a visual guide. displaying the synchronization of the log-periodic oscillations is pictured. Right: the estimated variance of the process (on 1024 realizations); the DSI appears clearly as log-periodic oscillations, superimposed on the overall growing in t^{2H} (dashed curve) of this nonstationary model.

We can restrict the discussion to first-order systems. Indeed, a filter of order p as in (22) is equivalent to p filters of order 1 in parallel [30], even if it is nonstationary. We will then focus on the order 1 to derive the discrete-time model (X_k is for $X(t_k)$)

$$X_k = a_k X_{k-1} + e_k. \tag{24}$$

The continuous-time model is written with coefficients $a(t)$ and $\sigma(t)$ (variance of the input noise) which are periodic functions in $\log t$, of period $\log \lambda$.

$$\frac{\mathrm{d}X}{\mathrm{d}t}(t) + \frac{a(t)}{t}X(t) = \frac{\sigma(t)\tilde{W}(t)}{t}. \tag{25}$$

The propagator G of the system is then

$$G(t,u) = \exp\left(-\int_u^t \frac{a(v)}{v}\mathrm{d}v\right) = \left(\frac{t}{u}\right)^{-c_0} \frac{F(t)}{F(u)}, \tag{26}$$

$$F(u) = \prod_{n\neq 0} \exp\left(\frac{ic_n \log\lambda}{2\pi n} u^{i2\pi n/\log\lambda}\right), \tag{27}$$

where c_0 and F derive from the expansion of $\frac{a(t)}{t}$ on a Mellin basis: $a(t)/t = \sum_{n\in\mathbb{Z}} c_n t^{-1-i2\pi n/\log\lambda}$. Comparing to strict EC systems solved by Noret [30], a log-periodic function F is added to the propagator, which accounts for the DSI.

Identifying (24) with (23) for order 1, with a discretization $t_k = t_{k-1}+\tau$, the parameter in the discrete-time model is $a_k = G(t_k, t_{k-1})$ and e_k is white noise given as: $e_k = \int_{t_{k-1}}^{t_k} G(t_k,u)\sigma(u)W(u)u^{H-1/2}\mathrm{d}u$, whose nonstationary variance comes from (23), r.h.s.

$$\mathbb{E}\{e_k\overline{e_k}\} = \int_{t_{k-1}}^{t_k} |G(t_k,u)|^2|\sigma(u)|^2 u^{2H-1}\mathrm{d}u. \tag{28}$$

The exact model which follows from this procedure has a time-varying coefficient a_k. When the sampling is regular, an approximation of a_k for the long times k, namely $a_k \simeq (1-c_0/k)F_k/F_{k-1}$, proves that the variation of a_k with time disappears in $1/k$ except for the DSI part (F). The H-ss and long-range dependance properties are in fact found mainly in the input noise e_k. Its variance is, for large k, $\mathbb{E}\{e_k\bar{e}_k\} \simeq {t_k}^{2H-1}\tau|\sigma_k|^2(1+\tau F_k'/2t_k)$. The H-ss behaviour (responsible, e.g., for a variance of X_k in k^{2H}) is then mainly imposed by this input. Inversely, the DSI is present only as a pertubation in the variance of e_k, but dominates the time variation of the coefficient a_k.

To prove the DSI of the model, we have to calculate the covariance of X_k. The correlation function is given as

$$R_X(t,s) = \int_{t_0}^{\min(t,s)} \left(\frac{t\,s}{u^2}\right)^{-c_0} |\sigma(u)|^2 \frac{F(t)\overline{F(s)}}{|F(u)|^2} u^{2H-1}\mathrm{d}u + G(\min(t,s),t_0)\mathbb{E}\left\{X(t_0)^2\right\}. \tag{29}$$

Assuming that the system is stable ($G(t,t_0) \longrightarrow 0$ when $(t-t_0) \longrightarrow +\infty$), the initial condition will be forgotten and the correlation function of the process will satisfy (2) for $k=\lambda$ if $(t,s) \longrightarrow \infty$. An alternative assumption might be that $X(t_0)$ has the convenient distribution (zero-mean and a specific variance) for t_0 to disappear in the expression of the covariance [30]. In both cases, as σ, a and F are log-periodic functions, we have $R_X(\lambda t, \lambda s) = \lambda^{2H} R_X(t,s)$ and the model has (H,λ)-DSI.

