Concurrent Games with Symmetry

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Context and motivations

1. What?
   Develop and extend the "truly concurrent" approach to game semantics.

2. Why?
   - Develop compositional partial order models of complex programming languages,
   - Get rid of artificialities in standard presentations of games, due to an excessive sequentiality.
   - Provide new, unified foundations for denotational semantics.

3. How?
   View strategies as **maps of event structures**, focus on **causality**.

In this talk, I will focus on the addition of symmetry, to model replication.

Game semantics

- Two players: P (Player/Program) and O (Opponent/Environment)
- Very expressive:
  - Models of logics/proof systems: λ-calculus, $\lambda\mu$-calculus, LL, System F, Induction/coinduction, etc.
  - Models of programming features: PCF, control operators, references, exceptions, names, concurrency, etc.
  - Presents higher-order computation as an interaction between two players exchanging first-order tokens.
- Algorithmic: verification procedures for third order Idealized Algol, fragments of ML or Parallel Algol. . .

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Models of concurrency

Two flavours:

- **Interleaving models**

$$[\![a \parallel b]\!] \quad = \quad ab \; + \; ba$$

  - More elementary/easier to manipulate
  - Subject to a combinatorial explosion problem.

- **Partial order/"Truly concurrent" models**.

$$[\![a \parallel b]\!] \quad = \quad a \;\; co \;\; b$$

  - More mathematically elegant/elaborate,
  - Avoids the state explosion,
  - Models are often at a more preliminary stage.

Game semantics for concurrency

Two flavours:

- **Interleaving models**
  - Laird: Game semantics for Idealized CSP.
  - Ghica, Murawski: Full abstraction for Parallel Algol.
    $\hookrightarrow$ Verification procedures for fragments of PA.
- **"Truly concurrent" models**
  - Abramsky, Melliès: based on closure operators, full completeness for MALL.
  - Melliès: based on asynchronous transition systems, full completeness for LL.
  - Melliès, Mimram: (non-alternating) asynchronous games.
  - Faggian, Piccolo: Concurrent strategies as partial orders,
  - Rideau, Winskel: Concurrent strategies as maps of event structures.
    $\hookrightarrow$ We will work with this formulation of concurrent games.

Motivations for true concurrency games models

Semantics/verification: Better structures for efficient program verification.

- Interleaving models are subject to the state explosion problem.
  $\hookrightarrow$ Partial order games models avoid it.

- Interleaved models characterize trace equivalence.
  $\hookrightarrow$ Partial order models can be more sensitive.

Logic/proof theory: Sheds light on positionality in game semantics.

- Interleaving models orders events chronologically, partial order models orders them causally.
  $\hookrightarrow$ Assimilates reachable states regardless of the path used to reach them.
  $\hookrightarrow$ Clean notion of positions, getting rid of artificialities in standard presentations (e.g. winning).

- Two strategies reaching the same states using different paths can be assimilated.
  $\hookrightarrow$ Possibility of ignoring excessive sequentialization, and obtain by quotients models of LL

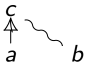
Outline

I. Concurrent Games

Event structures: a model of concurrent behaviour.

An event structure comprises $(E, \leqslant, \mathrm{Con})$ where

1. E is a set of **events**,
2. partially ordered by $\leqslant$ (the **causal dependency** relation),
3. together with a **consistency** relation Con.



Properties:

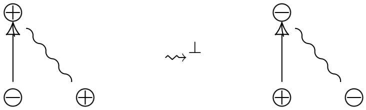
- States in an event structure are (consistent and $\leqslant$-downclosed) sets of events, called **configurations**: $\mathcal{C}(E)$.
- Events can carry information, e.g. labels, polarities ($pol : E \to \{+, -\}$), etc.

Notations:

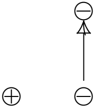
- Write $e_1 \rightarrowtriangle e_2$ iff $e_1 < e_2$ and if $e_1 \leqslant e \leqslant e_2$, then $e_1 = e$ or $e = e_2$.
- For $x \in \mathcal{C}(E)$, write $x \overset{e}{\longrightarrow\!\!\subset}$ iff $x \cup \{e\} \in \mathcal{C}(E)$.

