Learning to count up to symmetry

$\overline{\mathbb{N}}$-Rel :

objects : sets

morphisms : $(\alpha)_{a,b} \in \overline{\mathbb{N}}^{A \times B}$

composition :

$$(\beta \circ \alpha)_{a,c} = \sum_{b \in B} \alpha_{a,b} \times \beta_{b,c}$$

$$\mathrm{NPCF} \xrightarrow{[\![-]\!]} \overline{\mathbb{N}}\text{-Rel}$$

$$[\![M]\!]_{tt} = \#(M \Downarrow tt)$$

"What does
the weighted relational model
cant ?"

1. Counting, when life was simple

Definition: An event structure is

binary irreflexive **conflict** relation

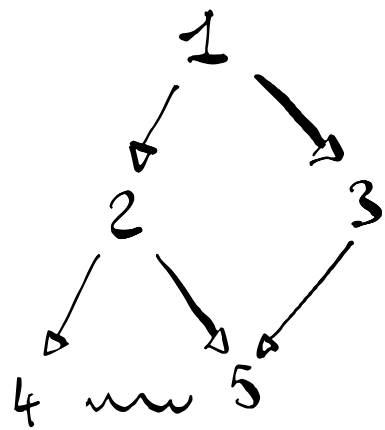
$$E = (|E|, \leq_E, \#_E)$$

set of **events**

partial order, **causality**

with a few axioms.

Example:

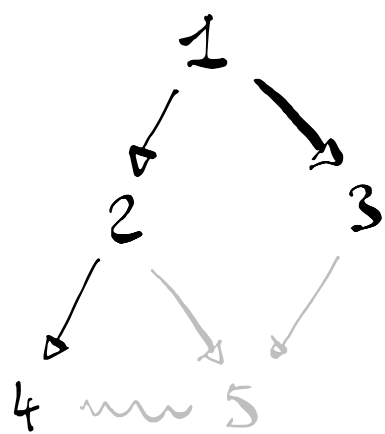


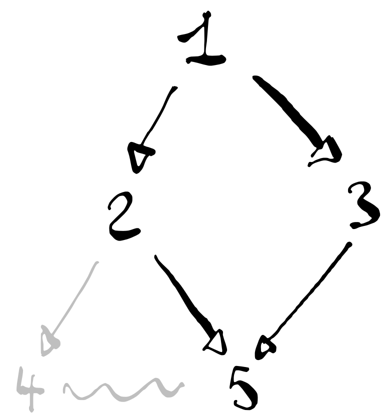
Definition: A configuration $x \in \mathscr{C}(E)$ is $x \subseteq_f |E|$

(1) down-closed: if $e \leq_E e' \in x$, then $e \in x$

(2) consistent: if $e_1, e_2 \in x$, then $\neg(e_1 \#_E e_2)$

Example:

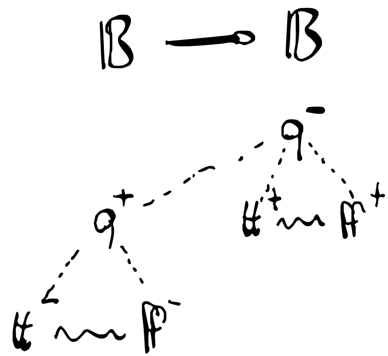


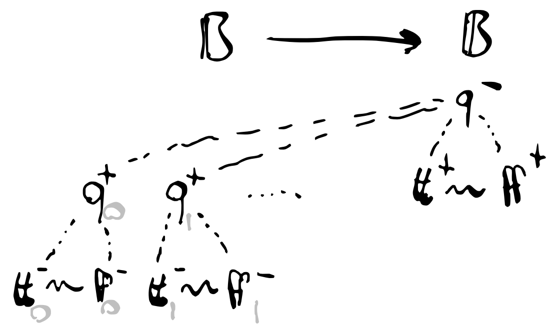


Definition: A *game* is $A = (|A|, \leq_A, \#_A, \mathrm{pol}_A)$

$$\mathrm{pol}_A : |A| \longrightarrow \{-, +\}$$

Examples:





Definition: A strategy $\sigma : A$ is

"display map"

$$(|\sigma|, \leq_\sigma, \#_\sigma, \partial_\sigma : |\sigma| \to |A|)$$

(1) $\forall x \in \mathscr{C}(\sigma), \quad \partial_\sigma(x) \in \mathscr{C}(A)$

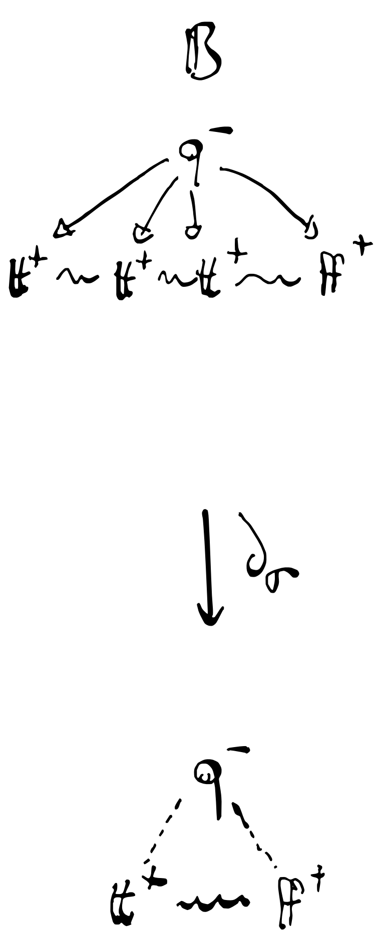
(2) $\forall s_1, s_2 \in x \in \mathscr{C}(\sigma),$