A representation is then valid for the discrete-time DSI system: it combines a discretized EC model with the parameters of [29] in cascade with an AR system with log-periodic coefficients to represent the DSI in the coefficients of the model. A snapshot is given in figure 3.

An intricate property of these models is that they have no kind of stationarity. Because of the input noise growing in $k^{H+1/2}$, no increment of any order is stationary (a remark made for order 1 in [30]). Furthermore, the wavelet methods, known to usually transform a H-sssi signal in a representation stationary at each scale, does not work this way for the discretized EC model. The wavelet coefficients at one scale are nonstationary and grow in time as k^{2H}. The method of estimation of H by wavelets fails then and a constant $H=1/2$ is found if this kind of estimation is tried. The model, both for H-ss and DSI, is then at the opposite of what we constructed in section 3. From the Weierstrass function we studied a model with some hidden stationarity. Here, the model has no disclosed stationarity.

4.3 Models from fractional differencing

Continuous-time models. A different model for H-ss signals is constructed from the theory of linear modelling. This is the Barnes-Allan model for a H-ss system

$$X_{BA}(t) = \int_0^t |t-u|^{H-1/2} W(u)\ \mathrm{d}u. \tag{30}$$

It has the same variance as the fBm [31] and the same power spectrum too [6], and it is H-ss. The model (30) is in fact a part of the moving average representation of the fBm which gives a special role to the origin, imposing that $X_{BA}(0)=0$. In fact it is a good approximation of a fBm and relates to the first-order EC models, see [31]. It is convenient to interpret this form as the expression of a fractional integral, in the sense of the Riemann-Liouville or the Grünwald calculus [32]. Then an equation on a fractional derivative of X_{BA} makes sense

$$\frac{\mathrm{d}^{H+1/2} X_{BA}(t)}{\mathrm{d}t^{H+1/2}} = W(t). \tag{31}$$

The fractional integral part in (30) is responsible for the H-ss behaviour. For a DSI process, we can introduce the DSI part by changing the input noise in $h(t)$ which has DSI for $H=1/2$ and the preferred scale ratio λ. The model is

$$\begin{cases} X_{BA}(t) = \displaystyle\int_0^t |t-u|^{H-1/2} h(u)\ \mathrm{d}u, \\ \mathbb{E}\left\{ h(\lambda t)\overline{h(\lambda s)} \right\} = \lambda^{-1}\, \mathbb{E}\left\{ h(t)\overline{h(s)} \right\}. \end{cases} \tag{32}$$

A possible h is a white noise with log-periodic variance. Its expression on a Mellin basis might be: $h(t) = W(t) \sum_{n\in\mathbb{Z}} c_n t^{i2\pi n/\log(\lambda)}$. From the fractional derivative of this continuous-time representation of a DSI process, it comes

$$\frac{\mathrm{d}^{H+1/2} X_{BA}(t)}{\mathrm{d}t^{H+1/2}} = h(t) = W(t) \sum_{n\in\mathbb{Z}} c_n t^{i2\pi n/\log(\lambda)}. \tag{33}$$

Discrete-time fractional differencing. This introduction validates the possibility to propose discrete-time models by discretizing the operator in (33), whereas in the previous section the solution was discretized. The basic scheme of the new model is then to have a fractional differencing filter applied in cascade with a log-periodic ARMA filter driven by ordinary white-noise, figure 4. The log-periodic ARMA model allows to have a stochastic equivalent of the r.h.s. of (33). The fractional difference is the filter which introduces a self-similarity in the system. This compares to the EC model up to the following: the representation of the fractional difference used is a stationary filter and the self-similar behaviour of the system will only be approximate because of this stationarity (incompatible with a strict self-similarity).

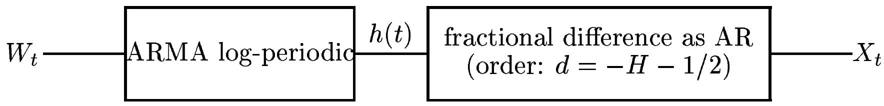

Figure 4: General block-diagram for models based on fractional differencing.

