1 Games and strategies

Question 1. Let $A$, $B$ and $C$ be three event structures with a common set of events $\{1, 2, 3\}$. In $A$, $B$ and $C$ the events are consistent with each other. In $C$ they have the trivial, identity relation as their causal dependency. In $A$, the only nontrivial causal dependency is $1 \rightarrow 3$ whereas in $B$ the only nontrivial causal dependency is $2 \rightarrow 3$. Let the maps $f : A \rightarrow C$ and $g : B \rightarrow C$ be the maps of event structures whose underlying relation is the identity relation on events. Using the pullback of stable families, derive the pullback of $f$ and $g$ in the category of event structures.

Question 2. For each instance of total map $\sigma$ of event structures with polarity below say whether $\sigma$ is a strategy and whether it is deterministic. In each case give a short justification for your answer. (Immediate causal dependency within the event structures is represented by an arrow $\rightarrow$ and inconsistency, or conflict, by a wiggly line $\sim \sim \sim$.)

\begin{itemize}
  \item[(i)] $S$ \quad $\oplus \rightarrow \oplus$
  \quad $\sigma$
  \quad $A$ \quad $\oplus \rightarrow \oplus$
  \item[(ii)] $S$ \quad $\oplus \rightarrow \oplus$
  \quad $\sigma$
  \quad $A$ \quad $\oplus \rightarrow \oplus$
  \item[(iii)] $S$ \quad $\oplus \rightarrow \oplus$
  \quad $\sigma$
  \quad $A$ \quad $\oplus \rightarrow \oplus$
  \item[(iv)] $S$ \quad $\oplus \rightarrow \oplus$
  \quad $\sigma$
  \quad $A$ \quad $\oplus \rightarrow \oplus$
  \item[(v)] $S$ \quad $\oplus \rightarrow \oplus$
  \quad $\sigma$
  \quad $A$ \quad $\oplus \rightarrow \oplus$
\end{itemize}
Question 3. Let $E$ be an event structure. Let $e, e' \in E$. Show
\[ \exists y, y_1 \in C^\infty(E). y \xrightarrow{e} y_1 \xrightarrow{e'} \iff e \rightarrow e' \text{ or } e \circ e'. \]

Question 4. Recall, the definition of a strategy as a total map of event structures with polarity which is receptive and courteous/innocent. Let $\sigma : S \rightarrow A$ be a total map of event structures. Show that $\sigma$ is a strategy iff the following three conditions hold:
(i) $\sigma x \xrightarrow{a} \& \ pol_A(a) = - \implies \exists ! s \in S. x \xrightarrow{s} \& \sigma(s) = a$, for all $x \in C(S), a \in A$,
(ii)(+) If $x \xrightarrow{e} x_1 \xrightarrow{e'} \& pol_S(e) = +$ in $C(S)$ and $\sigma x \xrightarrow{\sigma(e')} \in C(A)$, then $x \xrightarrow{e'}$ in $C(S)$, and
(ii)(−) If $x \xrightarrow{e} x_1 \xrightarrow{e'} \& pol_S(e') = -$ in $C(S)$ and $\sigma x \xrightarrow{\sigma(e')} \in C(A)$, then $x \xrightarrow{e'}$ in $C(S)$.

Question 5. Let $A$ be an event structure with polarity. Consider the empty map of event structures with polarity $\emptyset \rightarrow A$. Is it a strategy in $A$? Is it a deterministic strategy? Consider now the identity map $\text{id}_A : A \rightarrow A$ on an event structure with polarity $A$. Is it a strategy? Is it a deterministic strategy? [Your answer may depend on $A$. If so specify how.]
**Question 6.** (⋆) Say an event structure is *set-like* if its causal dependency relation is the identity relation and all pairs of distinct events are inconsistent. Let $A$ and $B$ be games with underlying event structures which are set-like event structures. In this case, can you see a simpler way to describe deterministic strategies $A \rightarrow B$? What does composition of deterministic strategies between set-like games correspond to? What do strategies in general between set-like games correspond to? [No proofs are required.]

**Question 7.** (⋆) Let $\sigma : S \rightarrow B$ be a strategy in a game $B$. Let $f : A \rightarrow B$ be a total map of event structures with polarity. Prove that $f^* \sigma$, the pullback of $\sigma$ along $f$, is a strategy in $A$.