Two simple operations on event structures with polarities

1. Dual.
   The dual, $E^{\perp}$, of an event structure with polarities, E comprises the same underlying event structure E but with a reversal of polarities.



2. Simple parallel composition.
   This operation juxtaposes two event structures E_1 and E_2 to form $E_1 \| E_2$.



where $E_1 = \oplus$ and $E_2 = \ominus \rightarrowtriangle \ominus$.

Maps of event structures and pre-strategies

Definition

A map $f : E \to E'$ of event structures is a function on events such that

- **Preservation of configurations.** For all $x \in \mathcal{C}(E)$, $f(x) \in \mathcal{C}(E')$,
- **Local injectivity.** For all $x \in \mathcal{C}(E)$, $e, e' \in x$, if $f(e) = f(e')$ then $e = e'$.

A map of event structures with polarities must additionally preserve polarities.

Definition (Pre-strategies)

A **pre-strategy** from A to B is a **total** map

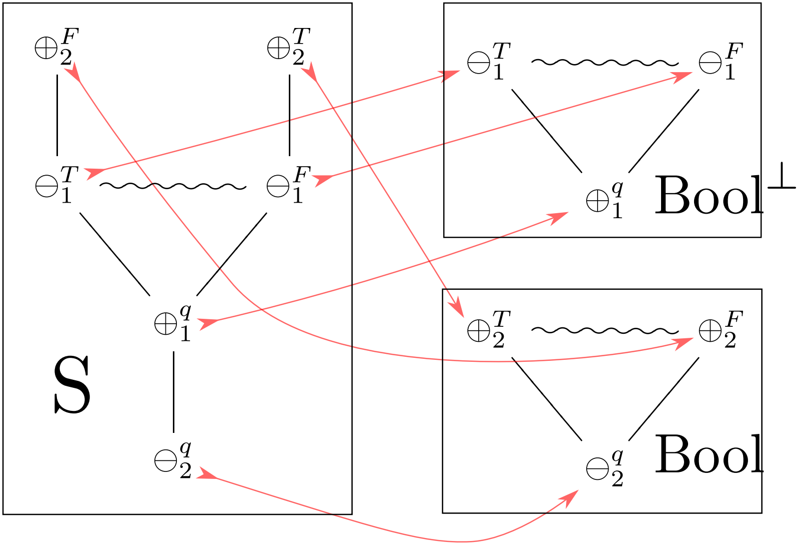
$$\sigma : S \to A^{\perp} \parallel B$$

of event structures with polarities.

We sometimes write $\sigma : A \mathrel{\longrightarrow\!\!\!\!\!\!+\!\!\!\!\!\!\!\!\!\longrightarrow} B$ for $\sigma : S \to A^{\perp} \parallel B$.

Example: the negation strategy

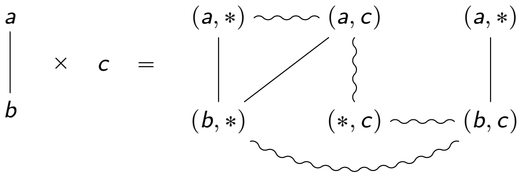
$$\mathrm{Neg} \;:\; \mathrm{S} \rightarrowtail \mathrm{Bool}^{\perp} \parallel \mathrm{Bool}$$



Two operations on event structures with polarities

Notation: $[e] = \{e' \in E \mid e' \leqslant e\}$

1. Product.



2. Restriction.
   If $(E, \leqslant, \mathrm{Con})$ is an event structure and $R \subseteq E$, then $(E, \leqslant, \mathrm{Con}) \upharpoonright R = (E', \leqslant', \mathrm{Con}')$ where:

$$\begin{array}{rcl} E' & = & \{e \in E \mid [e] \subseteq R\} \\ \leqslant' & = & \leqslant \cap (E' \times E') \\ \mathrm{Con}' & = & \{X \cap E' \mid X \in \mathrm{Con}\} \end{array}$$

Put together, these constructions build **pullbacks** on the category of event structures (with pol.) and maps between them.