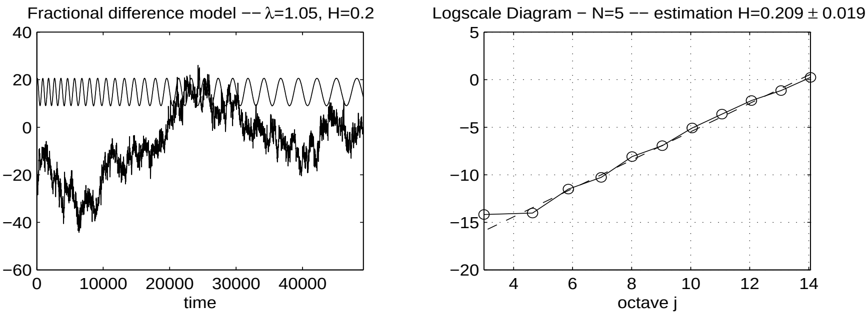

Figure 5: Left: a snapshot of model from 4.3; a visual guide displaying the synchronization of the log-periodic oscillations is pictured. Right: log-scale diagram estimated from wavelet methods, which shows oscillations of the DSI and a correct estimation of $\hat{H} = 0.209 \pm 0.019$ for $H = 0.2$.

Rao approach to discrete-time modelling. Some freedom exists to find a discrete-time equivalent to the operator $\mathrm{d}^{H+1/2}/\mathrm{d}t^{H+1/2}$. Two approaches are suggested here. The first dates back to the first FARIMA models [33] [34]. The idea is that the usual derivative admits a trivial discrete-time representation as $1 - B$ where B is the backward operator for sequences ($BX_k = X_{k-1}$). A fractional difference is then $(1 - B)^d$ with d any real number and a binomial expansion of this power gives a computable (but with an infinite memory) representation of the operator (Γ is the gamma function)

$$(1 - B)^d = 1 + \sum_{k=1}^{\infty} \frac{\Gamma(k - d)}{\Gamma(-d)\Gamma(k + 1)} B^k. \tag{34}$$

Another possibility is to adopt a more subtle discretization scheme [35]. The idea is to use a transformation which connects discrete-time processes and continuous-time processes in an invertible way. The purpose is to have a transformation best adapted to the dilation operator. This operator is expressed as simply in the continuous-time domain as in the frequency domain. Graduate level courses teach that transforms between discrete- and continuous-time processes exist: they are based on a warping transform $\Omega = f(\omega)$, bijective, continuous and anti-symmetric function from discrete-time frequencies $\omega \in [-\pi, \pi]$ to continuous-time frequencies $\Omega \in \mathbb{R}$. Consequently, the representation of the dilation operator is [36]:

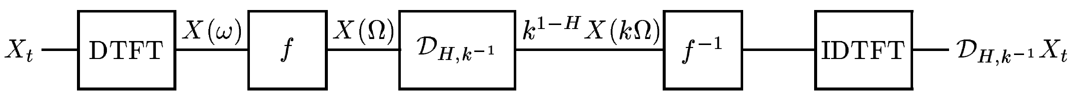

Figure 6: Dilation operator from bilinear transformation applied to a sequence. This general scheme to find the discrete-time equivalent of an operator is enlarged to the difference operator, and can be applied to any operator expressed as a combination of time- and/or frequency-operator.

In [36], a good choice is proposed, the bilinear transformation which reads

$$\Omega = f(\omega) \hat{=} 2\tan(\omega/2). \tag{35}$$

The useful property of the bilinear transform is that it links the Laplace operator p in continuous-time and the z operator in discrete-time. The equality is, for a sampling interval Δt,

$$p = \frac{2}{\Delta t}\frac{1 - z^{-1}}{1 + z^{-1}} \Longleftrightarrow z^{-1} = \frac{1 - p\Delta t/2}{1 + p\Delta t/2}. \tag{36}$$

