Subject 1: CCS, Petri nets, and basic categories

to be returned on the 04/10

The questions marked (⋆) are optional, some may be more difficult.

1 Models of concurrency

Question 1 (CCS). Let the CCS processes P_1, P_2, Q_1 and Q_2 be as follows.

$$P_1 \stackrel{\text{def}}{=} a.P_1 + \bar{b}.Q_1 \qquad Q_1 \stackrel{\text{def}}{=} b.P_1$$
$$P_2 \stackrel{\text{def}}{=} b.Q_2 \qquad Q_2 \stackrel{\text{def}}{=} c.Q_2 + \bar{b}.P_2$$

The transition system from $(P_1 \| P_2) \setminus \{b\}$ is as follows.

$$\begin{tikzcd}
(P_1\|P_2)\setminus\{b\} \arrow[loop above, "a"] \arrow[r, bend left, "\tau"] & (Q_1\|Q_2)\setminus\{b\} \arrow[loop above, "c"] \arrow[l, bend left, "\tau"]
\end{tikzcd}$$

Give full derivations for the two transitions that start from $(P_1 \| P_2) \setminus \{b\}$.

Question 2 (Petri nets). A *(basic) Petri net* is given by: a set of *conditions* B; a set of events E for which ${}^{\cdot}e$ and $e^{\cdot}$ are subsets of B specifying the *preconditions* and *postconditions* of $e \in E$; and $M_0 \subseteq B$ an *initial marking*.

Generally, a *marking* is a subset of conditions. Markings can change as events occur, precisely how being expressed by the transitions events e determine between markings M, M':

$$M \xrightarrow{e} M' \text{ iff (1) } {}^{\cdot}e \subseteq M \;\&\; (M \setminus {}^{\cdot}e) \cap e^{\cdot} = \emptyset \text{ (Concession), and}$$
$$\text{(2) } M' = (M \setminus {}^{\cdot}e) \cup e^{\cdot}\,.$$

So, an occurrence of the event ends the holding of its preconditions and begins the holding of its postconditions. There is an alternative characterisation of the transitions between markings induced by event occurrences: Let M, M' be markings and e an event of a net. Prove

$$M \xrightarrow{e} M' \text{ iff } {}^{\cdot}e \subseteq M \;\&\; e^{\cdot} \subseteq M' \;\&\; M \setminus {}^{\cdot}e = M' \setminus e^{\cdot}.$$

Draw a basic Petri net for which the token game gives the following transition system:

$$\begin{tikzcd}
& M_1 \arrow[dr, "b"] & \\
M_0 \arrow[ur, "a"] \arrow[rr, "c"] \arrow[dr, "b"'] & & M_3 \arrow[dl, bend right, "d"'] \\
& M_2 \arrow[ur, bend right, "a"'] &
\end{tikzcd}$$

The initial marking should be M_0 and your answer should have four events, a, b, c and d.

2 Basic categories

Question 3. The diagrams below represent categories: the objects are the natural numbers shown; the morphisms comprise those drawn, the identities, and any additional obtained by composition. Parallel morphisms are not assumed to be equal.

For each of those, say whether it has products? coproducts? equalizers? terminal objects?

(a)

$$1$$

(b)

$$\begin{tikzcd} 1 \arrow[r] & 2 \end{tikzcd}$$

(c)

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 2 \\ 3 & \end{tikzcd}$$

(d)

$$\begin{tikzcd} 1 \arrow[r, bend left] \arrow[r, bend right] & 2 \end{tikzcd}$$

Question 4. Let $\mathcal{C}$ be a category, $A, B \in \mathcal{C}$, and (P, π_A, π_B), (P', π'_A, π'_B) two products of A and B. Show that there exists a unique isomorphism $f : P \to P'$ making the diagram below commute:

$$\begin{tikzcd} & P \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] \arrow[d, "f"] & \\ A & P' \arrow[l, "\pi'_A"] \arrow[r, "\pi'_B"'] & B \end{tikzcd}$$

Question 5. In a cartesian category $\mathcal{C}$, prove that we have the following isomorphisms, for any A, B, C (where $A \times B$ is the object part of a product of A and B, and 1 is terminal).

$$\begin{array}{rcl} A \times 1 & \cong & A \\ 1 \times A & \cong & A \\ A \times (B \times C) & \cong & (A \times B) \times C \end{array}$$

Question 6. Consider the category $\mathsf{Set}_\perp$ of *sets and partial functions*. Its objects are all sets, and a morphism $f \in \mathsf{Set}_\perp[A, B]$, also written $f : A \rightharpoonup B$, is a function

$$f : A \to B + \{\perp\}$$

where $f(a) = \perp$ is interpreted as meaning that $f(a)$ is *undefined*. The identity morphism is the same as in Set, and composition is defined as, for $f : A \rightharpoonup B$, $g : B \rightharpoonup C$:

$$\begin{array}{rclll} g \circ f & : & A & \rightharpoonup & C & \\ & & a & \mapsto & \perp & \text{if } f(a) = \perp \\ & & a & \mapsto & g(f(a)) & \text{otherwise} \end{array}$$

Does $\mathsf{Set}_\perp$ have products? If so, describe them.

Question 7. Consider the category Div having natural numbers as objects, with a morphism from n to m if and only if n divides m, *i.e.* $\exists k \in \mathbb{N}$ such that $m = nk$ (all parallel morphisms are equal).

Does Div have products? If so, describe them.

Question 8. Find a universal property that, in Set, characterizes uniquely the empty set.

Writing 0 an object satisfying this property in a cartesian category $\mathcal{C}$, show that if 0 has no non-trivial incoming morphism (*i.e.* for all $f : X \to 0$, then $X = 0$), then for all $A \in \mathcal{C}$:

$$A \times 0 \quad \cong \quad 0 \times A \quad \cong \quad 0$$

Does this iso hold without the condition that 0 has no non-trivial incoming morphism?

Question 9. If $\mathcal{C}$ has finite products and equalizers, show that it has pullbacks.

Question 10. $(\star)$ If $\mathcal{C}$ has products and coproducts, do we always have the following isomorphism?

$$A \times (B + C) \quad \cong \quad (A \times B) + (A \times C)$$

Question 11. $(\star)$ We saw that a category with one object is exactly a monoid. In the same spirit, what is a category where any two parallel morphisms are equal?

For such a category, what does having all finite products and coproducts correspond to?