Composition via pullback

Take $\sigma : S \to A^{\perp} \parallel B$ and $\tau : T \to B^{\perp} \parallel C$. Temporarily forget polarities...

$$\begin{tikzcd}
S \parallel C \arrow[dr, "\sigma \parallel C"'] & & A \parallel T \arrow[dl, "A \parallel \tau"] \\
& A \parallel B \parallel C &
\end{tikzcd}$$

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1. **"Parallel interaction"** Take the pullback.

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Composition via pullback

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$$
\begin{tikzcd}
 & P \arrow[dl] \arrow[dr] \arrow[rr] \arrow[dd, phantom, "\vee" very near start] & & P \downarrow V = T \odot S \arrow[dd, "\tau \odot \sigma"'] \\
S \parallel C \arrow[dr, "\sigma \parallel C"'] & & A \parallel T \arrow[dl, "A \parallel \tau"] & \\
 & A \parallel B \parallel C \arrow[rr, "A \parallel \perp \parallel C"'] & & A \parallel C
\end{tikzcd}
$$

1. **"Parallel interaction"** Take the pullback.
2. **"Hiding"** Project to the outer events.

Copycat

Notation: if $a \in A^\perp \parallel A$, then $\overline{a}$ is its **dual**.

Definition

If A is a game, the event structure $\mathbb{C}\!\mathbb{C}_A$ is defined as follows:

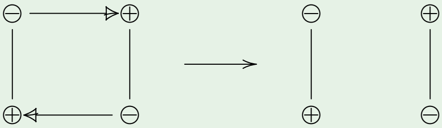
- Its events and consistency are inherited from $A^\perp \parallel A$,
- Its causality is:

$$((\leqslant_{A^\perp \parallel A}) \cup \{(a, \overline{a}) \mid a \in A^\perp \parallel A \ \& \ \mathrm{pol}(a) = -\})^*$$

Example

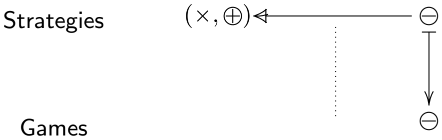
Consider $A = \ominus \rightarrowtriangle \oplus$

$$\mathbb{C}\!\mathbb{C}_A \xrightarrow{\quad \gamma_A \quad} A^\perp \parallel A$$

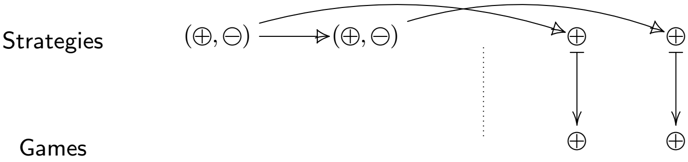


Not all pre-strategies compose well with copycat

1. Non-receptivity. Compose $\sigma : \varnothing \to \{\ominus\}$ with copycat.



2. Non-innocence. Compose $\sigma : (\oplus \rightarrowtriangle \oplus) \to (\oplus \parallel \oplus)$ with copycat.



Strategies

Notation: if $x \in \mathcal{C}(A)$, $x \stackrel{e}{\longrightarrow\!\!\!\subset}$ if $x \cup \{e\} \in \mathcal{C}(A)$.

Definition

A pre-strategy $\sigma : S \to A$ is a **strategy** iff it is:

1. Receptive If for all $x \in \mathcal{C}(S)$ and $a \in A$ with $\mathrm{pol}(a) = -$ and $\sigma x \stackrel{a}{\longrightarrow\!\!\!\subset}$, **then** there is a unique $s \in S$ such that $x \stackrel{s}{\longrightarrow\!\!\!\subset}$ and $\sigma(s) = a$.
2. Innocence If $s_1 \rightarrowtriangle s_2$ in S with $\mathrm{pol}(s_1) = +$ or $\mathrm{pol}(s_2) = -$, **then** $\sigma s_1 \rightarrowtriangle \sigma s_2$ in A.