![Diagram showing pullback]

[In fact this result also holds when $f$ is partial.]

Deduce that if $\sigma_1 : S_1 \rightarrow A$ and $\sigma_2 : S_2 \rightarrow A$ are strategies in a game $A$, then their pullback $\sigma_1 \land \sigma_2$ — see the diagram — is also a strategy in $A$.

![Diagram showing pullback of strategies]

The strategy $\sigma_1 \land \sigma_2$ is a form of conjunction between strategies. Can you describe its behaviour informally in terms of that of $\sigma_1$ and $\sigma_2$?

## 2 Cartesian closed categories

### 2.1 Cartesian structure

For this section, fix a cartesian category $C$. For each objects $A,B \in C$, we fix a product $(A \times B, \pi^A_{1,B}, \pi^A_{2,B})$ (the $A,B$ annotations on projections are often omitted when they can be recovered from the context). For $f : X \rightarrow A$ and $g : X \rightarrow B$, we write $\langle f, g \rangle : X \rightarrow A \times B$ for the pairing given by the universal property.

Recall also that the functorial action of $\times$ is then defined as

$$f_A \times f_B = \langle f_A \circ \pi_1, f_B \circ \pi_2 \rangle : A_1 \times B_1 \rightarrow A_2 \times B_2$$

for $f_A : A_1 \rightarrow A_2$ and $f_B : B_1 \rightarrow B_2$.

**Question 8.** Show that:

(a) For all $f : X \rightarrow A, g : X \rightarrow A$, and $h : Y \rightarrow X$, we have the following equation:

$$\langle f, g \rangle \circ h = \langle f \circ h, g \circ h \rangle$$
(b) For all \( f : X \to A \times B \), we have the following equation:
\[
f = (\pi_1 \circ f, \pi_2 \circ f)
\]
(c) For all \( f_B : A \to B, f_C : A \to C, g_B : B \to B', g_C : C \to C' \), we have the following equation:
\[
(g_B \times g_C) \circ (f_B, f_C) = (g_B \circ f_B, g_C \circ f_C)
\]

**Question 9.** Show that there is a unique \( \delta_A : A \to A \times A \) (the diagonal) such that \( \pi_1 \circ \delta_A = \text{id}_A \) and \( \pi_2 \circ \delta_A = \text{id}_A \). Show that for any \( f_B : A \to B, f_C : A \to C \), the following diagram commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{(f_B, f_C)} & B \times C \\
\delta_A \downarrow & & \downarrow \text{ev} \\
A \times A & \xrightarrow{f_B \times f_C} & B \times C
\end{array}
\]

Deduce that \( \delta_A \) is natural in \( A \).

**Question 10.** Show that the projections yield natural transformations:

\[
\pi_1^B : (-) \times B \to (-) \quad \pi_2^A : A \times (-) \to (-)
\]

between functors \( C \to C \).

### 2.2 Cartesian closed structure

**Question 11.** Let \( C \) be a cartesian category. Recall that an exponential of \( A \) to \( B \) is a pair \((E, \text{ev})\) such that \( E \in C \) and \( \text{ev} \in C[E \times A, B] \), satisfying the following universal property: for all object \( C \) and morphism \( f : C \times A \to B \), there is a unique \( h : C \to E \) such that the following diagram commutes:

\[
\begin{array}{ccc}
C \times A & \xrightarrow{f} & B \\
\downarrow h \times A & & \downarrow \text{ev} \\
E \times A & \xrightarrow{\text{ev}} & B
\end{array}
\]

Show that the exponentials of \( A \) to \( B \) are unique up to isomorphism, i.e. if \((E, \text{ev})\) and \((E', \text{ev}')\) are two exponentials of \( A \) to \( B \), then there is an isomorphism \( \phi : E \to E' \) such that the following diagram commutes:

\[
\begin{array}{ccc}
E \times A & \xrightarrow{\phi \times A} & E' \times A \\
\downarrow \text{ev} & & \downarrow \text{ev}' \\
B & \xrightarrow{\text{ev}} & B
\end{array}
\]

From now on, we assume that \( C \) is cartesian closed, and we fix, for any two objects \( A, B \), an exponential of \( A \) to \( B \) \((A \Rightarrow B, \text{ev}_{A,B})\). For \( f : B \times A \to C \), write \( \Lambda(f) : B \to A \Rightarrow C \) the morphism given by universal property.