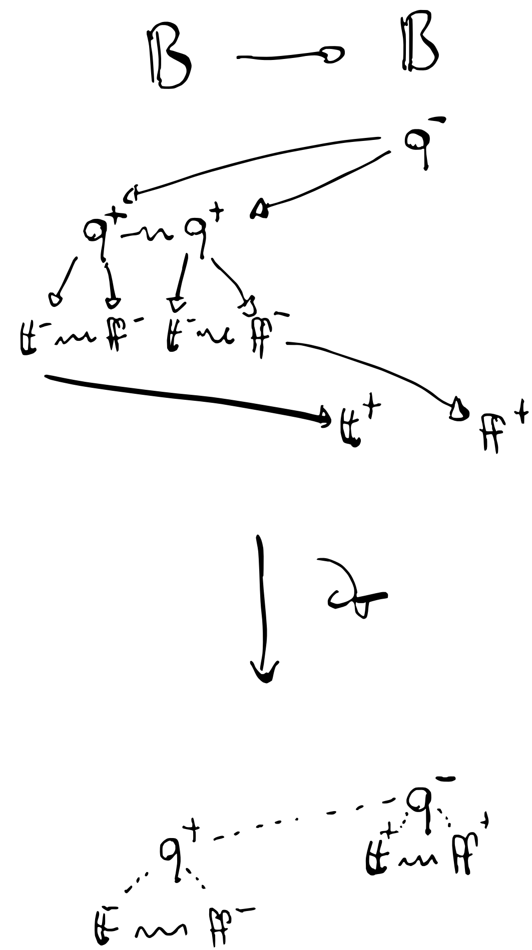
$$\partial_\sigma(s_1) = \partial_\sigma(s_2) \quad \text{implies} \quad s_1 = s_2$$

(3) receptivity

(4) courtesy

Examples





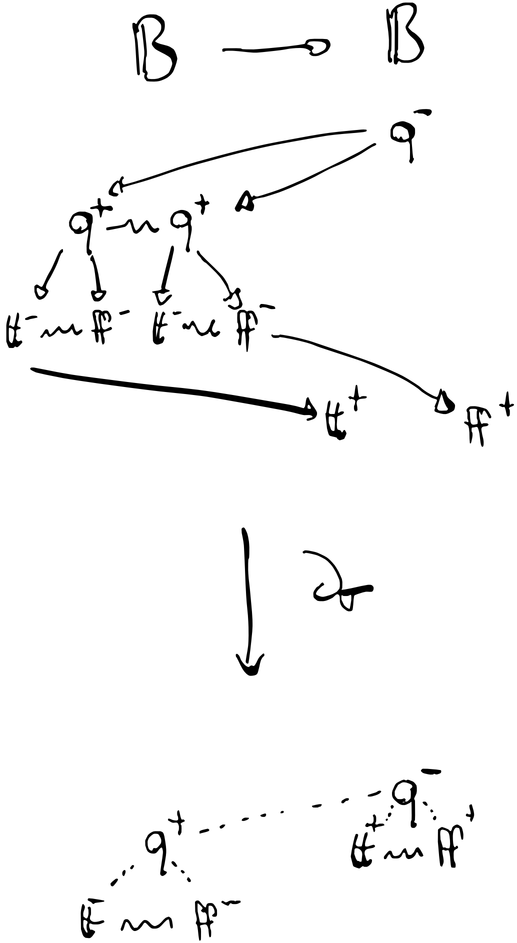
How many different ways to realize $x_A \in \mathcal{C}(A)$?

$$(\sigma)_{x_A} = \# \left\{ x^\sigma \in \mathcal{C}^+(\sigma) \;\middle|\; \partial_\sigma(x^\sigma) = x_A \right\}$$

$$= \# \, \mathrm{wit}_\sigma(x_A)$$

<u>Def</u>: $\mathcal{C}^+(\sigma)$ is the set of <u>+-covered</u> configurations

Examples



$$\# \mathrm{wit}_\sigma\left(\begin{array}{c} q^- \\ \vdots \\ q^+ \end{array}\right) =$$

$$\# \mathrm{wit}_\sigma\left(\begin{array}{ccc} & & q^- \\ q^+ & & ff^+ \\ tt^- & & \end{array}\right) =$$

$$\# \mathrm{wit}_\sigma\left(\begin{array}{ccc} & & q^- \\ q^+ & & tt^- \\ ff^- & & \end{array}\right) =$$