Theorem (Rideau & Winskel, 2011)

1. *A pre-strategy $\sigma : S \to A$ is a strategy iff there is an iso:*

$$\begin{tikzcd}
\mathbb{CC}_A \odot S \arrow[rr, bend left, ""{name=U}] \arrow[dr, "\gamma_A \odot \sigma"'] & & S \arrow[ll, bend left, ""{name=D}] \arrow[dl, "\sigma"] \arrow[from=U, to=D, phantom, "="] \\
& A \arrow[u, phantom, "="] &
\end{tikzcd}$$

2. *There is a bicategory of concurrent games and strategies.*

II. Concurrent games with symmetry

Motivations and background

1. Goals.
   - Express that several events are interchangeable
   - Build exponentials in concurrent games.
2. Replication and uniformity in games.
   - AJM games: partial equivalence relations on plays.
   - HO/N games: thread indexing by pointers.
   - Asynchronous games: groups of index permutations acting on plays.
   - Hyland exponential, Lamarche exponential and sequential algorithms, ...

We exploit the notion of symmetry on event structures.

Event structures with symmetry

- **Open maps**[1] provide an abstract notion of bisimulation.
- Defined by a path lifting property.

Definition

An **event structure with symmetry** is an event structure E with a **span**

$$\begin{tikzcd} & \widetilde{E} \arrow[dl, "l_E"'] \arrow[dr, "r_E"] & \\ E & & E \end{tikzcd}$$

where l_E, r_E are **open**, **jointly monic** and form an **equivalence relation**.

[1] A. Joyal, M. Nielsen and G. Winskel, Bisimulation from Open Maps, LICS'93

Concrete presentation on event structures: isomorphism families

A symmetry on E can be presented as an **isomorphism family** on E, i.e. a family of bijections $x \overset{\theta}{\cong}_E y$ between pairs $x, y \in \mathcal{C}(E)$ satisfying:

1. Equivalence relation.

$$
\begin{array}{rcl}
x \in \mathcal{C}(E) & \Longrightarrow & x \overset{\mathrm{id}_x}{\cong}_E x \\
x \overset{\theta}{\cong}_E y & \Longrightarrow & y \overset{\theta^{-1}}{\cong}_E x \\
x \overset{\theta_1}{\cong}_E y \ \&\ y \overset{\theta_2}{\cong} z & \Longrightarrow & x \overset{\theta_2 \circ \theta_2}{\cong}_E z
\end{array}
$$

2. Restriction.

$$
\begin{array}{ccc}
x & \overset{\theta}{\cong}_E & y \\
\cup\!| & & \\
x' & &
\end{array}
\quad \Longrightarrow \quad
\begin{array}{ccc}
x & \overset{\theta}{\cong}_E & y \\
\cup\!| & \cup\!| & \cup\!| \\
x' & \overset{\theta'}{\cong}_E & y'
\end{array}
$$

3. Extension.

$$
\begin{array}{ccc}
x' & & \\
\cup\!| & & \\
x & \overset{\theta}{\cong}_E & y
\end{array}
\quad \Longrightarrow \quad
\begin{array}{ccc}
x' & \overset{\theta'}{\cong}_E & y' \\
\cup\!| & \cup\!| & \cup\!| \\
x & \overset{\theta}{\cong}_E & y
\end{array}
$$

Event structures with symmetry

Event structures with symmetry are pairs $(E, \widetilde{E})$.

Definition

If $f : E \to F$ is a map, and $\theta : x \cong y$ is a bijection, define:

$$f\ \theta = \{(f\ a, f\ b) \mid (a, b) \in \theta\}$$

Then, f **preserves symmetry** iff

$$x \overset{\theta}{\cong}_E y \implies f\ x \overset{f\ \theta}{\cong}_F f\ y$$

Definition

Two maps $f, g : E \to F$ are **symmetric** iff for all $x \in \mathcal{C}(E)$, the bijection

$$\{(f\ e, g\ e) \mid e \in x\}$$

is in the isomorphism family.

We get a category $\mathcal{ES}$ of event structures with symmetry, enriched on equivalence relations.

