Positional Injectivity for Innocent Strategies

Concurrent Games Café

$$
\begin{tikzcd}[column sep=small]
 & \text{PROGRAMS} \arrow[dl] \arrow[dr] & \\
\text{GAME SEMANTICS} \arrow[rr, "\text{position}"'] & & \text{RELATIONAL SEMANTICS}
\end{tikzcd}
$$

Dynamic

Static

What happens when we forget temporal information?

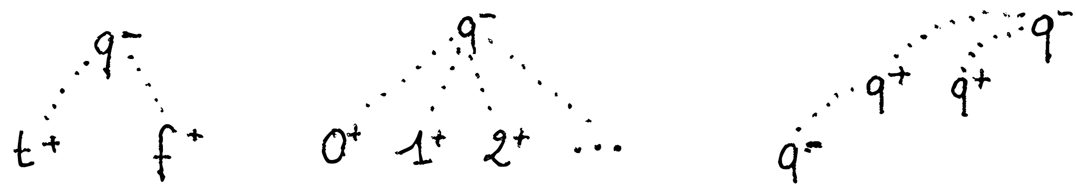
Thm: Total finite innocent strategies are positionally injective.

Thm: Total finite innocent strategies are positionally injective.

Thm: Finite causal strategies are positionally injective.

1. Causal Games

A) Arenas (Types)



bool nat $(\alpha \to \alpha) \to \alpha \to \alpha$

<u>**Arena:**</u> $A = \langle |A|, \leq_A, \lambda_A \rangle$ where $\langle |A|, \leq_A \rangle$ is a partial order and $\lambda_A : |A| \to \{+,-\}$ is a polarity function.

- <u>finitary</u>: $\forall a \in |A|$, $[a]_A = \{a' \in |A| \mid a' \leq_A a\}$ is finite,
- <u>forestial</u>: $\forall a_1, a_2 \leq_A a$, then $a_1 \leq_A a_2$ or $a_2 \leq_A a_1$,
- <u>alternating</u>: $\forall a_1 \xrightarrow[A]{} a_2$, $\lambda_A(a_1) \neq \lambda_A(a_2)$,
- <u>well-opened</u>: $\# \min(A) = 1$,
- <u>negative</u>: $\forall a \in \min(A)$, $\lambda_A(a) = -$.

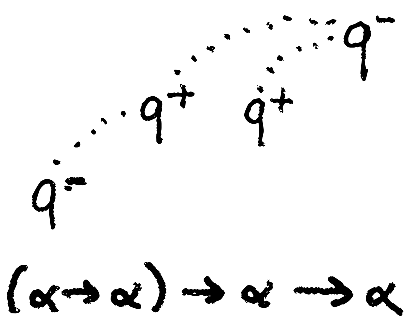
<u>Configurations</u> : $x \in \mathscr{C}(A)$: $x = \langle |x|, \leq_x, \partial_x \rangle$ such that

- $\langle |x|, \leq_x \rangle$ is a finite tree,
- $\partial_x : |x| \to |A|$ is an arena-labelling function s.t.:
  - <u>minimality preserving</u> : $\forall a \in |x|$, a min for $\leq_x \Leftrightarrow \partial_x(a)$ min for $\leq_A$;
  - <u>causality preserving</u> : $\forall a_1, a_2 \in |x|$, $a_1 \rightarrow_x a_2 \Rightarrow \partial_x(a_1) \rightarrow_A \partial_x(a_2)$.

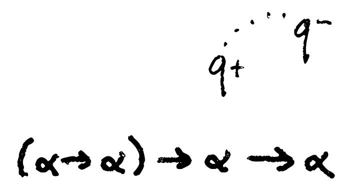
Configurations : $x \in \mathscr{C}(A)$: $x = \langle |x|, \leq_x, \partial_x \rangle$ such that

- $\langle |x|, \leq_x \rangle$ is a finite tree,
- $\partial_x : |x| \to |A|$ is an arena-labelling function s.t.:
  - minimality preserving : $\forall a \in |x|$, a min for $\leq_x \Leftrightarrow \partial_x(a)$ min for $\leq_A$;
  - causality preserving : $\forall a_1, a_2 \in |x|$, $a_1 \to_x a_2 \Rightarrow \partial_x(a_1) \to_A \partial_x(a_2)$.

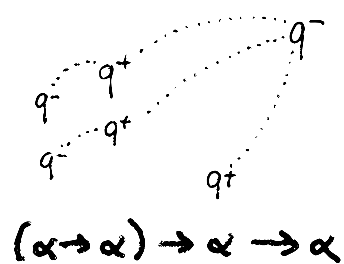
A :



x_0 :



x_2 :



Isomorphism of configurations: $\varphi : x \cong y$

$\varphi : |x| \cong |y|$ a bijection such that:

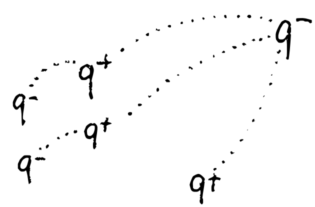
- arena-preserving: $\forall a \in |x|,\ \partial_y(\varphi(a)) = \partial_x(a)$
- causality-respecting: $\forall a_1, a_2 \in |x|,\ a_1 \rightarrow_x a_2 \iff \varphi(a_1) \rightarrow_y \varphi(a_2)$

Isomorphism of configurations: $\varphi : x \cong y$

$\varphi : |x| \cong |y|$ a bijection such that:

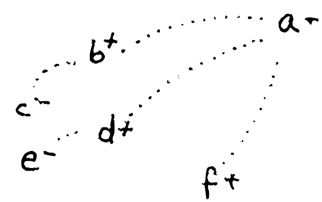
- arena-preserving: $\forall a \in |x|,\ \partial_y(\varphi(a)) = \partial_x(a)$
- causality-respecting: $\forall a_1, a_2 \in |x|,\ a_1 \to_x a_2 \iff \varphi(a_1) \to_y \varphi(a_2)$

x_2:



$(\alpha \to \alpha) \to \alpha \to \alpha$

y_2:



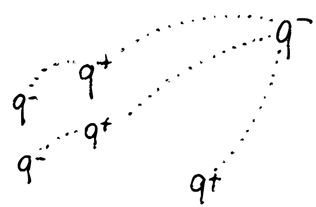
$(\alpha \to \alpha) \to \alpha \to \alpha$

Isomorphism of configurations: $\varphi : x \cong y$

$\varphi : |x| \cong |y|$ a bijection such that:

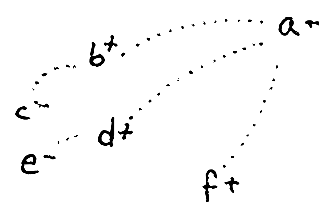
- arena-preserving: $\forall a \in |x|,\ \partial_y(\varphi(a)) = \partial_x(a)$
- causality-respecting: $\forall a_1, a_2 \in |x|,\ a_1 \rightarrow_x a_2 \iff \varphi(a_1) \rightarrow_y \varphi(a_2)$

x_2:



$(\alpha \to \alpha) \to \alpha \to \alpha$

y_2:



$(\alpha \to \alpha) \to \alpha \to \alpha$

Position: $x \in \mathscr{C}(A)$. The position of x is the isomorphism class of x, noted $\langle x \rangle \in \langle A \rangle$.