$(A^\perp \parallel B)$

<u>Composition</u>: if $\sigma : A \vdash B$ and $\tau : B \vdash C$, then

$$\tau \odot \sigma : A \vdash C$$

<u>Th</u>: if $\sigma : A \vdash B$ and $\tau : B \vdash C$ do not deadlock,

$$(\tau \odot \sigma)_{x_A, x_C} = \sum_{x_B \in \mathscr{C}(B)} (\sigma)_{x_A, x_B} \cdot (\tau)_{x_B, x_C}$$

Fact: for any $\sigma : A \vdash B$, $\tau : B \vdash C$,
there exists a strategy, unique up to iso, s.t.

$$\mathscr{C}^+(\tau \odot \sigma)$$

$$\cong$$

$$\left\{ (x^\sigma, x^\tau) \in \mathscr{C}^+(\sigma) \times \mathscr{C}^+(\tau) \;\middle|\; \begin{array}{l} \text{matching: } x^\sigma_B = x^\tau_B \\ \text{deadlock-free} \end{array} \right\}$$

an order-isomorphism preserving display maps

<u>Theorem</u>: For deadlock-free strategies,

$$\mathrm{wit}_{\tau \odot \sigma}(x_A, x_C) = \sum_{x_B \in \mathscr{C}(B)} \mathrm{wit}_\sigma(x_A, x_B) \times \mathrm{wit}_\tau(x_B, x_C)$$

"What does the weighted relational model count?"

$$\begin{tikzcd}
\mathrm{ANPCF} \arrow[d, "\llbracket - \rrbracket_G"'] \arrow[dr, "\llbracket - \rrbracket_{Rel}"] & \\
\mathrm{Games} \arrow[r, "(-)"'] & \overline{\mathbb{N}}\text{-}\mathrm{Rel}
\end{tikzcd}$$

↳ It counts concurrent game <u>witnesses</u>

2. A quick refresher on symmetry

Definition: An isomorphism family on A is a set

$\mathcal{S}(A)$ of bijections $\theta : x \cong_A y$

between configurations, such that:

(1) it is a groupoid

(2)
$$\begin{array}{ccc} x & \overset{\theta}{\cong}_A & y \\ \rotatebox{-90}{$\subseteq$} & \rotatebox{-90}{$\subseteq$} & \rotatebox{-90}{$\subseteq$} \\ x' & \underset{\theta'}{\cong}_A & y' \end{array}$$

(3)
$$\begin{array}{ccc} x & \overset{\theta}{\cong}_A & y \\ \rotatebox{90}{$\subseteq$} & \rotatebox{90}{$\subseteq$} & \rotatebox{90}{$\subseteq$} \\ x' & \underset{\theta'}{\cong} & y' \end{array}$$

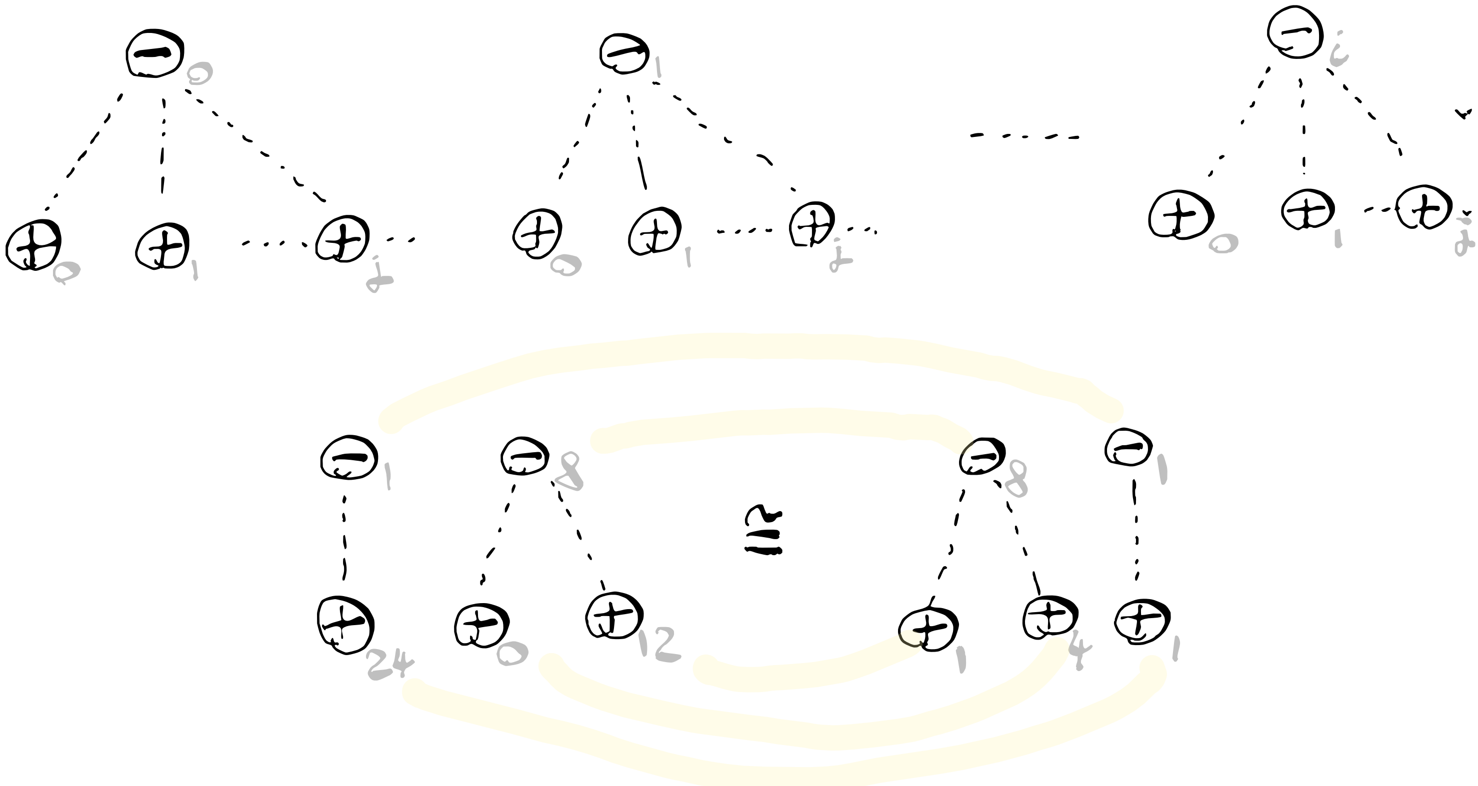
Terminology:

- $A = (|A|, \leq_A, \#_A, \mathcal{S}(A))$
  is an event structure with symmetry

- $\theta \in \mathcal{S}(A)$ is a symmetry

- dom : $\mathcal{S}(A) \longrightarrow \mathcal{C}(A)$
  cod: $\mathcal{S}(A) \longrightarrow \mathcal{C}(A)$

Example : $!(!\alpha \multimap \alpha)$



symmetries corresponds to reindexing

Definition: if A is a game with symmetry,

- $|!A| = \mathbb{N} \times |A|$
- $\mathcal{S}(!A)$ comprises those

$$\vartheta : \coprod_{i \in I} x_i \cong \coprod_{j \in J} y_j$$

such that $\vartheta(i, a) = (\pi(i), \vartheta_i(a))$

with $\pi : I \cong J$

$(\vartheta_i) : x_i \cong_A y_{\pi(i)}$

Definition: $\mathcal{C}_{\cong}(A)$ is the set of symmetry classes

$$\mathcal{C}_{\cong}(!A) \cong \mathcal{M}_f(\mathcal{C}_{\cong}(A))$$

Notation: we use $x, y \in \mathcal{C}_{\cong}(A)$

<u>Definition</u>: A strategy with symmetry on A is

$$\sigma = (|\sigma|, \leq_\sigma, \#_\sigma, \mathcal{S}(\sigma), \partial_\sigma)$$

such that

⋮

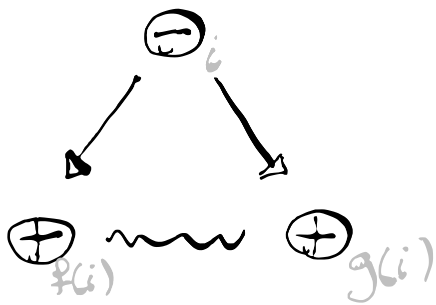
- ∂_σ preserves symmetry:

$$\partial_\sigma : \mathcal{S}(\sigma) \longrightarrow \mathcal{S}(A)$$

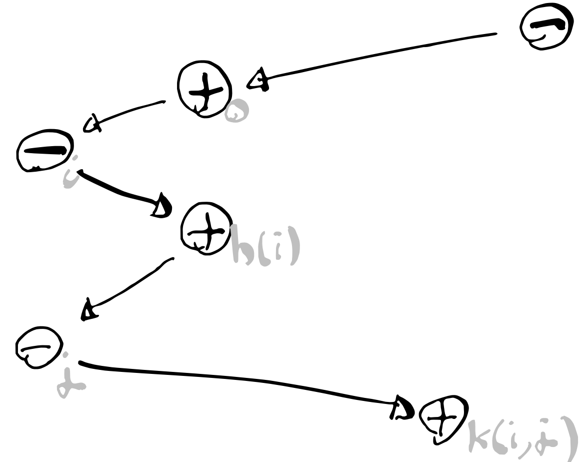
- $\partial_\sigma : \mathcal{S}(\sigma) \longrightarrow \mathcal{S}(A)$ is receptive

Example

$!(!\alpha \multimap \alpha)$



$!(!\alpha \multimap \alpha) \vdash !\alpha \multimap \alpha$



3. Counting with symmetry

Fact: for any $\sigma : A \vdash B$, $\tau : B \vdash C$,
there exists a strategy, unique up to iso, s.t.