A different representation of the fractional differencing is obtained as the equivalent, for the previous equality, of the operator p^d, because p acts on the Laplace transform as a derivative in time. The impulse response of this representation as a filter is given after a binomial expansion of $\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^d$. If we note this filter $\sum_{n=0}^{\infty} l_1[n]z^{-n}$, the filter is [36]

$$l_1[n] = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(d+k)\Gamma(-d+n-k)}{\Gamma(k+1)\Gamma(n-k+1)\Gamma(d)\Gamma(-d)}. \tag{37}$$

According to our general block diagram 4, this discrete-time derivative is put in cascade with a log-periodic ARMA and gives a model of sequence with DSI. The particularity is that those sequences have some kind of stationarity if examined through a wavelet lens. If the DSI does not disturb too much a wavelet method of estimation of H (because of the oscillations it induces in a scale diagram), those methods are efficient. The representation in scale by the wavelet transform has then the usual property of stationarizing the coefficients. Comparing to the model in 4.2, this model has DSI mainly driven by the input wheras the self-similar part results from fractional differencing (for $H \neq 1/2$).

A sample path is shown in figure 5. On the right, the log-scale diagram estimated by discrete wavelet transfrom shows two things: the oscillations due to the DSI; the mean slope which estimates H correctly as 0.209 ± 0.019.

5 Some remarks concerning tree-based synthesis techniques

The methods described so far fit well in the theoretical framework of DSI we have proposed. In particular, methods based on generalized difference equations are ad-hoc models that generate DSI signals.

However, there is another general class of synthesis methods which leads to DSI : the class of tree-based constructions. A famous example of this type of construction is the Cantor set. Many known synthesis techniques are tree-based : stochastic process on the dyadic tree of wavelets, constructions of Benassi *et. al.* ... ; in these techniques, the generated signals inherit the DSI property from the tree structure. This property is called "semi-selfsimilarity" by Benassi and his co-workers [37].

For deterministic fractals, a popular method is the so-called Iterated Function System (IFS) approach, a method which is fundamentally tree-based (see [38, 39] and references therein). In the sequel, the IFS we consider act in function spaces. They have been examined by many authors and share as many names as IFSM (with Maps, or Place Dependent IFSM in full generality) [40], Read-Bajraktarević operator [39], self-similar functions in [41] are fixed points of these IFS (see also [42]) ...

Let $\mathbb{X}$ be a compact subset of $\mathbb{R}$, say $\mathbb{X} = [0,1]$, and consider the space of function $L_p(\mathbb{X})$ with the usual distance $d_p(f,g) = (\int |f(t)-g(t)|^p \mathrm{d}t)^{1/p}$. Define N contractive functions $w_i : \mathbb{X} \to \mathbb{X}$ such that $\cup_i w_i(\mathbb{X}) = \mathbb{X}$, and $w_i(\mathbb{X}) \cap w_j(\mathbb{X}) = \emptyset$ if $i \neq j$. Then, an operator is defined as

$$\left\{ \begin{array}{lcl} (Tf)(x) & = & \sum_{i=1}^{N} (T_i f)(x) \\ (T_i f)(x) & = & \varphi_i\left(f(w_i^{-1}(x)), w_i^{-1}(x)\right) \mathbf{1}_{w_i(\mathbb{X})}(x) \end{array} \right. \tag{38}$$

where $\mathbf{1}_I(x) = 1$ if $x \in I$ and 0 if not. The functions $\varphi_i(x,y)$ are supposed to be Lipschitz in variable x, for some choice of the norm ${d_p}^p$ (maybe $p = \infty$). Then it can be shown that T is contractive, and by the fixed point theorem in Banach spaces, the series $T^n f_0$ admits a unique limit $f_\star$ whose graph is usually a fractal set. The system $(\mathbb{X}, \mathbb{Y}, \{w_i\}, \{\varphi_i\})$ defines a so-called IFS systems.