Structure in $\mathcal{ES}$

No pullbacks! Instead:

Definition (Pseudo-pullbacks)

$$
\begin{tikzcd}
 & X \arrow[dl, bend right] \arrow[d, dotted] \arrow[dr, bend left] & \\
 = & P \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & = \\
A \arrow[dr] & \sim & B \arrow[dl] \\
 & C &
\end{tikzcd}
$$

Definition

From A, we can get $\widetilde{A}$ as an event structure with symmetry, by forming:

$$
\begin{tikzcd}
 & \widetilde{A} \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & \\
A \arrow[dr, "\mathrm{id}_A"'] & \sim & A \arrow[dl, "\mathrm{id}_A"] \\
 & A &
\end{tikzcd}
$$

Towards concurrent games with symmetry

How to extend concurrent games with symmetry?

- **Games.** Event structures with polarities and symmetry.
- **Pre-strategies.** Morphism $\sigma : S \to A$ preserving symmetry.
- **Composition.** ~~Pullbacks~~ $\longrightarrow$ **pseudo-pullbacks.**

What strategies compose well with copycat?

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What strategies compose well with copycat?

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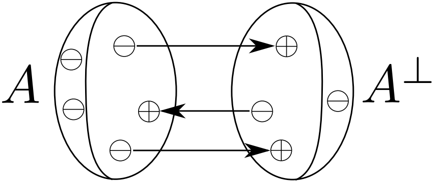
What is copycat?

- Let us try the same definition.

$$x_1 \supseteq^- x_2 \overset{\overline{(-)}}{\mapsto} \overline{x_2} \subseteq^- \overline{x_3}$$



- But what is $\widetilde{\mathbb{C}\mathbb{C}}_A$?

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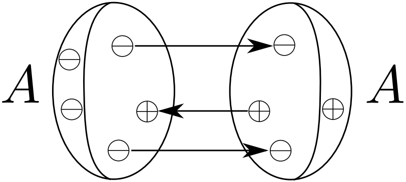
$$x_1 \supseteq^- x_2 = x_2 \subseteq^+ x_3$$



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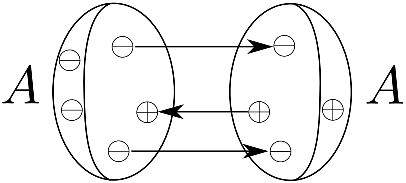

A configuration of $\mathbb{C}\!\mathbb{C}_A$ corresponds to a pair $x_1 \sqsubseteq x_3$ in the Scott order, defined by:

$$\sqsubseteq \quad = \quad \supseteq^- \cdot \subseteq^+$$

(strategies can be reformulated as fibrations on the Scott order)

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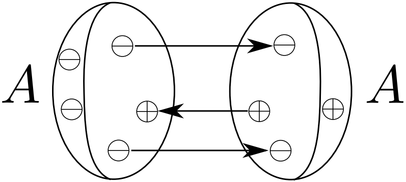

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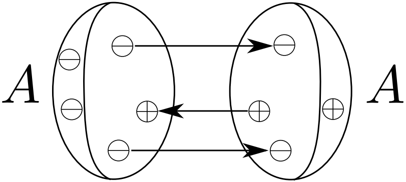

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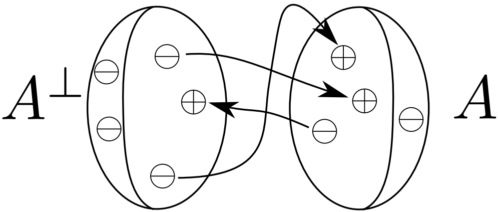
- But what is $\widetilde{\mathbb{C}\!\mathbb{C}_A}$?

  All the natural candidates fail, unless one assumes polarization. . .

What strategies compose well with copycat?

We need a **saturated** version of copycat.

$$x_1 \supseteq^- x_2 \overset{\theta}{\cong}_A x_2 \subseteq^+ x_3$$



"Definition"

A **configuration** *of* $\mathbb{C}_A$ *is a situation:*

$$x_1 \supseteq^- x_2 \overset{\theta}{\cong}_A \overline{x_3} \subseteq^+ \overline{x_4}$$

where $x_1, x_2, x_3, x_4 \in \mathcal{C}(A)$.