B) Augmentations (Executions)

Augmentation: $q = \langle |q|, \leqslant_{\triangleleft q \triangleright}, \leqslant_q, \partial_q \rangle \in Aug(A)$

where $\triangleleft q \triangleright = \langle |q|, \leqslant_{\triangleleft q \triangleright}, \partial_q \rangle \in \mathcal{C}(A)$

and $\langle |q|, \leq_q \rangle$ is a tree satisfying:

- rule-abiding: $\forall a_1, a_2 \in |q|$, if $a_1 \leq_{\triangleleft q \triangleright} a_2$, then $a_1 \leqslant_q a_2$,
- courteous: $\forall a_1 \rightarrowtriangle_q a_2$, if $\lambda(a_1) = +$ or $\lambda(a_2) = -$, then $a_1 \rightarrowtriangle_{\triangleleft q \triangleright} a_2$,
- deterministic: $\forall a^- \rightarrowtriangle_q a_1^+$, $a^- \rightarrowtriangle_q a_2^+$, then $a_1 = a_2$.

B) Augmentations (Executions)

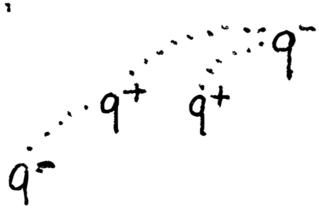
Augmentation: $q = \langle |q|, \leqslant_{\langle\!\langle q \rangle\!\rangle}, \leqslant_q, \partial_q \rangle \in \mathrm{Aug}(A)$

where $\langle\!\langle q \rangle\!\rangle = \langle |q|, \leqslant_{\langle\!\langle q \rangle\!\rangle}, \partial_q \rangle \in \mathcal{C}(A)$

and $\langle |q|, \leq_q \rangle$ is a tree satisfying:

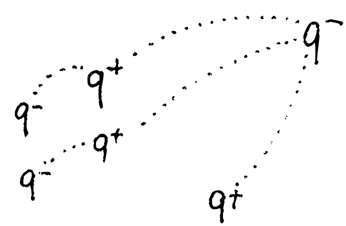
- rule-abiding: $\forall a_1, a_2 \in |q|$, if $a_1 \leq_{\langle\!\langle q \rangle\!\rangle} a_2$, then $a_1 \leqslant_q a_2$,
- courteous: $\forall a_1 \rightarrowtriangle_q a_2$, if $\lambda(a_1) = +$ or $\lambda(a_2) = -$, then $a_1 \rightarrowtriangle_{\langle\!\langle q \rangle\!\rangle} a_2$,
- deterministic: $\forall a^- \rightarrowtriangle_q a_1^+$, $a^- \rightarrowtriangle_q a_2^+$, then $a_1 = a_2$.

A,



$(\alpha \to \alpha) \to \alpha \to \alpha$

x_2:



$(\alpha \to \alpha) \to \alpha \to \alpha$

B) Augmentations (Executions)

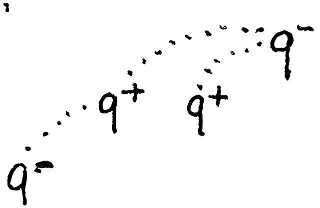
Augmentation: $q = \langle |q|, \leqslant_{\lhd q \rhd}, \leqslant_q, \partial_q \rangle \in \mathrm{Aug}(A)$

where $\lhd q \rhd = \langle |q|, \leqslant_{\lhd q \rhd}, \partial_q \rangle \in \mathcal{C}(A)$

and $\langle |q|, \leq_q \rangle$ is a tree satisfying:

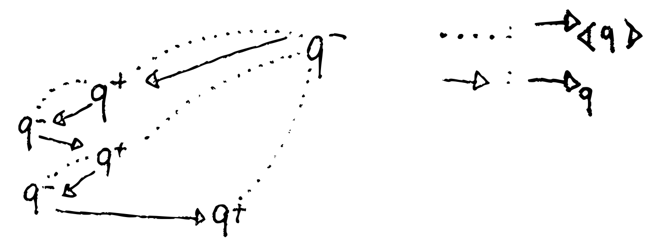
- rule-abiding: $\forall a_1, a_2 \in |q|$, if $a_1 \leq_{\lhd q \rhd} a_2$, then $a_1 \leqslant_q a_2$,
- courteous: $\forall a_1 \rightarrowtriangle_q a_2$, if $\lambda(a_1) = +$ or $\lambda(a_2) = -$, then $a_1 \rightarrowtriangle_{\lhd q \rhd} a_2$,
- deterministic: $\forall a^- \rightarrowtriangle_q a_1^+$, $a^- \rightarrowtriangle_q a_2^+$, then $a_1 = a_2$.

A,



$(\alpha \to \alpha) \to \alpha \to \alpha$

q_2 :



$(\alpha \to \alpha) \to \alpha \to \alpha$

B) Augmentations (Executions)

Augmentation: $q = \langle |q|, \leqslant_{\langle\!\langle q \rangle\!\rangle}, \leqslant_q, \partial_q \rangle \in \mathrm{Aug}(A)$

where $\langle\!\langle q \rangle\!\rangle = \langle |q|, \leqslant_{\langle\!\langle q \rangle\!\rangle}, \partial_q \rangle \in \mathcal{C}(A)$

and $\langle |q|, \leq_q \rangle$ is a tree satisfying:

- **rule-abiding**: $\forall a_1, a_2 \in |q|$, if $a_1 \leq_{\langle\!\langle q \rangle\!\rangle} a_2$, then $a_1 \leqslant_q a_2$,
- **courteous**: $\forall a_1 \rightarrowtriangle_q a_2$, if $\lambda(a_1) = +$ or $\lambda(a_2) = -$, then $a_1 \rightarrowtriangle_{\langle\!\langle q \rangle\!\rangle} a_2$,
- **deterministic**: $\forall a^- \rightarrowtriangle_q a_1^+$, $a^- \rightarrowtriangle_q a_2^+$, then $a_1 = a_2$.

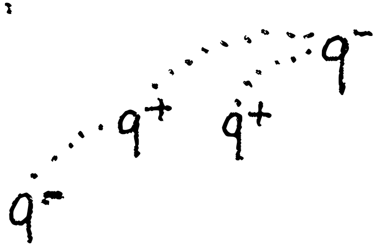
Position: $q \in \mathrm{Aug}(A)$. The position of q is the position of $\langle\!\langle q \rangle\!\rangle$, noted $\langle q \rangle \in \langle A \rangle$.

C) Stategies (Programs)

Causal strategy: $\sigma \in \mathrm{Aug}(A)$ such that:

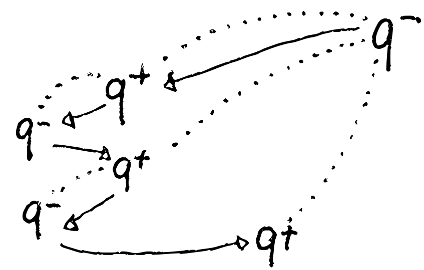
- receptive: $\forall a \in |\sigma|$, if $\partial_\sigma(a) \rightarrowtriangle_A b^-$, $\exists\, a \rightarrowtriangle_\sigma b'$ st $\partial(b') = b$,
- -- linear: $\forall\, a \rightarrowtriangle_\sigma a_1^-$, $a \rightarrowtriangle_\sigma a_2^-$, if $\partial_\sigma(a_1) = \partial_\sigma(a_2)$ then $a_1 = a_2$.