**Question 12.** Show that for all \( f : B_2 \times A \to C \), for all \( g : B_1 \to B_2 \), we have:

\[
\Lambda(f) \circ g = \Lambda(f \circ (g \times A))
\]
Question 13. Assume we have typed terms:

\[
\Gamma, x : A \vdash M : B \\
\Gamma \vdash N : A
\]

Recall from the lecture that:

\[
\llbracket \Gamma \vdash M \vdash N : B \rrbracket = \text{ev}_{\llbracket A \rrbracket, \llbracket B \rrbracket} \circ (\Lambda(\llbracket M \rrbracket), \llbracket N \rrbracket) : \llbracket \Gamma \rrbracket \to \llbracket B \rrbracket
\]

(note that since we assume here that \( x \) is the last variable in the typing context of \( M \), there is no need to reorder the context via a \( \gamma \) isomorphism when computing \( \llbracket \lambda x. M \rrbracket \))

Show that:

\[
\llbracket (\lambda x. M) N \rrbracket = \llbracket M[N/x] \rrbracket
\]

using the substitution lemma seen in the lecture.

2.3 Optional: isomorphisms in cartesian closed categories

Question 14. (*) Let \( C \) be a category, and \( A, B \) be objects.

(a) Show that for all \( h : A \to B \), the function

\[
\varphi_X : C[X, A] \to C[X, B] \\
f \mapsto h \circ f
\]

is natural in \( X \).

(b) Show that reciprocally, any natural transformation

\[
\varphi : C[-, A] \to C[-, B]
\]

has the form \( \varphi(f) = h \circ f \) (for some \( h : A \to B \)).

(c) Deduce that for any \( A, B \in C \), \( A \cong B \) iff the functors \( C[-, A] \) and \( C[-, B] \) are naturally isomorphic.

Question 15. (*) Show that for any \( A, B, C \), we have the following isomorphisms:

\[
A \Rightarrow 1 \cong 1 \\
1 \Rightarrow A \cong A \\
(A \times B) \Rightarrow C \cong A \Rightarrow (B \Rightarrow C) \\
A \Rightarrow (B \times C) \cong (A \Rightarrow B) \times (A \Rightarrow C)
\]

If \( \mathcal{V} \) is a set of variables, we consider the following arithmetic expressions on \( \mathcal{V} \):

\[
e, e' ::= x \mid e \cdot e' \mid e/e' \mid 1
\]

where \( x \in \mathcal{V} \). Given a valuation, i.e. some \( v : \mathcal{V} \to \mathbb{N} \), we define \( \{e\}_v \in \mathbb{N} \) by \( \{x\}_v = v(x) \), \( \{1\}_v = 1 \), \( \{e \cdot e'\}_v = \{e\}_v \times \{e'\}_v \), and \( \{e/e'\}_v = \{e\}_v^{\{e'\}_v} \). We say that

\[
\mathbb{N} \models e = e' \iff \forall v : \mathcal{V} \to \mathbb{N}, \{e\}_v = \{e'\}_v
\]

We recall the following theorem:
Theorem 1 (Martin, 1972). For $e, e'$ arithmetic expressions as above, we have $\mathbb{N} \models e = e'$ iff $e$ and $e'$ are convertible using the following “high school algebra” equations:

\[
\begin{align*}
1 \cdot x &= x \\
x^1 &= x \\
(x \cdot y) \cdot z &= x \cdot (y \cdot z) \\
(x \cdot y)^z &= x^z \cdot y^z
\end{align*}
\]

\[1^x = 1 \quad x \cdot y = y \cdot x \quad x^y^z = (x^y)^z\]

But we may also, given a valuation $\rho : V \to C_0$, interpret arithmetic expressions as objects in a cartesian closed category $C$, with $[1]_{\rho} = 1$, $[x]_{\rho} = \rho(x)$, $[e \cdot e']_{\rho} = [e]_{\rho} \times [e']_{\rho}$, $[e^e']_{\rho} = [e']_{\rho} \Rightarrow [e]_{\rho}$.

Question 16. (**) Using Martin’s theorem, show that for all arithmetic expressions $e, e'$, $\mathbb{N} \models e = e'$ iff for all cartesian closed category $C$, for all valuation $\rho : V \to C_0$, we have $[e]_{\rho} \cong [e']_{\rho}$. 