What strategies compose well with copycat?

$$x_0 \quad \supseteq^- \quad x_1 \quad \stackrel{\theta}{\cong}_A \quad x_2 \quad \subseteq^+ \quad x_3$$

$$x_0' \quad \supseteq^- \quad x_1' \quad \stackrel{\theta'}{\cong}_A \quad x_2' \quad \subseteq^+ \quad x_3'$$

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$$
\begin{array}{ccccccc}
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Remarks.

- We use the symmetry on the symmetry.
- This definition amounts to:

$$\widetilde{\mathbb{CC}_A} \quad = \quad \mathbb{CC}_{\tilde{A}}$$

- The "Scott order":

$$x_0 \sqsubseteq x_2 \qquad = \qquad x_0 \supseteq^- x_1 \subseteq^+ x_2$$

is replaced by a "Scott category":

$$x_0 \xrightarrow{\theta} x_3 \qquad = \qquad x_0 \supseteq^- x_1 \overset{\theta}{\cong}_A x_2 \subseteq^+ x_3$$

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$$x_0 \sqsubseteq x_2 \quad = \quad x_0 \supseteq^- x_1 \subseteq^+ x_2$$

is replaced by a "Scott category":

$$x_0 \xrightarrow{\theta} x_3 \quad = \quad x_0 \supseteq^- x_1 \overset{\theta}{\cong}_A x_2 \subseteq^+ x_3$$

What strategies compose well with copycat?

$$\begin{array}{ccccccc}
x_0 & \supseteq^- & x_1 & \overset{\theta}{\cong}_A & x_2 & \subseteq^+ & x_3 \\
\overset{\theta_0}{\cong}_A & \supseteq^- & \overset{\theta_1}{\cong}_A & \overset{\phi}{\cong}_{\tilde{A}} & \overset{\theta_2}{\cong}_A & \subseteq^+ & \overset{\theta_3}{\cong}_A \\
x_0' & \supseteq^- & x_1' & \overset{\theta'}{\cong}_A & x_2' & \subseteq^+ & x_3'
\end{array}$$

Remarks.

- We use the symmetry on the symmetry.
- This definition amounts to:

$$\begin{array}{ccc}
\widetilde{\mathbb{C}\mathbb{C}_A} & = & \mathbb{C}\mathbb{C}_{\tilde{A}} \\
\widetilde{T \odot S} & = & \tilde{T} \odot \tilde{S}
\end{array}$$

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What strategies compose well with copycat?

What equivalence on pre-strategies with symmetry?

Definition (Equivalent pre-strategies)

$$\begin{tikzcd} S \arrow[rr, bend left, "\sim"] \arrow[dr, "\sigma"'] & & T \arrow[ll, bend left] \arrow[dl, "\tau"] \\ & A \arrow[u, phantom, "="] & \end{tikzcd}$$

We write:

$$\sigma \simeq \tau$$

What strategies compose well with copycat?

Definition (Action of the symmetry)

Take the pseudo-pullback:

$$\begin{tikzcd}
 & S \times_A A \arrow[dl] \arrow[dr, "\pi_2"] \arrow[d, phantom, "\lrcorner" rotate=-45] & \\
S \arrow[dr, "\sigma"'] & \sim & A \arrow[dl, "\mathrm{id}_A"] \\
 & A &
\end{tikzcd}$$

Definition (Saturated strategy)

A strategy $\sigma : S \to A$ is **saturated** if the canonical map $\eta_S : S \to S \times_A A$ has an adjoint $act_S : S \times_A A \to S$ making an equivalence:

$$\begin{tikzcd}
S \times_A A \arrow[rr, bend left, "act_S"] \arrow[dr, "\pi_2"'] & \sim & S \arrow[ll, bend left, "\eta_S"] \arrow[dl, "\sigma"] \\
 & A \arrow[u, phantom, "="] &
\end{tikzcd}$$

Summary of results

Definition

A pre-strategy $\sigma : S \to A$ is **strong-receptive** if $\tilde{\sigma} : \tilde{S} \to \tilde{A}$ is receptive.

Theorem

- *A **strategy** $\sigma : S \to A$ is a strong-receptive, innocent and saturated pre-strategy. We have:*

$$\gamma_A \odot \sigma \simeq \sigma$$

- *Concurrent games with symmetry and strategies form a $\sim$-bicategory (coherence laws satisfied up to symmetry) Strat.*

Theorem

The quotient $\mathrm{Strat}/\simeq$ of this $\sim$-bicategory by equivalence is a compact closed category, hence a model of MLL.