$$\mathscr{C}^+(\tau \odot \sigma)$$

$$\cong$$

$$\left\{ (x^\sigma, x^\tau) \in \mathscr{C}^+(\sigma) \times \mathscr{C}^+(\tau) \;\middle|\; \begin{array}{l} \text{matching: } x^\sigma_B = x^\tau_B \\ \text{deadlock-free} \end{array} \right\}$$

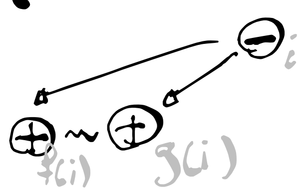
an order-isomorphism preserving display maps

Wait, but...

points of the web are symmetry classes!

$$\mathrm{wit}_\sigma(\mathbf{X}) \overset{?}{=} \left\{ x \in \mathcal{C}(\sigma) \mid \partial_\sigma(x) \in \mathbf{X} \right\}$$

$\sigma : \; !(!\alpha \multimap \alpha)$



$\rightsquigarrow$ $\mathrm{wit}_\sigma(\{\ominus_0, \oplus_0\}_{/\cong})$ infinite

$$\mathrm{wit}_\sigma(\mathbf{x}) = \left\{ y \in \mathcal{C}_{\cong}(\sigma) \mid \partial_\sigma(y) = \mathbf{x} \right\}$$

Ok, but do we still have:

$$\mathrm{wit}_{\tau \odot \sigma}(x_A, x_C) \cong \sum_{x_B \in \mathscr{C}_{\cong}(B)} \mathrm{wit}_\sigma(x_A, x_B) \times \mathrm{wit}_\tau(x_B, x_C)$$

?

$$\hookrightarrow [x^\tau \odot x^\sigma]_{\cong} \longmapsto \left([x_B]_{\cong}, [x^\sigma]_{\cong}, [x^\tau]_{\cong} \right)$$

$$\hookleftarrow$$

$$\mathrm{wit}_\sigma(x_A, x_B) \times \mathrm{wit}_\tau(x_B, x_C) \longrightarrow \mathrm{wit}_{\tau \odot \sigma}(x_A, x_C) \quad ?$$

$$\mathrm{wit}_{\sigma}(x_A, x_B) \times \mathrm{wit}_{\tau}(x_B, x_C) \longrightarrow \mathrm{wit}_{\tau\odot\sigma}(x_A, x_C)$$

$$[x^{\sigma}]_{\cong} \qquad\qquad [x^{\tau}]_{\cong}$$

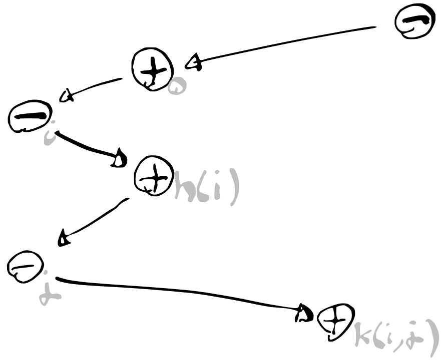
$$\begin{array}{ccc} x^{\sigma} & & x^{\tau} \\ {\scriptstyle \partial_{\sigma}}\big\downarrow & & \big\downarrow{\scriptstyle \partial_{\tau}} \\ x^{\sigma}_A \parallel x^{\sigma}_B & & x^{\tau}_B \parallel x^{\tau}_C \end{array} \longmapsto \left[x^{\tau} \overset{(\theta)}{\odot} x^{\tau} \right]_{\cong}$$

$$[x^{\sigma}_B]_{\cong} = [x^{\tau}_B]_{\cong}$$

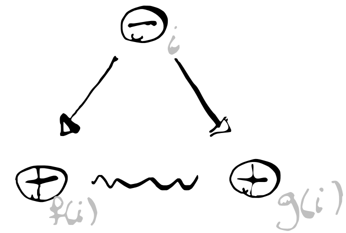
$$x^{\sigma}_B \overset{\theta}{\cong}_B x^{\tau}_B$$

invariant under the choice of x^{σ}, x^{τ}, θ ?

$!(!\alpha \multimap \alpha) \vdash !\alpha \multimap \alpha$ $\qquad\qquad$ $!(!\alpha \multimap \alpha)$