A,



$(\alpha \to \alpha) \to \alpha \to \alpha$

σ_2:



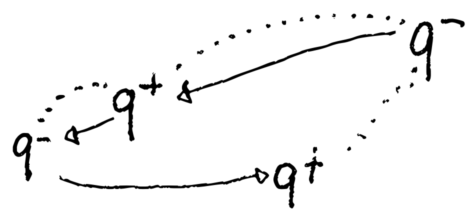
$(\alpha \to \alpha) \to \alpha \to \alpha$

D) Expansions (Executions)

Morphism: $\varphi : q \to p$ is a function $\varphi : |q| \to |p|$ such that:

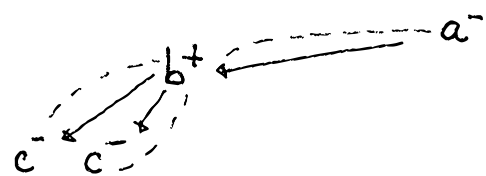
- arena-preserving: $\partial_p \circ \varphi = \partial_q$,
- causality-preserving: $\forall a_1, a_2 \in |q|,\ a_1 \rightarrowtriangle_q a_2 \Rightarrow \varphi(a_1) \rightarrowtriangle_p \varphi(a_2)$,
- configuration-preserving: $\forall a_1, a_2 \in |q|,\ a_1 \rightarrowtriangle_{\langle q \rangle} a_2 \Rightarrow \varphi(a_1) \rightarrowtriangle_{\langle p \rangle} \varphi(a_2)$.

σ_1 :



$(\alpha \to \alpha) \to \alpha \to \alpha$

q :



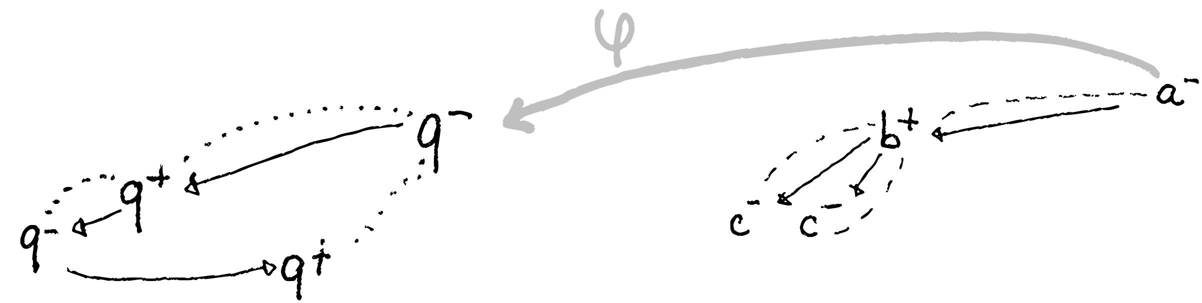
$(\alpha \to \alpha) \to \alpha \to \alpha$

D) Expansions (Executions)

Morphism: $\varphi : q \to p$ is a function $\varphi : |q| \to |p|$ such that:

- arena-preserving: $\partial_p \circ \varphi = \partial_q$,
- causality-preserving: $\forall a_1, a_2 \in |q|,\ a_1 \rightarrowtriangle_q a_2 \Rightarrow \varphi(a_1) \rightarrowtriangle_p \varphi(a_2)$,
- configuration-preserving: $\forall a_1, a_2 \in |q|,\ a_1 \rightarrowtriangle_{\langle q \rangle} a_2 \Rightarrow \varphi(a_1) \rightarrowtriangle_{\langle p \rangle} \varphi(a_2)$.

σ_1 :

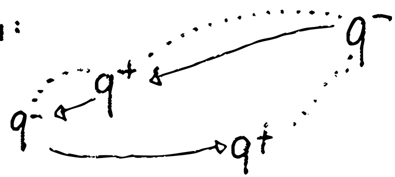


$(\alpha \to \alpha) \to \alpha \to \alpha$ $\qquad\qquad$ $(\alpha \to \alpha) \to \alpha \to \alpha$

Expansion: $q \in Aug(A)$ is an expansion of $p \in Aug(A)$ iff

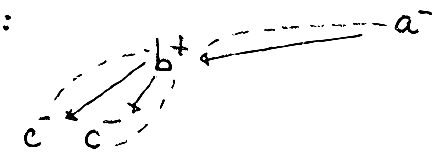
- simulation: $\exists\ \varphi : q \to p$,
- +-obsessional: $\forall a^- \in |q|$, if $\varphi(a^-) \to_p b^+$ then $\exists\ a^- \to_q a'$ st $\varphi(a') = b^+$.

σ_1:



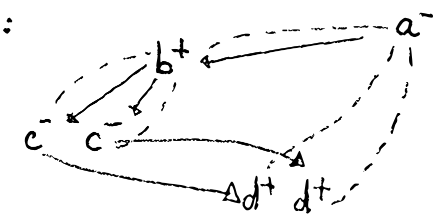
$(\alpha \to \alpha) \to \alpha \to \alpha$

q:



$(\alpha \to \alpha) \to \alpha \to \alpha$

q':



$(\alpha \to \alpha) \to \alpha \to \alpha$

Positions of a strategy : $\sigma \in \mathrm{Aug}(A)$ causal strategy.

$$\langle\!\langle \sigma \rangle\!\rangle = \{ \langle q \rangle \mid q \in \exp(\sigma) \}$$

Question : Given $\sigma, \tau \in \mathrm{Aug}(A)$ causal strategies,

if $\langle\!\langle \sigma \rangle\!\rangle = \langle\!\langle \tau \rangle\!\rangle$,

do we have $\sigma \cong \tau$?

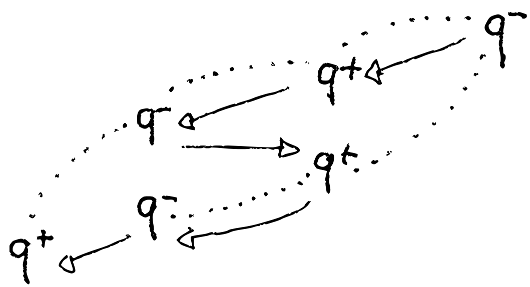
2. Positional Injectivity

$$\blacktriangleleft\!\!|\, \sigma \,|\!\!\blacktriangleright = \blacktriangleleft\!\!|\, \tau \,|\!\!\blacktriangleright \Rightarrow \sigma \cong \tau \quad ?$$

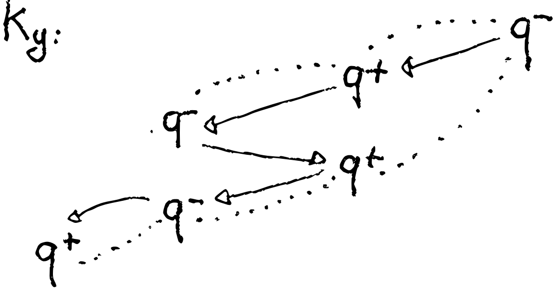
A) Proof Idea

$$((\alpha \to \alpha) \to \alpha) \to \alpha \qquad\qquad ((\alpha \to \alpha) \to \alpha) \to \alpha$$

K_x:



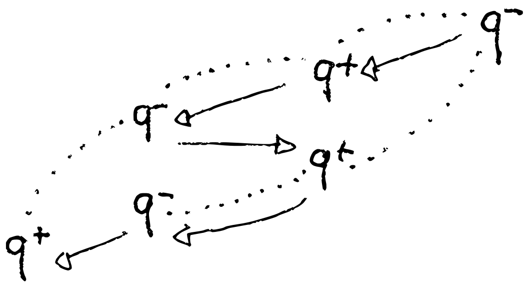
K_y:



A) Proof Idea

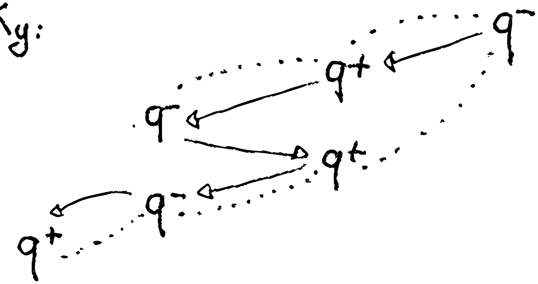
$$((\alpha \to \alpha) \to \alpha) \to \alpha$$

