Subject 2: Maps of event structures, and natural transformations

to be returned on the 25/10

1 Event structures and stable families

The required definitions for these exercises appear in the slides for lecture 3, available at `http://perso.ens-lyon.fr/pierre.clairambault/enseignement/CR07-slides-lecture3.pdf`.

Question 1. Let $f : A \to B$ be a map of event structures.
(i) Show f locally reflects causal dependency, in the following sense. Let $e, e' \in x$, a configuration of A. Show if $f(e) \leq_B f(e')$ (with both $f(e)$ and $f(e')$ defined) then $e \leq_A e'$.
(ii) Show f preserves the concurrency relation, *i.e.* if $e \, co_A \, e'$ and $f(e)$ and $f(e')$ are both defined, then $f(e) \; co_B \; f(e')$.

Question 2. Let $(A, \leq_A, \mathrm{Con}_A), (B, \leq_B, \mathrm{Con}_B)$ be event structures. Let $f : A \rightharpoonup B$ be a partial function. Show f is a map of event structures, $f : (A, \leq_A, \mathrm{Con}_A) \to (B, \leq_B, \mathrm{Con}_B)$, iff

$$\begin{aligned} &(i)\ \forall a \in A, b \in B.\ b \leq_B f(a) \implies \exists a' \in A.\ a' \leq_A a \ \&\ f(a') = b\,, \text{ and} \\ &(ii)\ \forall X \in \mathrm{Con}_A.\ fX \in \mathrm{Con}_B \ \&\ \forall a_1, a_2 \in X.\ f(a_1) = f(a_2) \implies a_1 = a_2\,. \end{aligned}$$

Question 3. Verify that the finite configurations of an event structure form a stable family.

Question 4. Say an event structure A is *tree-like* when its concurrency relation is empty (so two distinct events are either causally related or inconsistent).

Suppose B is tree-like and $f : A \to B$ is a total map of event structures (a map of event structures is *total* if the underlying function between events is total). Show A must also be tree-like, and moreover that the map f is rigid, *i.e.* preserves causal dependency.

Question 5. Show a map $f : A \to B$ of event structures is *stable*, *i.e.*

$$x \uparrow y \implies f(x \cap y) = (fx) \cap (fy)\,.$$

Question 6. Let $\mathcal{F}$ be a nonempty family of finite sets satisfying the Completeness axiom in the definition of stable families. Show $\mathcal{F}$ is coincidence-free iff

$$\forall x, y \in \mathcal{F}.\ x \subsetneq y \implies \exists x_1, e_1.\ x \overset{e_1}{\longrightarrow\!\!\!\subset} x_1 \subseteq y\,.$$

By $x \overset{e_1}{\longrightarrow\!\!\!\subset} x_1$, for configurations x, x_1 and event e_1, is meant that $e_1 \notin x$ & $x \cup \{e_1\} = x_1$. [Hint: For 'only if' use induction on the size of $y \setminus x$.]

Question 7. Let $f : \mathcal{F} \to \mathcal{G}$ be a map of stable families. Let $e, e' \in x$, a configuration of $\mathcal{F}$. Show if $f(e) \leq_{fx} f(e')$ (with both $f(e)$ and $f(e')$ defined) then $e \leq_x e'$.

(the definition of $\leq_x$ for $x \in \mathcal{F}$ in a stable family $\mathcal{F}$ appears on slide 17)

2 Natural transformations and equivalence of categories

Question 8. Which of the following functions correspond to natural transformations?

(a) Is this a natural transformation between functors from Set to Set?

$$\begin{array}{rrcl} \mathrm{fst}_A & : \quad \mathsf{List}(A) & \to & A + \{*\} \\ & [a_1, \ldots, a_n] & \mapsto & (1, a_1) \\ & [] & \mapsto & (2, *) \end{array}$$

where $A + B = (\{1\} \times A) \cup (\{2\} \times B)$ is the disjoint union.

(b) Is this a natural transformation between functors from Set to Set?

$$\begin{array}{rrcll} \mathrm{equal}_A & : \quad \mathsf{List}(A) & \to & \mathbb{B} & \\ & l & \mapsto & \texttt{true} & \text{if all elements of } l \text{ are equal} \\ & & & \texttt{false} & \text{otherwise} \end{array}$$

(note: an object $A \in \mathcal{C}$ may be regarded as the constant functor $\mathcal{C} \to \mathcal{C}$ which to any $X \in \mathcal{C}$ associates A, and to any $f : X \to Y$ associates id_A).

(c) Is this a natural transformation between functors from Set to Set?

$$\begin{array}{rrcl} \mathrm{length}_A & : \quad \mathsf{List}(A) & \to & \mathbb{N} \\ & [] & \mapsto & 0 \\ & [a_1, \ldots, a_n] & \mapsto & n \end{array}$$

(d) Is this a natural transformation between functors from Ford to Set?

$$\begin{array}{rrcl} \mathrm{sort}_A & : \quad \mathsf{List}(A) & \to & \mathsf{List}(A) \\ & l & \mapsto & \mathrm{sort}(l) \end{array}$$

where Ford is the category of totally ordered finite sets and order-preserving functions? What if the morphisms in Ford were not required to preserve the order?

Question 9. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors, and $\eta : F \Rightarrow G$ be a natural transformation. Show that η has an inverse iff for all $A \in \mathcal{C}$, η_A has an inverse.

Question 10. Consider the category Fam having

- *Objects:* families of sets $(A_i)_{i \in I}$ (where $I \in \mathsf{Set}$, and for all $i \in I, A_i \in \mathsf{Set}$).
- *Morphisms* from $(A_i)_{i \in I}$ to $(B_j)_{j \in J}$: pairs $(f, (g_i)_{i \in I})$ where $f : I \to J$, and for all $i \in I$, $g_i : A_i \to B_{f(i)}$.

Show that Fam is equivalent to the presheaf category $[1 \to 2, \mathsf{Set}]$, where $1 \to 2$ is the category having two objects $1, 2$ and only one morphism from 1 to 2 (plus identities).

Question 11. Show that if $F : \mathcal{C} \to D$ is part of an equivalence of categories, then it is full, faithful and essentially surjective on objects. [Hint: prove faithful first]

Question 12. Show that equivalence of categories is an equivalence relation [you may use results in the lecture].