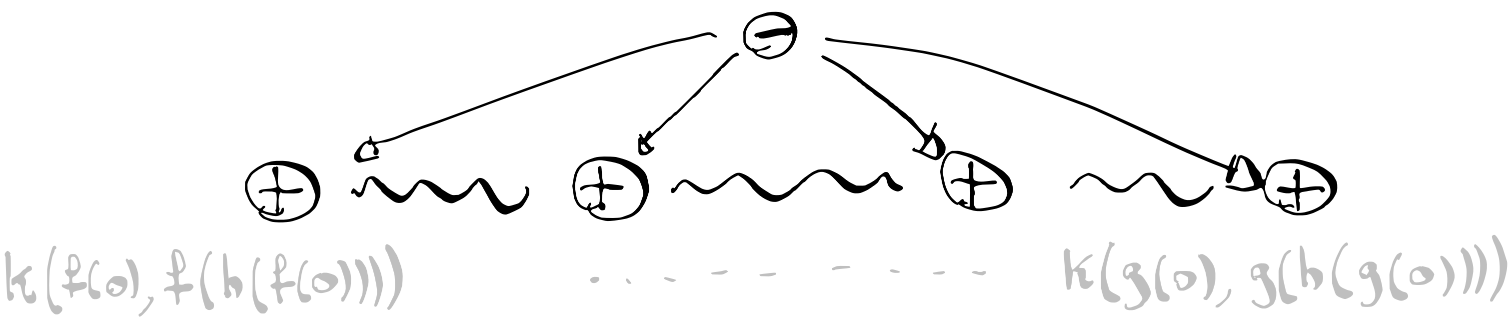


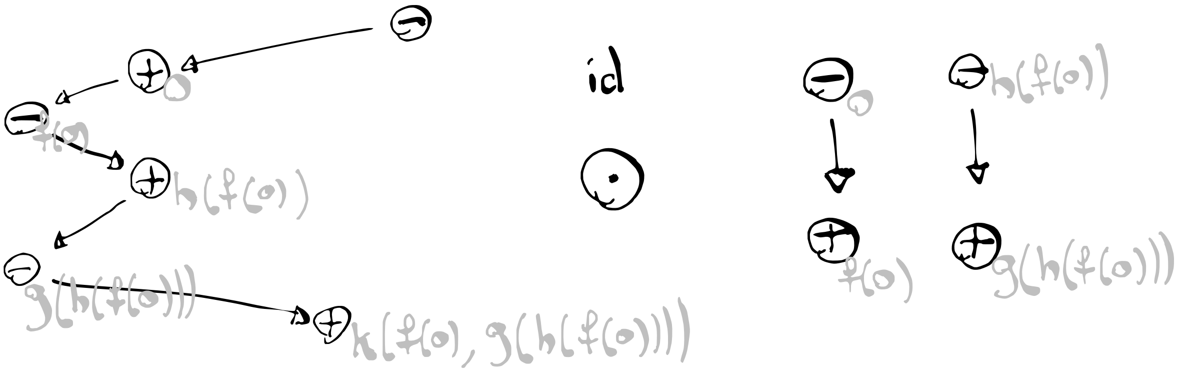
$\odot$



$=$

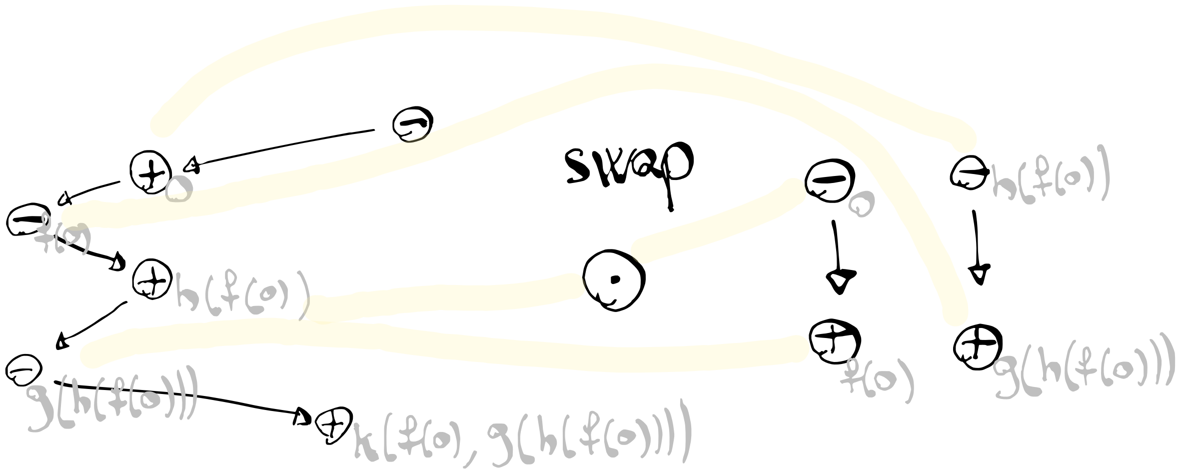
$!\alpha \multimap \alpha$




0
f(0)
h(f(0))
g(h(f(0)))
k(f(0), g(h(f(0))))
id
0
f(0)
h(f(0))
g(h(f(0)))

=


k(f(0), g(h(f(0))))



$$=$$

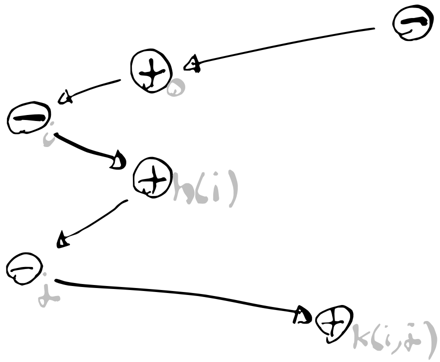
$$\oplus\ k(f(0), f(h(g(0))))$$

<u>Fact</u>: There is, in general, no bijection

$$\mathrm{wit}_{\tau\odot\sigma}(x_A, x_C) \cong \sum_{x_B \in \mathcal{C}_{\mathbf{t}}(B)} \mathrm{wit}_{\sigma}(x_A, x_B) \times \mathrm{wit}_{\tau}(x_B, x_C)$$