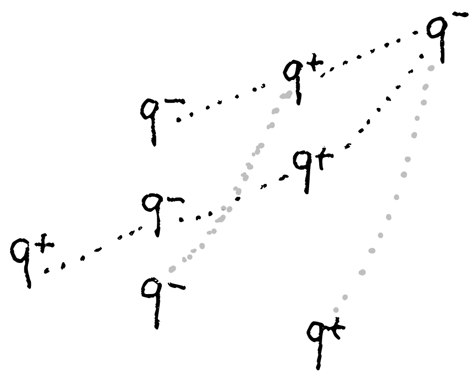
K_x:

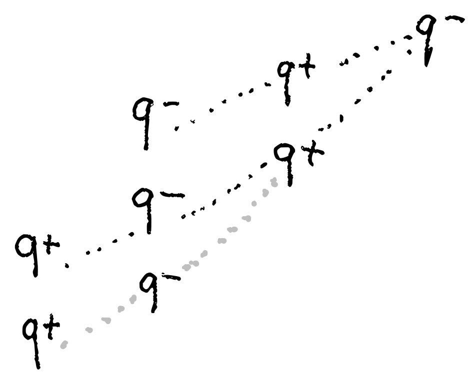


$$((\alpha \to \alpha) \to \alpha) \to \alpha$$

K_y:







Proof idea: σ, τ such that $⟪\sigma⟫ = ⟪\tau⟫$

$$\begin{tikzcd} \sigma & \\ & q \arrow[ul, "\varphi"'] \end{tikzcd}$$

expansion
where we
duplicate
some $\ominus$'s

Proof idea: σ, τ such that $\langle\!\langle \sigma \rangle\!\rangle = \langle\!\langle \tau \rangle\!\rangle$

$$
\begin{tikzcd}
\sigma & & & & \tau \\
& q \arrow[ul, "\varphi"'] & \langle q \rangle \cong \langle p \rangle & p \arrow[ur, "\Psi"'] &
\end{tikzcd}
$$

expansion
where we
duplicate
some $\ominus$'s

Proof idea: σ, τ such that $\langle\!\langle \sigma \rangle\!\rangle = \langle\!\langle \tau \rangle\!\rangle$

$$\sigma \xleftarrow{\varphi} q \qquad \langle q \rangle \cong \langle p \rangle \qquad p \xrightarrow{\Psi} \tau$$

expansion where we duplicate some $\ominus$'s

we want to reconstruct the causal structure of p:

$\oplus \to \ominus$: courtesy

$\ominus \to \oplus$: ?

Proof idea: σ, τ such that $⟪\sigma⟫ = ⟪\tau⟫$

$$\sigma \xleftarrow{\varphi} q \qquad ⟪q⟫ \cong ⟪p⟫ \qquad p \xrightarrow{\Psi} \tau$$

expansion
where we
duplicate
some $\ominus$'s

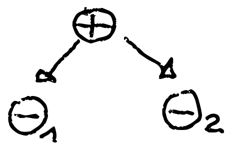
we want to reconstruct the causal structure of p:

$\oplus \to \ominus$: courtesy

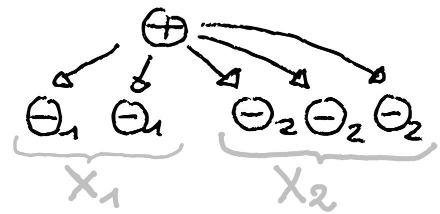
$\ominus \to \oplus$: multiplicity of the duplications

B) Twin Sets

r:

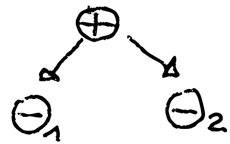


q:

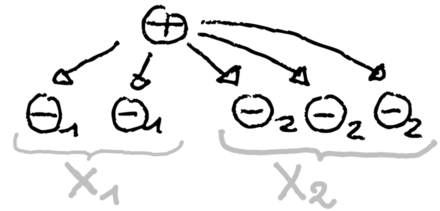


B) Twin Sets

r:



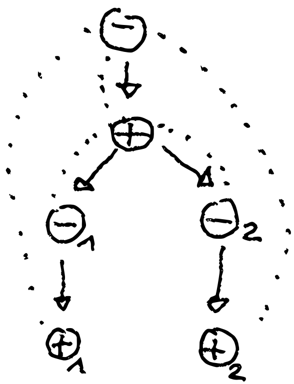
q:



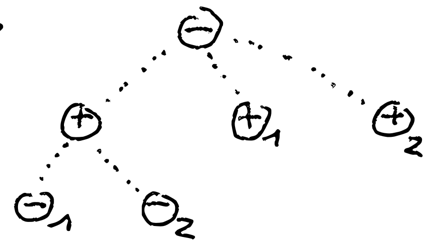
Twin Set: $X \subseteq |q|$ maximal non empty such that:

- negative: $\forall a \in X,\ \lambda(a) = -$
- sibling: $X = \min(q)$ or $\exists b \in |q|,\ \forall a \in X,\ b \rightarrow_q a$
- identical: $\forall a_1, a_2 \in X,\ \partial_q(a_1) = \partial_q(a_2)$

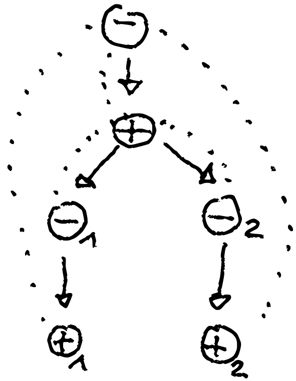
T :



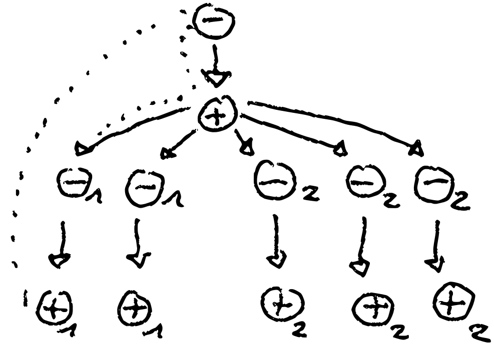
$\langle T \rangle$



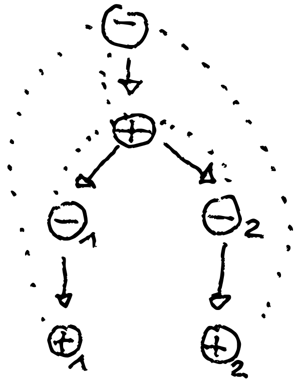
r:



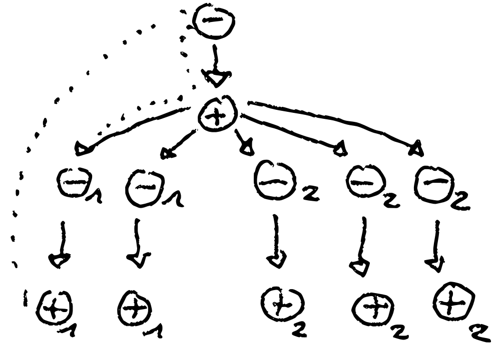
q:



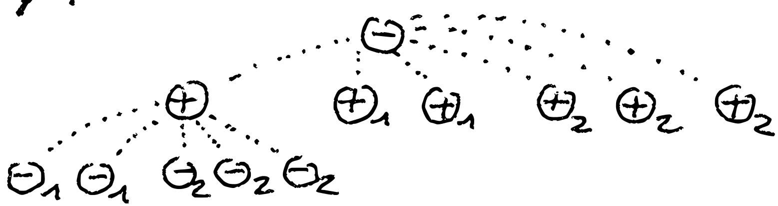
r:



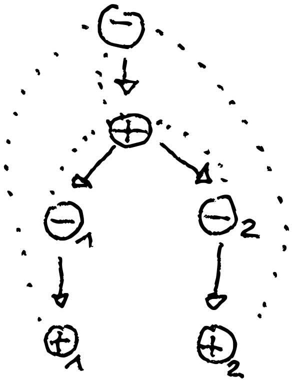
q:



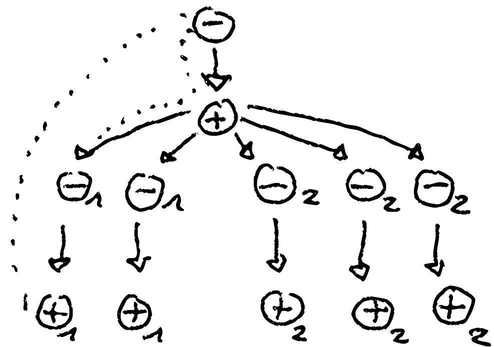
$\langle q \rangle$:



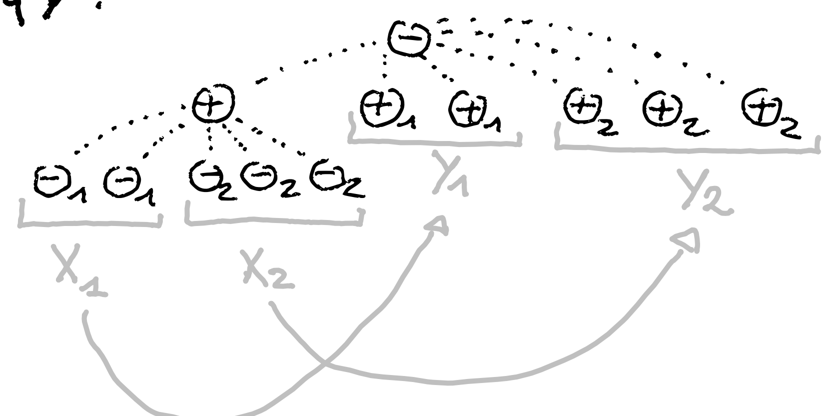
r:



q:



$\langle q \rangle$:



C) Characteristic Expansions

Def: A characteristic expansion of τ is $q \in \exp(\tau)$ s.t.:

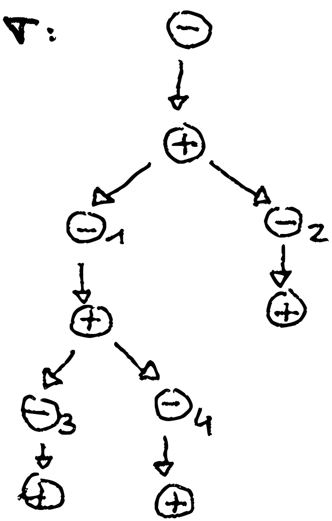
- injective: for $X, Y \in Twin(q)$, $\# Y = \# X \Rightarrow X = Y$
- well-powered: $\forall X \in Twin(q)$, $\# X = 2^n$ for $n \in \mathbb{N}$
- --obsessional: $\forall a^+ \in |q|$, if $\partial_q(a^+) \to_A b^-$, $\exists\, a^+ \to_q b'$, $\partial_q(b') = b$

C) Characteristic Expansions

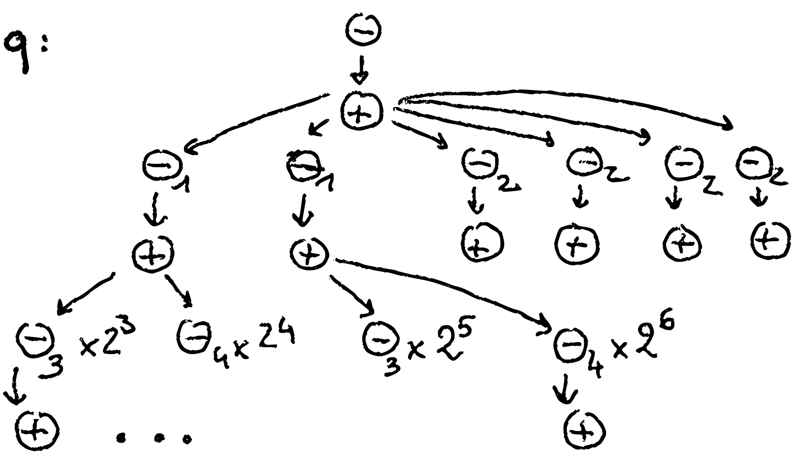
Def: A characteristic expansion of τ is $q \in \exp(\tau)$ s.t.:

- injective: for $X, Y \in Twin(q)$, $\#Y = \#X \Rightarrow X = Y$
- well-powered: $\forall X \in Twin(q)$, $\#X = 2^n$ for $n \in \mathbb{N}$
- --obsessional: $\forall a^+ \in |q|$, if $\partial_q(a^+) \rightarrow_A b^-$, $\exists\, a^+ \rightarrow_q b'$, $\partial_q(b') = b$

τ:



q:



Fact: $q \in \exp(T)$ "is a charac. exp." is a property of $\langle q \rangle$.

Fact: $q \in \exp(\tau)$ "is a charac. exp." is a property of $\langle q \rangle$.

Corollary: $q \in \exp(\tau)$, $p \in \exp(\tau)$, $\langle q \rangle \cong \langle p \rangle$.

Then q charac. exp. $\Rightarrow$ p charac. exp.

Thm (not proved yet): σ, τ strategies

$$\langle\!\langle \sigma \rangle\!\rangle = \langle\!\langle \tau \rangle\!\rangle \Rightarrow \sigma \cong \tau$$

Thm (not proved yet): σ, τ strategies

$$\langle\!\langle \sigma \rangle\!\rangle = \langle\!\langle \tau \rangle\!\rangle \Rightarrow \sigma \cong \tau$$

$$\begin{tikzcd}
\sigma & & & \tau \\
q \arrow[u, "\text{charac. exp.}"] \arrow[dr] & & & p \arrow[u] \arrow[dl] \\
& \langle q \rangle \arrow[r, phantom, "\cong"] & \langle p \rangle &
\end{tikzcd}$$

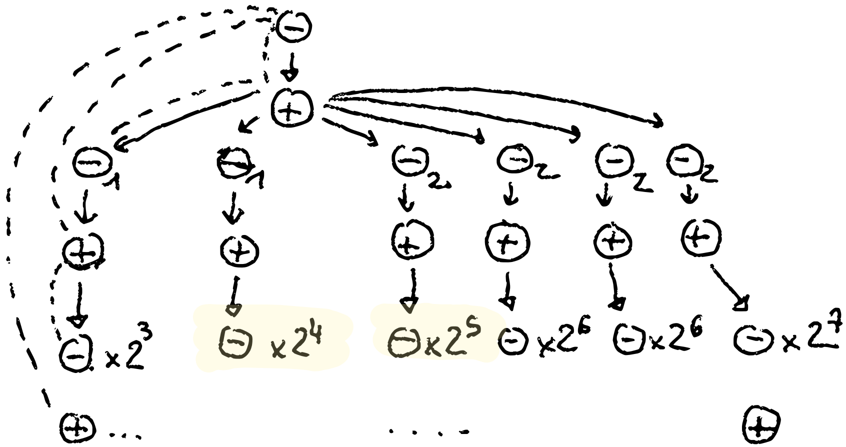
p is a charac. exp. of τ

1. What can we deduce about p from $\langle q \rangle \cong \langle p \rangle$?

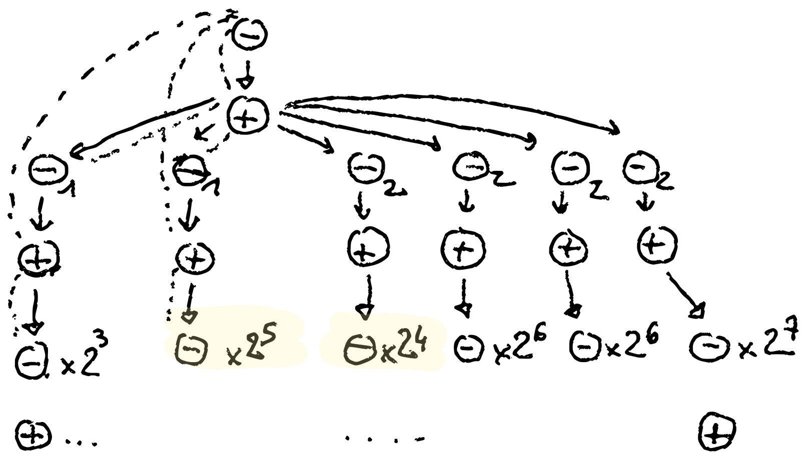
2. What can we deduce about τ from p charac. exp. of τ?