The generalization of IFS to the random case has been done by Falconer, Graf, Mauldin & Williams, ... in the mid-eighties, especially for the case of random sets and random measures [43, 44, 45]. More recently, Hutchinson & Ruschendorf proved several convergence results in the case of random IFS for functions ([46] and ref. therein), defined as above except for the single valued $\varphi_i(x,y) = \varphi_i(x)$. The generalization of deterministic IFS to random IFS relies on the tree underlying the IFS. Indeed, a fractal function is defined as the limit of $T^n(f_0)$; but of course recursivity is present since $(T^n f_0) = T(T^{n-1}(f_0))$. Since $T = \sum_i T_i$ the N-ary tree structure of the IFS construction is clear if we write $(T^n f_0) = \sum_{i=1}^{N}(T_i(T^{n-1} f_0))$. For the random contruction, each T_i at iteration n is a random Lipschitz function of the form (38) (the w_is are however deterministic) independently drawn at each iteration.

Precisely, at the first iteration $T^1 f_0 = \sum_{i_1} T_{i_1} f_0$ where the T_{i_1}s are drawn from the random operators $T_{i=1\ldots N}$. At iteration 2, $T^2 f_0 = \sum_{i_1,i_2} T_{i_1} \circ T_{i_2}^{i_1} f_0$ where the $T_{i_2}^{i_1}$ $(i_{1,2} = 1 \ldots N)$ are drawn from the random operators $T_{i=1\ldots N}$, independently of the T_{i_1}, and the $T_{i_2}^{i}$ and $T_{i_2}^{j}$ are also independent if $i \neq j$. At the nth iteration we write $T^n f_0 = \sum_{i_1,i_2,\ldots,i_n} T_{i_1} \circ T_{i_2}^{i_1} \circ \ldots \circ T_{i_n}^{i_1,i_2,\ldots,i_{n-1}} f_0$ where the $T_{i_n}^{i_1,i_2,\ldots,i_{n-1}}$ are drawn from the random operators $T_{i=1\ldots N}$, independently of the previous iterations. Furthermore, $T_{i_n}^{i_1,i_2,\ldots,i_{n-1}}$ and $T_{i_n}^{j_1,j_2,\ldots,j_{n-1}}$ are independent if $i \neq j$. To illustrate this, we depict in figure 7 the construction for a binary tree and three iterations.

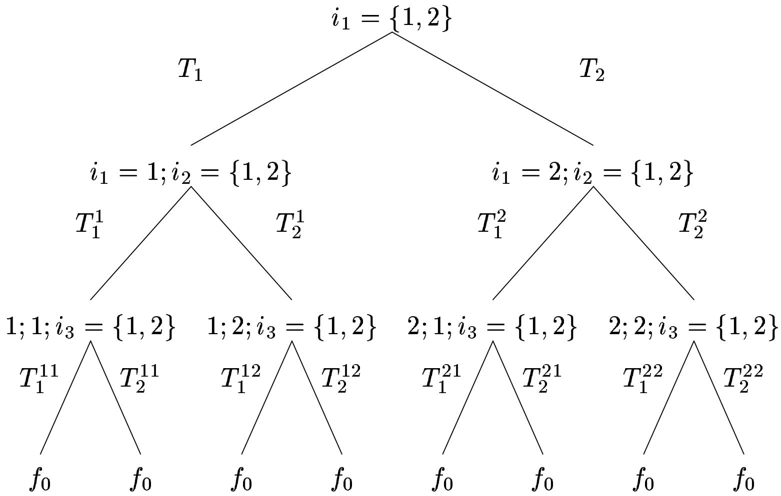

Figure 7: Construction tree of a random IFS. Illustration for a binary tree and three iteration of the construction process.

The main result of Hutchinson & Ruschendorf, proved for single valued φ_is ($\varphi_i(x,y) = \varphi_i(x)$) is summed up in

Theorem 3 ([46]) *Let* $\{(\varphi_i(x)\}_{i=1,\ldots,N}\}$ *be* N *random Lipschitz functions, with Lipschitz (random) constants* r_i*; let* N *contractive functions* w_i *that provide a partition of* $\mathbb{X} = [0,1]$*, with contractivity factor* p_i*; suppose that* $\lambda_p = \mathbb{E}\{\sum_i p_i r_i^p\} < 1$ *and* $\mathbb{E}\{\sum p_i |\varphi_i(0)|^p\} < \infty$ *for some* $1 < p < \infty$*. Then,* $\forall f_0 \in L_p(\mathbb{X})$

$$(\mathbb{E}\{d_p(T^n f_0, f_\star)^p\})^{1/p} \quad \leq \quad \frac{\lambda_p^{n/p}}{1-\lambda_p^{1/p}} (\mathbb{E}\{d_p(f_0, Tf_0)^p\})^{1/p} \longrightarrow 0 \tag{39}$$

when $n \to \infty$*. Furthermore, in the distribution sense* $f_\star$ *is a fixed point of* T*.*