$!(!\alpha \multimap \alpha) \vdash !\alpha \multimap \alpha$

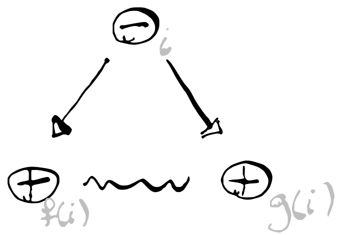
τ



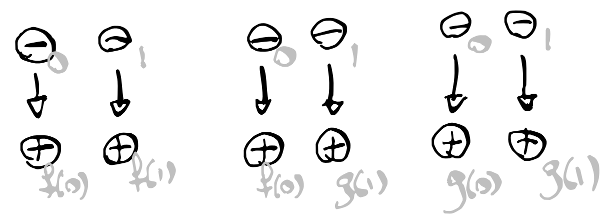
$\# \mathrm{wit}\left(\begin{smallmatrix}\ominus & \ominus \\ \vdots & \vdots \\ \oplus & \oplus\end{smallmatrix}, \begin{smallmatrix}\ominus \\ \vdots \\ \oplus\end{smallmatrix}\right) = 1$

$!(!\alpha \multimap \alpha)$

σ

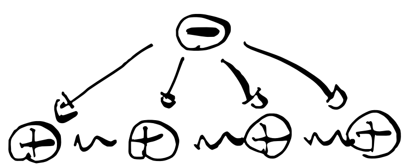


$\# \mathrm{wit}\left(\begin{smallmatrix}\ominus & \ominus \\ \vdots & \vdots \\ \oplus & \oplus\end{smallmatrix}\right) = 3$



$!\alpha \multimap \alpha$

$\tau \odot \sigma$



$\# \mathrm{wit}\left(\begin{smallmatrix}\ominus \\ \vdots \\ \oplus\end{smallmatrix}\right) = 4$

So, the weighted relational model

does <u>not</u> count symmetry classes.

Then, what ?

4. What indeed ?

<u>Def</u>: A thin concurrent game is

$$(|A|, \leq_A, \#_A, \mathcal{S}(A), \mathcal{S}_+(A), \mathcal{S}_-(A))$$

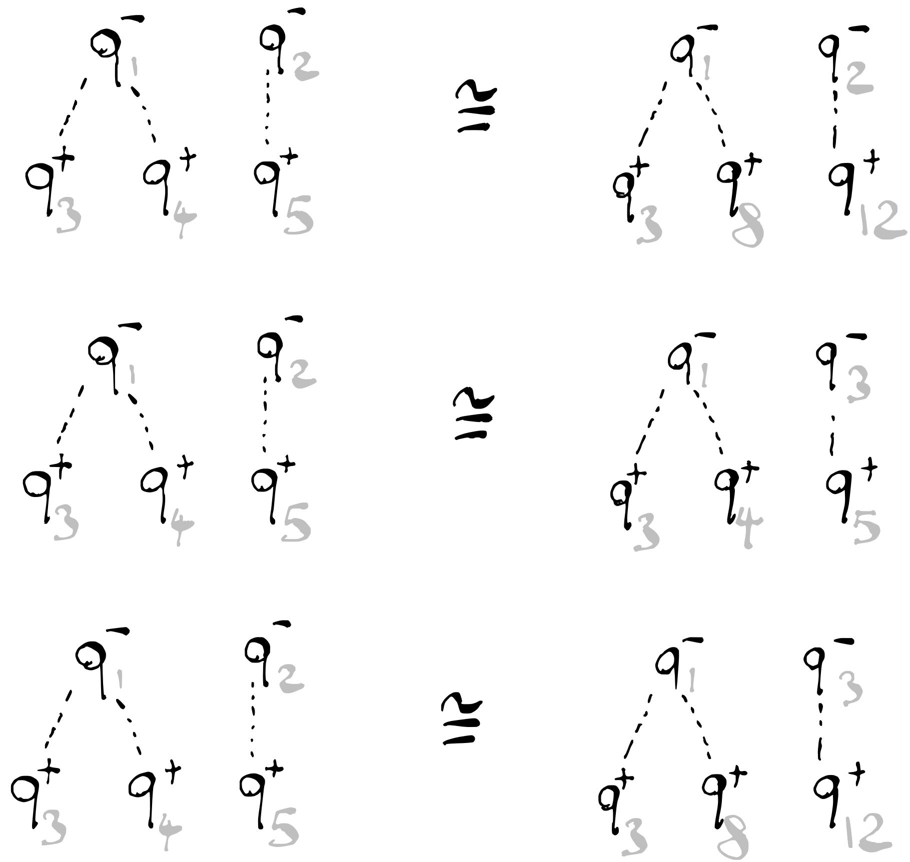
positive symmetries

negative symmetries

such that $\mathcal{S}_-(A) \subseteq \mathcal{S}(A)$, $\mathcal{S}_+(A) \subseteq \mathcal{S}(A)$,

+ axioms...