$$\langle q \rangle \cong \langle p \rangle \not\Rightarrow q \cong p$$

q:



p:



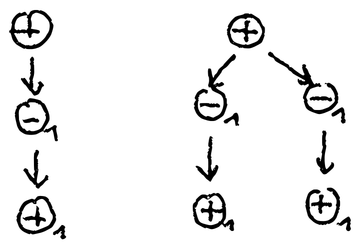
D) Bisimulation

Goal: $\langle q \rangle \cong^{\varphi} \langle p \rangle \Rightarrow q \sim^{\varphi} p$

Idea: $a \in |q|$, $b \in |p|$

"$a \sim^{\varphi}_{\Gamma} b$": a and b have "the same follow-up, up to multiplicity"

Ex 1:



$\oplus \sim^{\varphi} \oplus$

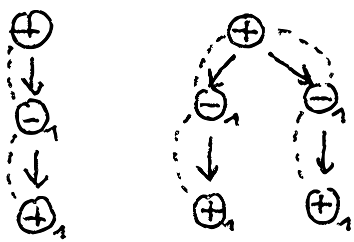
D) Bisimulation

Goal: $\triangleleft q \triangleright \cong^{\varphi} \triangleleft p \triangleright \Rightarrow q \sim^{\varphi} p$

Idea: $a \in |q|$, $b \in |p|$

"$a \sim^{\varphi}_{\Gamma} b$": a and b have "the same follow-up, up to multiplicity"

Ex 2:



$$\oplus_1 \sim^{\varphi}_{\Gamma} \oplus_1$$

$$\Gamma = \{(\ominus_1, \ominus_1)\}$$

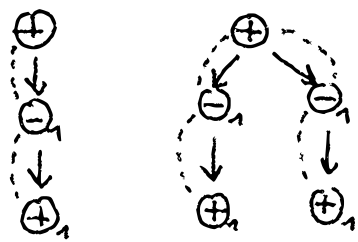
D) Bisimulation

Goal: $\lhd q \rhd \cong^{\varphi} \lhd p \rhd \quad \Rightarrow \quad q \sim^{\varphi} p$

Idea: $a \in |q|$, $b \in |p|$

"$a \sim^{\varphi}_{\Gamma} b$": a and b have "the same follow-up, up to multiplicity"

Ex 2:



$\oplus_1 \sim^{\varphi}_{\Gamma} \oplus_1$

$\Gamma = \{(\ominus_1, \ominus_1)\}$

Context: A context Γ between $q, p \in Aug(A)$

is a bijection such that

$dom(\Gamma) \subseteq |q|^-$, $cod(\Gamma) \subseteq |p|^-$

and for all $a^- \in dom(\Gamma)$, $\partial_q(a) = \partial_p(\Gamma(a))$

Bisimulation: $q, p \in Aug(A)$. $\lhd q \rhd \cong^{\varphi} \lhd p \rhd$.

$a \in |q|$, $b \in |p|$, Γ a context. We define $a \sim^{\varphi}_{\Gamma} b$ if:

(a) well-formed: $\partial_q(a) = \partial_p(b)$ and pointers compatible via φ, Γ;

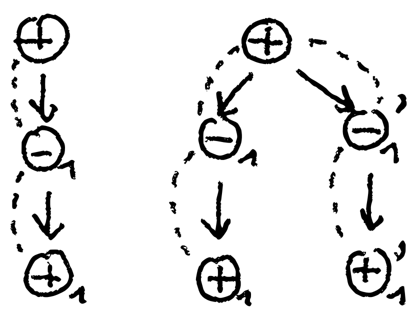
(1) if $a^{+} \rightarrow_q a'$, then $\exists b' \in |p|$ with $b^{+} \rightarrow_p b'$

and $a' \sim^{\varphi}_{\Gamma \cup \{(a', b')\}} b'$, and symmetrically,

(2) if $a^{-} \rightarrow_q a'$, then $\exists b' \in |p|$ with $b^{-} \rightarrow_p b'$

and $a' \sim^{\varphi}_{\Gamma} b'$, and symmetrically,

Ex:



$\oplus \sim^{\varphi} \oplus$

$\ominus_1 \sim^{\varphi}_{\{(\ominus_1, \ominus_1)\}} \ominus_1$ $\quad$ $\ominus_1 \sim^{\varphi}_{\{(\ominus_1, \ominus_1')\}} \ominus_1'$

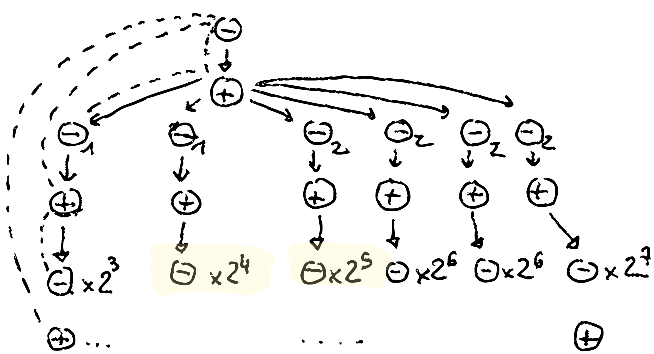
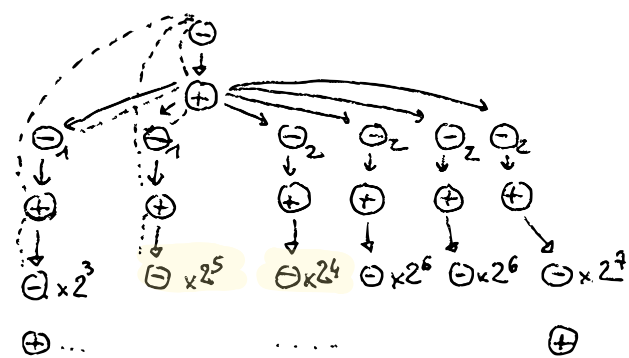
$\oplus_1 \sim^{\varphi}_{\{(\ominus_1, \ominus_1)\}} \oplus_1$ $\quad$ $\oplus_1 \sim^{\varphi}_{\{(\ominus_1, \ominus_1')\}} \oplus_1'$

Notations: $a \sim^{\varphi} b$ for $a \sim^{\varphi}_{\emptyset} b$ $\qquad$ $a \sim_{\Gamma} b$ for $a \sim^{\mathrm{id}}_{\Gamma} b$

Notations: $a \sim^{\varphi} b$ for $a \sim^{\varphi}_{\emptyset} b$ $\qquad$ $a \sim_{\Gamma} b$ for $a \sim^{id}_{\Gamma} b$