To prove that result, Hutchinson & Ruschendorf use the fact that the space

$$\mathbb{L}_p = \left\{ f(\omega,t), \omega \in \Omega \;/\; \mathbb{E}\left[\int_{\mathbb{X}} |f(\omega,t)|^p \,\mathrm{d}t\right] < +\infty \right\} \tag{40}$$

with the distance $\{\mathbb{E}[d_p(f,g)^p]\}^{1/p}$ is a complete metric space, and that the random operator T as defined above is contractive in that space with this distance.

A very simple example of such a construction occurs for operators of the form $\varphi_i(x,y) = sx + \varepsilon\lambda_i(y)$ where $|s| < 1$ and where $\varepsilon = \pm 1$ with some discrete probability law. Note that only one ε is chosen for all $i = 1, \ldots, N$. Then it can be shown that the fixed point $f_\star$ of T reads

$$f_\star(x) \quad = \quad \sum_{\nu=1}^{+\infty} s^{\nu-1} \varepsilon^{\nu}_{q_1\ldots q_{\nu-1}} \lambda_{q_\nu}(w_{\boldsymbol{q}}^{-1}(x)) \tag{41}$$

where $w_{\boldsymbol{q}} = w_1 \circ \ldots w_{q_\nu}(x)$, where the q_ν are related to the coefficients of x in its N-ary representation, precisely

$$x \quad = \quad \sum_{\nu=1}^{n_x} \frac{q_\nu - 1}{N^\nu}, \text{ where } q_\nu \in \{1,\ldots,N\} \tag{42}$$

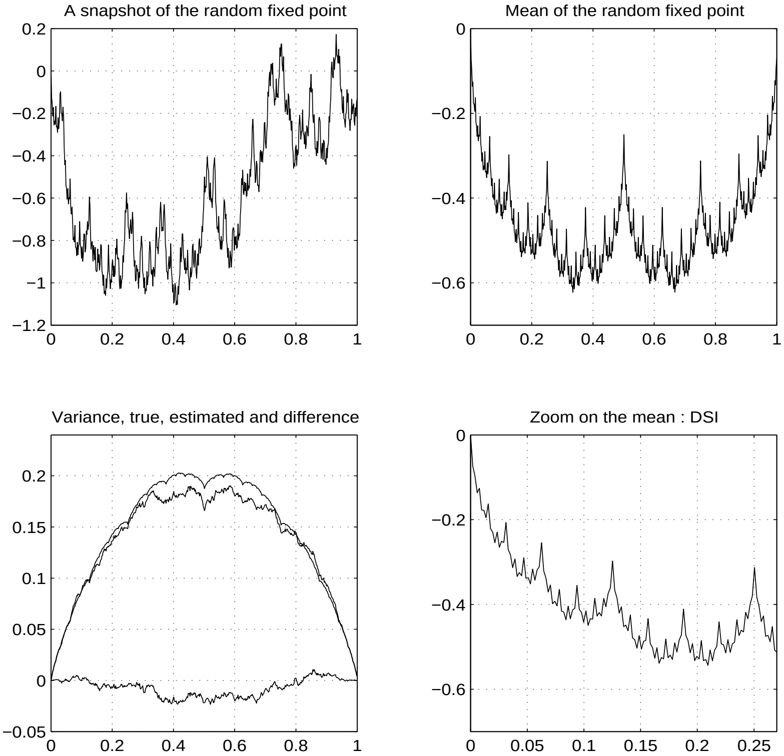

Figure 8: Illustration of DSI in random function which are fixed point of IFS

Note that in (41) we have explicitly indexed ε as $\varepsilon^{\nu}_{q_1\ldots q_{\nu-1}}$ to reflect the tree-based construction as explained above. Indeed, to respect the construction process, the sequence $\varepsilon^{\nu}_{q_1\ldots q_{\nu-1}}$ is an i.i.d. sequence. For one point statistics, this notation is heavy, but it is needed when one evaluate multipoint statistics.

