Subject 2: Maps of event structures, and natural transformations
to be returned on the 25/10

1 Event structures and stable families


Question 1. Let \( f : A \to B \) be a map of event structures.
(i) Show \( f \) locally reflects causal dependency, in the following sense. Let \( e, e' \in x, a \) configuration of \( A \). Show if \( f(e) \leq_B f(e') \) (with both \( f(e) \) and \( f(e') \) defined) then \( e \leq_A e' \).
(ii) Show \( f \) preserves the concurrency relation, i.e. if \( e \co_A e' \) and \( f(e) \) and \( f(e') \) are both defined, then \( f(e) \co_B f(e') \).

Question 2. Let \((A, \leq_A, \text{Con}_A), (B, \leq_B, \text{Con}_B)\) be event structures. Let \( f : A \rightsquigarrow B \) be a partial function. Show \( f \) is a map of event structures, \( f : (A, \leq_A, \text{Con}_A) \to (B, \leq_B, \text{Con}_B) \), iff

\[
\begin{align*}
(i) \quad & \forall a \in A, b \in B. \quad b \leq_B f(a) \implies \exists a' \in A. \quad a' \leq_A a \ \& \ f(a') = b, \\
(ii) \quad & \forall X \in \text{Con}_A. \quad fX \in \text{Con}_B \ \& \ \forall a_1, a_2 \in X. \quad f(a_1) = f(a_2) \implies a_1 = a_2.
\end{align*}
\]

Question 3. Verify that the finite configurations of an event structure form a stable family.

Question 4. Say an event structure \( A \) is tree-like when its concurrency relation is empty (so two distinct events are either causally related or inconsistent).
Suppose \( B \) is tree-like and \( f : A \to B \) is a total map of event structures (a map of event structures is total if the underlying function between events is total). Show \( A \) must also be tree-like, and moreover that the map \( f \) is rigid, i.e. preserves causal dependency.

Question 5. Show a map \( f : A \to B \) of event structures is stable, i.e.
\[ x \uparrow y \implies f(x \cap y) = (fx) \cap (fy). \]

Question 6. Let \( F \) be a nonempty family of finite sets satisfying the Completeness axiom in the definition of stable families. Show \( F \) is coincidence-free iff
\[ \forall x, y \in F. \quad x \subseteq y \implies \exists x_1, e_1. \quad x \leftarrow^{e_1} x_1 \subseteq x_1 \subseteq y. \]
By \( x \leftarrow^{e_1} x_1 \), for configurations \( x, x_1 \) and event \( e_1 \), is meant that \( e_1 \notin x \) & \( x \cup \{e_1\} = x_1 \). [Hint: For ‘only if’ use induction on the size of \( y \setminus x \).]

Question 7. Let \( f : F \to G \) be a map of stable families. Let \( e, e' \in x \), a configuration of \( F \). Show if \( f(e) \leq_{fx} f(e') \) (with both \( f(e) \) and \( f(e') \) defined) then \( e \leq_x e' \).
(the definition of \( \leq_x \) for \( x \in F \) in a stable family \( F \) appears on slide 17)
2 Natural transformations and equivalence of categories

Question 8. Which of the following functions correspond to natural transformations?

(a) Is this a natural transformation between functors from $\text{Set}$ to $\text{Set}$?

\[
\text{fst}_A : \text{List}(A) \to A + \{\ast\} \\
[a_1, \ldots, a_n] \mapsto (1, a_1) \\
[] \mapsto (2, \ast)
\]

where $A + B = (\{1\} \times A) \cup (\{2\} \times B)$ is the disjoint union.

(b) Is this a natural transformation between functors from $\text{Set}$ to $\text{Set}$?

\[
\text{equal}_A : \text{List}(A) \to \mathbb{B} \\
l \mapsto \begin{cases} 
\text{true} & \text{if all elements of } l \text{ are equal} \\
\text{false} & \text{otherwise}
\end{cases}
\]

(note: an object $A \in C$ may be regarded as the constant functor $C \to C$ which to any $X \in C$ associates $A$, and to any $f : X \to Y$ associates $\text{id}_A$).

(c) Is this a natural transformation between functors from $\text{Set}$ to $\text{Set}$?

\[
\text{length}_A : \text{List}(A) \to \mathbb{N} \\
[ ] \mapsto 0 \\
[a_1, \ldots, a_n] \mapsto n
\]

(d) Is this a natural transformation between functors from $\text{Ford}$ to $\text{Set}$?

\[
\text{sort}_A : \text{List}(A) \to \text{List}(A) \\
l \mapsto \text{sort}(l)
\]

where $\text{Ford}$ is the category of totally ordered finite sets and order-preserving functions. What if the morphisms in $\text{Ford}$ were not required to preserve the order?

Question 9. Let $F, G : C \to D$ be functors, and $\eta : F \Rightarrow G$ be a natural transformation. Show that $\eta$ has an inverse iff for all $A \in C$, $\eta_A$ has an inverse.

Question 10. Consider the category $\text{Fam}$ having

- **Objects:** families of sets $(A_i)_{i \in I}$ (where $I \in \text{Set}$, and for all $i \in I, A_i \in \text{Set}$).

- **Morphisms** from $(A_i)_{i \in I}$ to $(B_j)_{j \in J}$: pairs $(f, (g_i)_{i \in I})$ where $f : I \to J$, and for all $i \in I$, $g_i : A_i \to B_{f(i)}$.

Show that $\text{Fam}$ is equivalent to the presheaf category $[1 \to 2, \text{Set}]$, where $1 \to 2$ is the category having two objects $1, 2$ and only one morphism from $1$ to $2$ (plus identities).

Question 11. Show that if $F : C \to D$ is part of an equivalence of categories, then it is full, faithful and essentially surjective on objects. [Hint: prove faithful first]

Question 12. Show that equivalence of categories is an equivalence relation [you may use results in the lecture].