Def: $q \sim^{\varphi} p \iff \mathrm{init}(q) \sim^{\varphi} \mathrm{init}(p)$

Ex: q: $\qquad\qquad$ p:

$$q \sim^{\varphi} p$$

Proof idea :

$$\begin{tikzcd}
\sigma \arrow[rr, phantom, "\cong"] & & \tau \\
q \arrow[u, "\substack{\text{charac}\\ \text{exp}}"] \arrow[rr, phantom, "\sim^{\varphi}"] & & p \arrow[u, "\substack{\text{charac}\\ \text{exp}}"'] \\
\lhd q \rhd \arrow[rr, phantom, "\cong^{\varphi}"] & & \lhd p \rhd
\end{tikzcd}$$

1) $\lhd q \rhd \cong^{\varphi} \lhd p \rhd \quad \Rightarrow \quad q \sim^{\varphi} p$

2) $q \sim^{\varphi} p \quad \Rightarrow \quad \sigma \cong \tau$

Proof idea :

$$\begin{tikzcd}
\tau \arrow[r, phantom, "\cong"] & \sigma \\
q \arrow[u, "\text{charac exp}"] \arrow[r, phantom, "\sim^{\varphi}"] & p \arrow[u, "\text{charac exp}"'] \\
\langle q \rangle \arrow[r, phantom, "\cong^{\varphi}"] & \langle p \rangle
\end{tikzcd}$$

1) $\langle q \rangle \cong^{\varphi} \langle p \rangle \Rightarrow q \sim^{\varphi} p$

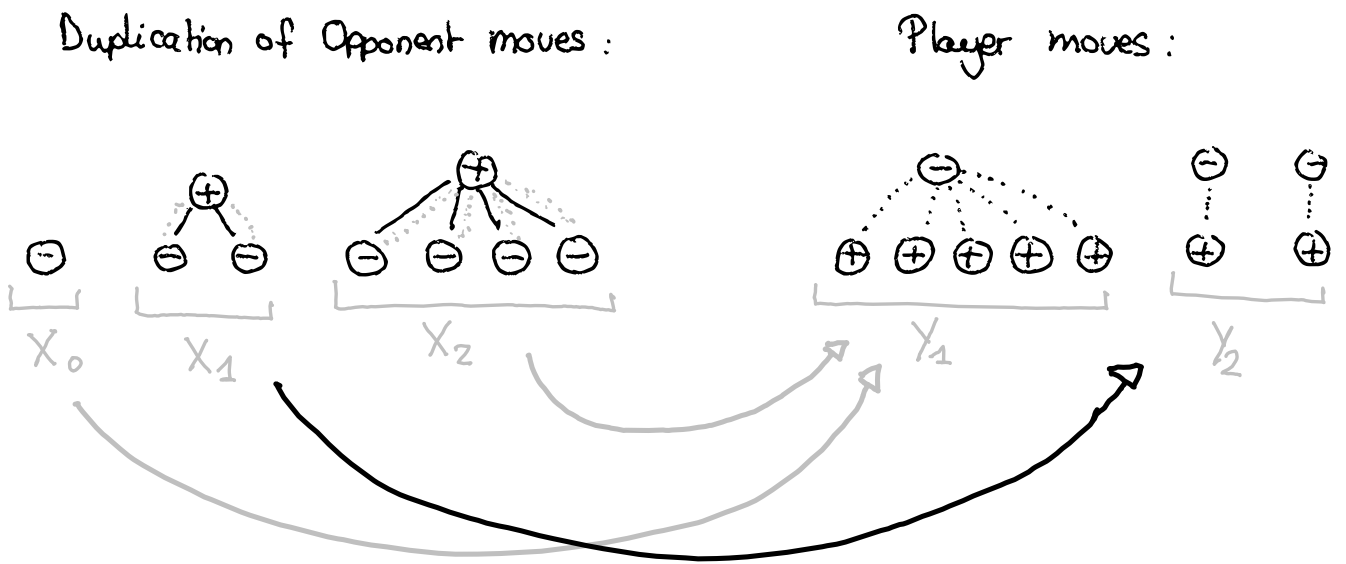
2) $q \sim^{\varphi} p \Rightarrow \tau \cong \sigma$

Pour 2) : $\tau \sim q \sim^{\varphi} p \sim \sigma \Rightarrow \tau \cong \sigma$

3. Injectivity Proof

Why does $\lhd q \rhd \overset{\varphi}{\cong} \lhd p \rhd$ implies $q \sim^{\varphi} p$?

Proof Idea:



"Y_i": equivalence class of positive events.

A) Clones (One last notion of bisimulation)

Clones: $q, p \in Aug(A)$, $\varphi : \langle q \rangle \cong \langle p \rangle$, $a \in |q|$, $b \in |p|$.

a and b are clones through φ : $a \approx^{\varphi} b$ iff

$a \sim^{\varphi}_{\Gamma} b$ and Γ preserves pointers.

Equivalence properties: $a \in |q|, b \in |p|, c \in |r|$

$\varphi : \langle q \rangle \cong \langle p \rangle$ $\quad\quad$ $\Psi : \langle p \rangle \cong \langle r \rangle$

- reflexivity: $a \approx^{id} a$,
- transitivity: $a \approx^{\varphi} b \wedge b \approx^{\Psi} c \Rightarrow a \approx^{\Psi \circ \varphi} c$,
- Symmetry: $a \approx^{\varphi} b \Rightarrow b \approx^{\varphi^{-1}} a$.

B) Partition Lemma

Lemme: $q \in exp(\nabla)$, q -- obsessional, $X \in Twin(q)$.

$\forall a_1, a_2 \in X$, $\forall a_1 \rightarrowtriangle_q b_1 \wedge a_2 \rightarrowtriangle_q b_2$, $b_1 \approx b_2$.

$$q: \quad \begin{tikzcd} & a^+ \arrow[dl] \arrow[dr] & \\ a_1^- \arrow[d] & & a_2^- \arrow[d] \\ b_1^+ & & b_2^+ \end{tikzcd} \qquad\qquad \begin{tikzcd} a^+ \arrow[d] \\ a_1^- \arrow[d] \\ b_1^+ \end{tikzcd} \quad : \nabla$$

B) Partition lemma

Lemme : $q \in \exp(\triangledown)$, q -- obsessional, $X \in \mathrm{Twin}(q)$.

$\forall\, a_1, a_2 \in X$, $\forall\, a_1 \rightarrow_q b_1 \wedge a_2 \rightarrow_q b_2$, $b_1 \approx b_2$.

$$
q: \quad
\begin{tikzcd}[column sep=small]
 & a^+ \arrow[dl] \arrow[dr] & \\
a_1^- \arrow[d] & & a_2^- \arrow[d] \\
b_1^+ & & b_2^+
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
a^+ \arrow[d] \\
a_1^- \arrow[d] \\
b_1^+
\end{tikzcd}
\quad : \triangledown
$$

Lemma : q charac exp of $\triangledown$, Y clone class of positive events

$$\# Y = \sum_{i \in I} 2^i \qquad \text{for } I \subseteq \mathbb{N} \text{ finite.}$$

Then $\forall i \in \mathbb{N}$, $i \in I \Leftrightarrow \begin{cases} \exists X_i \in \mathrm{Twin}(q), \ \# X_i = 2^i, \\ \exists\, a \in X_i,\, b \in Y,\ a \rightarrow_q b. \end{cases}$

c) Key lemma

Lemma: $q, p \in \mathrm{Aug}(A)$ charac exp of σ, τ. $\varphi : \langle\!\langle q \rangle\!\rangle \cong \langle\!\langle p \rangle\!\rangle$

Then for all $a^+ \in |q|$, $\quad a \approx^{\varphi} \varphi(a)$

Proof : Induction on depth of a.