Example on $!(!\alpha \multimap \alpha)$



__Fact__: if A is a tcg,

any $\theta : x \cong_A y$ factors uniquely as

$$\begin{array}{ccccc} & \theta^+ & & \theta^- & \\ x & \cong_A^+ & z & \cong_A^- & y \end{array}$$

and much more...

$$\mathrm{wit}^{+}_{\sigma}(X_A) = \left\{ x^{\sigma} \in \mathcal{C}(\sigma) \;\middle|\; \partial_{\sigma}(x^{\sigma}) \simeq^{+}_{A} \underline{X}_A \right\}$$

concrete configuration

chosen concrete
canonical
representative

Theorem: if $\sigma : A \vdash B$, $\tau : B \vdash C$ deadlock-free and B representable, then for all $x_A \in \mathscr{C}_{\cong}(A)$, $x_C \in \mathscr{C}_{\cong}(C)$,

$$\# \mathrm{wit}^{+}_{\tau \odot \sigma}(x_A, x_C) = \sum_{x_B \in \mathscr{C}_{\cong}(B)} \left(\# \mathrm{wit}^{+}_{\sigma}(x_A, x_B)\right) \times \left(\# \mathrm{wit}^{+}_{\tau}(x_B, x_C)\right)$$

$\hookrightarrow$ $\overline{\mathbb{N}}$-Rel counts concrete witnesses up to $S_+(A)$

5. Want more ?

In fact, there is a bijection

$$\left\{ \begin{array}{ccccc} & & x^\sigma & & \\ & \theta^-_A & \Big\downarrow \partial^\sigma & \theta^+_B & \\ \underline{X}_A \cong^-_A x^\sigma_A & & & & x^\sigma_B \cong^+_B \underline{X}_B \end{array} \right\} \times \left\{ \begin{array}{ccccc} & & x^\tau & & \\ & \theta^-_B & \Big\downarrow \partial^\tau & \theta^+_C & \\ \underline{X}_B \cong^-_B x^\tau_B & & & & x^\tau_C \cong^+_C \underline{X}_C \end{array} \right\}$$

$$\cong$$

$$\left\{ \begin{array}{ccccc} & & y^\tau \circledast y^\sigma & & \\ & \theta^-_A & \Big\downarrow \partial & \theta^+_C & \\ \underline{X}_A \cong^-_A x_A & & x_B & & x_C \cong^+_C \underline{X}_C \\ & & \cap & & \\ & & x_B & & \end{array} \right\} \times \zeta(\underline{X}_B)$$

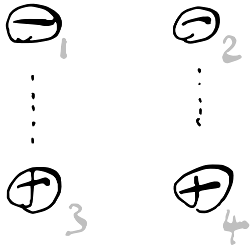
$$\#\left(\xi(\underline{x}_B)\right) = \#\left(\xi^-(\underline{x}_B)\right) \times \#\left(\xi^+(\underline{x}_B)\right)$$

Need to choose $\underline{x}_B$ <u>canonical</u>, such that any

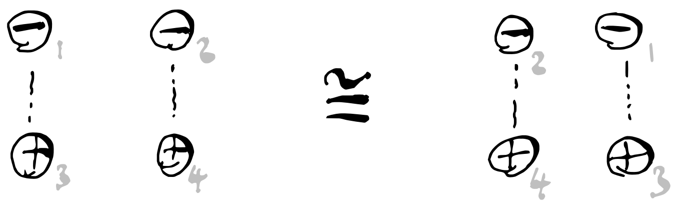
$$\underline{x}_B \overset{\theta}{\cong}_B \underline{x}_B$$

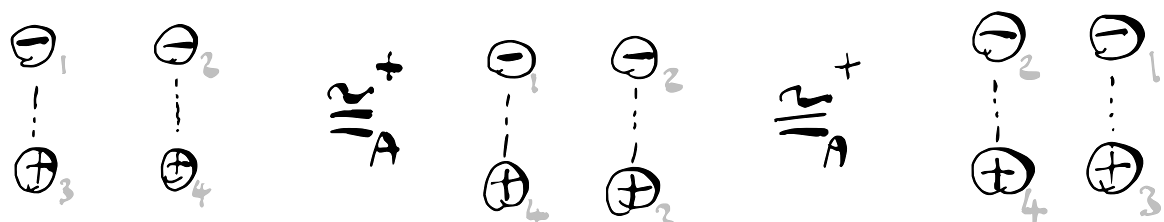
factors as

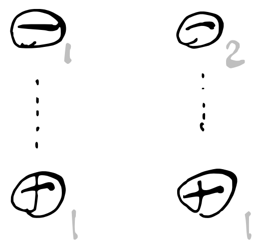
$$\underline{x}_B \overset{\theta^+}{\cong}{}^{+}_{B} \underline{x}_B \overset{\theta^-}{\cong}{}^{-}_{B} \underline{x}_B$$



not canonical







canonical

<u>Fact</u>: games interpreting types are <u>representable</u>,

i.e. there are always canonical representatives.

The End