Then it easy to show for example that the nth-order cumulant of $f_\star$ is, up to a multiplicative constant, the fixed point of a deterministic IFS. Indeed, we have the result

$$\mathrm{Cum}_n[f_\star(x)] \quad = \quad \mathrm{Cum}_n[\varepsilon]\sum_{\nu=1}^{+\infty} s^{n(\nu-1)}\lambda^n_{q_\nu}(w_{\boldsymbol{q}}^{-1}(x)) \tag{43}$$

Therefore, the statistics of $f_\star(x)$ are selfsimilar functions in the sense of being the fixed point of an IFS. This is illustrated in figure 8. For this figure we have chosen $w_1^{-1}(x) = 2x$, $w_2^{-1}(x) = 2x - 1$, $\lambda_1(x) = x/2, \lambda_2(x) = (1-x)/2$, $s = 3/4$ and $\varepsilon = (-1, +1)$ with probability (0.75,0.25). We have plotted one snapshot of the random fixed point $f_\star(x)$ (top left panel); the theoretical mean (top right panel); the variance of the fixed point, the variance being estimated by averaging on 1000 snapshots (bottom left panel). Furthermore, we plot in this last panel the difference between the theoretical and estimated cumulant. Finally, the bottom right panel depicts a zooming of the theoretical mean: the DSI property can be seen in the mean where log-periodic patterns clearly appear.

The IFS approach provides an alternative definition of DSI. The main difference between this approach and the general framework proposed in the paper is the restriction of IFS based random functions to compact sets of the line. Therefore, the DSI property cannot be defined classically for IFS fixed points and is indeed replaced by the fixed point condition $f_\star = Tf_\star$ in a distribution sense. Finally, let us mention that if the DSI property comes from the tree structure, the value of the log-period is not restricted to the degree of the tree. In fact, the log-period is also related to the parameters of the contractions w_is. For a binary tree, one can choose the contractions so that the lengths of $w_1(\mathbb{X})$ and of $w_2(\mathbb{X})$ are different, leading to a log-period different of 2.

6 Conclusion

We have detailed here some efficient methods to generate random sequences with the property of Discrete Scale Invariance. More could be said on this. We have only described one possibility to generate DSI with tree-based synthesis. Considering a tree is indeed a good means to put a stochastic behaviour on a system with a preferred scale factor – the geometric interpretation shows clearly the DSI. When one wanted H-ss processes, this property was hidden as much as possible, but the property is here put forward when one studies DSI. Wavelet-based synthesis, models of cascades with DSI might be envisaged – the majority of the work is already done elsewhere, see for example [47] – laying the emphasis on the preferred scale factor induced by the tree structure.

The Weierstrass model, formulated above from an original point of view, was also seen as a way to generate approximate H-ss signals [20] (or as ARMA systems with frequencies distributed in λ^n [48]) which are in fact true DSI processes. The purpose here was to propose and illustrate different ways to generate DSI in random processes and describe their theoretical rootings.

One application of the proposed framework for stochastic DSI, and consequently of the proposed methods of synthesis, is to refine the ways of analysis of DSI and estimation of H and λ. The approaches described elsewhere [7] are not sufficient even if they work in simple situations. Comparing to the works of Sornette and co-authors, the proposed framework to seek DSI does not apply to the same kind of situations. They considered first pure deterministic DSI and ways to analyze it, then used adapted processing to transform time signals in integrated quantities (such as distribution functions or moments) which are supposed to show DSI in a deterministic manner and display a few oscillations only. A refined example of such a method is exposed in [4]. The point of view coming from the Lamperti transformation is, on the contrary, that the signal with time is directly used to find some scale periodicity. We hope then that the number of log-periods in the signal will be higher than on integrated quantities, as is the case in the synthetic sequences shown here. Efficient tools might be envisaged with the extension of the Lamperti transformation directly on discrete-time processes; or with the combination of Mellin analysis with suitable averaging methods not entirely based on stationarity.

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