$$\begin{array}{l}
a^+ \\
\downarrow \\
b^- \in X_i \in \mathrm{Twin}(q) \\
\downarrow \\
\boxed{c}
\end{array}$$

Y_i

avec $2^i \in \# Y_i$

c) Key lemma

Lemma: $q, p \in \mathrm{Aug}(A)$ charac exp of σ, τ. $\varphi : \langle\!\langle q \rangle\!\rangle \cong \langle\!\langle p \rangle\!\rangle$

Then for all $a^+ \in |q|$, $a \approx^{\varphi} \varphi(a)$

Proof: Induction on depth of a.

$$a^+ \to b^- \in X_i \in \mathrm{Twin}(q) \to c^+$$

Y_i

avec $2^i \in \# Y_i$

$$\varphi(a)^+ \to \varphi(b)^- \in X_i \to d^+$$

Z_i avec $2^i \in \# Z_i$

c) Key lemma

Lemma: $q, p \in \mathrm{Aug}(A)$ charac exp of σ, τ. $\varphi : \langle q \rangle \cong \langle p \rangle$

Then for all $a^+ \in |q|$, $a \approx^{\varphi} \varphi(a)$

Proof: Induction on depth of a.

$$a^+ \to b^- \in X_i \in \mathrm{Twin}(q) \to c^+$$

Y_i

avec $2^i \in \# Y_i$

$$\varphi(a)^+ \to \varphi(b)^- \in X_i \to d^+$$

Z_i avec $2^i \in \# Z_i$

$\varphi(c)$

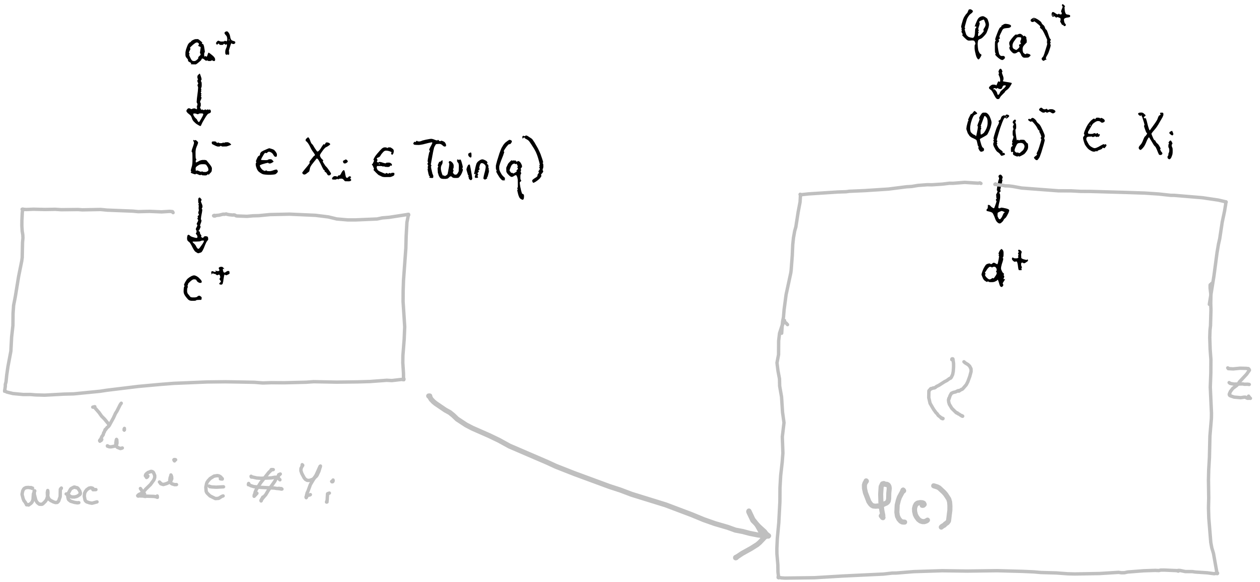
$\varphi(Y_i)$ avec $2^i \in \# \varphi(Y_i)$

c) Key lemma

Lemma: $q, p \in \mathrm{Aug}(A)$ charac exp of σ, τ. $\varphi : \langle\!\langle q \rangle\!\rangle \cong \langle\!\langle p \rangle\!\rangle$

Then for all $a^+ \in |q|$, $a \approx^{\varphi} \varphi(a)$

Proof: Induction on depth of a.



__Conclusion__: $\varphi : \lhd q \rhd \cong \lhd p \rhd \;\Rightarrow\; \forall a^{+} \in |q|,\ a \approx^{\varphi} \varphi(a)$

$\Rightarrow\; \mathrm{init}(q) \sim^{\varphi} \mathrm{init}(p)$

$\Rightarrow\; q \sim^{\varphi} p\,.$

__Thm__: $\forall\, \sigma, \tau$ causal strategies,

$$\blacktriangleleft \sigma \blacktriangleright = \blacktriangleleft \tau \blacktriangleright \;\Rightarrow\; \sigma \cong \tau\,.$$

Merci pour votre attention !

Lemma: q charac exp of ∇, Y clone class of positive events

$$\# Y = \sum_{i \in I} 2^i \qquad \text{for } I \subseteq \mathbb{N} \text{ finite.}$$

Then $\forall i \in \mathbb{N}$, $i \in I \iff \exists X_i \in \mathrm{Twin}(q)$, $\# X_i = 2^i$,
$\exists\, a \in X_i,\ b \in Y,\ a \to_q b$

Pf: $J = \{ j \mid j \in \mathbb{N},\ \exists X_j \in \mathrm{Twin}(q),\ \exists\, a \in X_j,\ b \in Y,\ a \to_q b \}$

By Lemma, $\quad Y = \biguplus_{j \in J} \mathrm{succ}(X_j)$. (*)

So $\# Y = \sum_{j \in J} \# \mathrm{succ}(X_j) = \sum_{j \in J} \# X_j = \sum_{j \in J} 2^j$ $\square$

(*) $\qquad$ determinism $\qquad$ def X_j

Lemme : q charac exp de $\tau \iff q \sim \tau$

Proof : $\boxed{\Rightarrow}$ Let $\varphi : q \to \tau$. Then for all $a \in |q|$,

$$a \sim_{\Gamma(a)} \varphi(a)$$

with $\Gamma(a) = [a]_{\bar{q}} \cong [\varphi(a)]_{\bar{p}}$

$\boxed{\Leftarrow}$ We construct $\varphi : q \to \tau$ by induction on $\leqslant_q$.

$\forall a \in |q|$, $\varphi(a)$ is provided by the bisimulation,

and unique by determinism and --linearity of τ. $\square$

$$\tau \sim q \sim^{\varphi} p \sim \zeta \Rightarrow \tau \sim \zeta \Rightarrow \tau = \zeta$$

uniqueness of $\varphi : \tau \to \tau$
by linearity and determinism

(1) $\forall\ a \xrightarrow[q]{} b^- \xrightarrow[q]{} c^+$, we have $\varphi(a) \xrightarrow[p]{} \varphi(b)^- \xrightarrow[p]{} d^+$ and $c \approx^{\varphi} d$.

(2) $\forall\ \varphi(a) \xrightarrow[p]{} \varphi(b) \xrightarrow[p]{} d$, we have $a \xrightarrow[q]{} b \xrightarrow[q]{} c^+$ with $d \approx^{\varphi^{-1}} c$

The definition of $\approx^{\varphi}$ allows all contexts for $c \approx^{\varphi} d$ to be compatible: if $c \sim^{\varphi}_{\Gamma} d$ and $c' \sim^{\varphi}_{\Gamma'} d'$, Γ, Γ' canonical, then if $e \in \mathrm{dom}(\Gamma) \cap \mathrm{dom}(\Gamma')$, $\Gamma(e) = \Gamma'(e)$.