III. Applications

The AJM exponential

Definition

From a game with symmetry A, form $!A$ having:

- **Events**, pairs $(i, a) \in \mathbb{N} \times A$
- **Causality**,
$$(i_1, a_1) \leqslant_{!A} (i_2, a_2) \Leftrightarrow i_1 = i_2 \ \&\ a_1 \leqslant_A a_2$$
- **Consistency**,
$$\mathrm{Con}_{!A} = \bigcup_{i \in I} \{i\} \times X_i$$
- **Isomorphism family**,
$$\bigcup_{i \in I} \{i\} \times x_i \quad \overset{\theta}{\cong}_{!A} \quad \bigcup_{j \in J} \{j\} \times x_j$$
when there is a bijection $\pi : I \to J$ and isomorphisms $x_i \overset{\theta_i}{\cong}_A x_j$ with, for all $(i, a) \in \bigcup_{i \in I} \{i\} \times x_i$,
$$\theta(i, a) = (\pi(i), \theta_i(a))$$

AJM games and Classical Linear Logic

We recover (and extend) the model of [2].

Theorem

Concurrent games with symmetry form a model of classical linear logic

Proof.

We have natural maps preserving symmetry:

$$\begin{array}{llrcl}
\mu_A & : & !!A & \to & !A \\
 & & (i,(j,a)) & \mapsto & (\langle i,j\rangle, a) \\
 & & & & \\
\eta_A & : & A & \to & !A \\
 & & a & \mapsto & (0,a)
\end{array}
\qquad
\begin{array}{llrcl}
m_A & : & !A \parallel !A & \to & !A \\
 & & (1,(i,a)) & \mapsto & (2i,a) \\
 & & (2,(i,a)) & \mapsto & (2i+1,a) \\
 & & & & \\
e_A & : & 1 & \to & !A
\end{array}$$

satisfying monad/monoid laws up to symmetry. Those are lifted to strategies with a general construction, we get an exponential by self-duality. □

[2] P. Baillot, V. Danos, T. Ehrhard and L. Regnier, Believe it or not, AJM's games model is a model of classical linear logic, LICS'97

Concurrent HO games: arenas

Definition

An **arena** is a forest $(A, \leqslant)$ with polarity that is **negative**, in the sense that all minimal events have negative polarity.

Definition (Constructions on arenas)

From two arenas A and B, form:

1. The arena $A \parallel B$ as usual,
2. The arena $A \multimap\!\!\boxplus B$, having:
   - Events, those of $(\|_{b \in \min(B)} A) \parallel B$
   - Causality, that of $(\|_{b \in \min(B)} A) \parallel B$, plus:

$$\{((2, b), (1, (b, a))) \mid b \in \min(B) \;\&\; a \in A\}$$

where $\min(B)$ is the set of minimal events of B.

Those are the usual constructions $\times$ and $\Rightarrow$ on arenas in HO games.

Concurrent games "with pointers"

Definition

From any arena A, form a concurrent game with symmetry $\wn A$ having:

- Events, pairs (α, a) where $a \in A$ and $\alpha : [a] \to \mathbb{N}$.
- Causality,

$$(\alpha_1, a_1) \leqslant_{\wn A} (\alpha_2, a_2)$$

  iff $\alpha_1 \sqsubseteq \alpha_2$, i.e.
  - $a_1 \leqslant a_2$,
  - for all $a \leqslant a_1$, $\alpha_1(a) = \alpha_2(a)$.
- Consistency, all finite sets of events are consistent.
- Symmetry, the family consisting of order-preserving bijections $\theta : x \cong y$ such that for all $(\alpha, a) \in x$,

$$\theta(\alpha, a) = (\alpha', a)$$

We could say that (α, a) "points to" (α', a') iff $\alpha' \sqsubseteq \alpha$, and $a' \rightarrowtriangle a$.

Example

Take the arena of booleans $\mathrm{Bool} = \begin{tikzcd}[column sep=small] \oplus^T \arrow[dr, dash] & & \oplus^F \arrow[dl, dash] \\ & \ominus^q & \end{tikzcd}$.

Example

The following is a configuration $x \in \mathcal{C}(\text{‽}\mathrm{Bool})$.

$$\begin{tikzcd}[column sep=small] (2, \oplus^T) \arrow[d, dash] & (0, \oplus^T) \arrow[dr, dash] & & (1, \oplus^F) \arrow[dl, dash] & (5, \oplus^F) \arrow[d, dash] \\ (1, \ominus^q) & & (4, \ominus^q) & & (7, \ominus^q) \end{tikzcd}$$

So, all events can be duplicated deep in the arena.

- For now, no well-bracketing – eventually, only one copy of $\oplus^T$ or $\oplus^F$ should be allows by copy of $\ominus^q$ (unless one wants to model $\mathrm{call/cc}$).
- In a play with pointers, all the different copies of moves would appear in some chronological order. Here, they are causally unrelated (like in Boudes' notion of thick subtrees).

Well-threaded strategies

Definition

A **HO-strategy** on arena A is a (strong-receptive, innocent, saturated) strategy

$$\sigma : S \to \text{‽}A$$

that is well-threaded and negative, in the sense that for each $s \in S$, $[s]$ has a unique (negative) minimal event.

Lemma

There is a one-to-one correspondence between:

- *HO-strategies* $\sigma : S \to \text{‽}(A \mathbin{-\!\!\boxplus} B)$,
- *Well-threaded, negative strategies* $\sigma' : S \to \text{‽}A^{\perp} \parallel \text{‽}B$

Using this, HO-strategies $\sigma : S \to \text{‽}(A \mathbin{-\!\!\boxplus} B)$ *and* $\tau : T \to \text{‽}(B \mathbin{-\!\!\boxplus} C)$ *can be composed in* $\mathrm{Strat}/\simeq$.

Using this, we build a SMCC called CHO of arenas and HO-strategies.

The CCC of concurrent HO-strategies

Lemma

There are negative, well-threaded strategies:

$$d_A : {?}A \twoheadrightarrow {?}A \parallel {?}A \qquad\qquad e_A : {?}A \twoheadrightarrow 1$$

making each arena A a comonoid in CHO*.*

Proof.

As for AJM games, the strategies are automatically lifted from the monoid in the category of maps of event structures with polarity and symmetry. □

Proposition

Negative, well-threaded strategies are comonoid morphisms, so the category CHO *is cartesian closed.*

HO-innocence

Definition

A HO-strategy $\sigma : S \to \wn A$ is

- **HO-innocent** iff for all $s \in S$, if $s_1, s_2 \leqslant s$ have negative polarity, then $s_1 \leqslant s_2$ or $s_2 \leqslant s_1$.
- **Sequential** iff for each $s \in S$, if $s_1 \neq s_2$ have positive polarity and $[s] \stackrel{s_1}{\longrightarrow\!\!\subset}, [s] \stackrel{s_2}{\longrightarrow\!\!\subset}$, then $\{s_1, s_2\} \notin \mathrm{Con}_S$.

Theorem

The subcategory of CHO *having alternating arenas as objects and HO-innocent, sequential, deterministic strategies is* <u>*isomorphic*</u> *to the standard category of arenas and innocent strategies.*

Proof.

For $\sigma : S \to A$ HO-innocent, the prime configurations $[s] \in \mathcal{C}(S)$ are **P-views**:

$$s_1^- \rightarrowtriangle s_2^+ \rightarrowtriangle s_3^- \rightarrowtriangle \ldots \rightarrowtriangle s_n^+$$

that are O-branching (up to symmetry) by sequentiality and determinism. $\square$

IV. Conclusion

Conclusion

Achievements:

1. General framework for concurrent games with symmetry,
2. Able to express and extend the approach of replication of AJM and HO games.

Research directions:

1. Relating concurrent game semantics and operational semantics for a higher-order concurrent language,
2. Modeling state in the category CHO, relationship with the model of Ghica and Murawski,
3. "Folded" version of this model, possibly based on Petri nets, and applications to program verification.