Subject 3: Games, Strategies and Cartesian Closed Categories

to be returned on the 13/12

The questions marked $(\star)$ are optional, some may be more difficult.

1 Games and strategies

Question 1. Let A, B and C be three event structures with a common set of events $\{1, 2, 3\}$. In A, B and C the events are consistent with each other. In C they have the trivial, identity relation as their causal dependency. In A, the only nontrivial causal dependency is $1 \rightarrowtriangle 3$ whereas in B the only nontrivial causal dependency is $2 \rightarrowtriangle 3$. Let the maps $f : A \to C$ and $g : B \to C$ be the maps of event structures whose underlying relation is the identity relation on events. Using the pullback of stable families, derive the pullback of f and g in the category of event structures.

Question 2. For each instance of total map σ of event structures with polarity below say whether σ is a strategy and whether it is deterministic. In each case give a short justification for your answer. (Immediate causal dependency within the event structures is represented by an arrow $\rightarrowtriangle$ and inconsistency, or conflict, by a wiggly line $\sim\!\sim\!\sim$.)

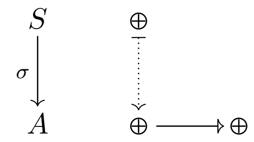
(i)
$$\begin{tikzcd} S \arrow[d, "\sigma"'] & \ominus \arrow[r, -Triangle] \arrow[d, dotted, mapsto] & \oplus \arrow[d, dotted, mapsto] \\ A & \ominus & \oplus \end{tikzcd}$$

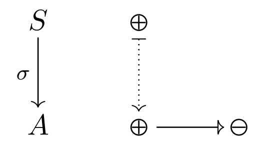
(ii)
$$\begin{tikzcd} S \arrow[d, "\sigma"'] & \oplus \arrow[r, -Triangle] \arrow[d, dotted, mapsto] & \ominus \arrow[d, dotted, mapsto] \\ A & \oplus & \ominus \end{tikzcd}$$

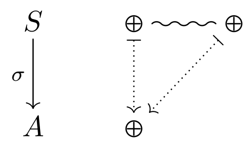
(iii)
$$\begin{tikzcd} S \arrow[d, "\sigma"'] & \oplus \arrow[r, -Triangle] \arrow[d, dotted, mapsto] & \oplus \arrow[d, dotted, mapsto] \\ A & \oplus & \oplus \end{tikzcd}$$

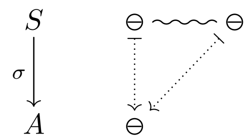
(iv)
$$\begin{tikzcd} S \arrow[d, "\sigma"'] & \ominus \arrow[r, -Triangle] \arrow[d, dotted, mapsto] & \ominus \arrow[d, dotted, mapsto] \\ A & \ominus & \ominus \end{tikzcd}$$

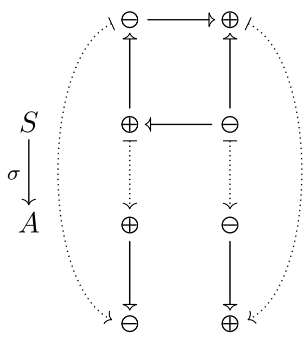
(v)
$$\begin{tikzcd} S \arrow[d, "\sigma"'] & \ominus \arrow[d, dotted, mapsto] & \\ A & \ominus \arrow[r, -Triangle] & \oplus \end{tikzcd}$$

(vi) 

(vii) 

(viii) 

(ix) 

(x) 

Question 3. Let E be an event structure. Let $e, e' \in E$. Show

$$\exists y, y_1 \in \mathcal{C}^\infty(E).\ y \overset{e}{\longrightarrow\!\!\!\!\subset} y_1 \overset{e'}{\longrightarrow\!\!\!\!\subset} \iff e \rightarrowtriangle e' \ \text{ or } \ e\ co\ e'\,.$$

Question 4. Recall, the definition of a strategy as a total map of event structures with polarity which is receptive and courteous/innocent. Let $\sigma : S \to A$ be a total map of event structures. Show that σ is a strategy iff the following three conditions hold:
(i) $\sigma x \overset{a}{\longrightarrow\!\!\!\!\subset}\ \&\ pol_A(a) = - \Rightarrow \exists! s \in S.\ x \overset{s}{\longrightarrow\!\!\!\!\subset}\ \&\ \sigma(s) = a$, for all $x \in \mathcal{C}(S)$, $a \in A$,
(ii)(+) If $x \overset{e}{\longrightarrow\!\!\!\!\subset} x_1 \overset{e'}{\longrightarrow\!\!\!\!\subset}\ \&\ pol_S(e) = +$ in $\mathcal{C}(S)$ and $\sigma x \overset{\sigma(e')}{\longrightarrow\!\!\!\!\subset}$ in $\mathcal{C}(A)$, then $x \overset{e'}{\longrightarrow\!\!\!\!\subset}$ in $\mathcal{C}(S)$, and
(ii)(−) If $x \overset{e}{\longrightarrow\!\!\!\!\subset} x_1 \overset{e'}{\longrightarrow\!\!\!\!\subset}\ \&\ pol_S(e') = -$ in $\mathcal{C}(S)$ and $\sigma x \overset{\sigma(e')}{\longrightarrow\!\!\!\!\subset}$ in $\mathcal{C}(A)$, then $x \overset{e'}{\longrightarrow\!\!\!\!\subset}$ in $\mathcal{C}(S)$.

Question 5. Let A be an event structure with polarity. Consider the empty map of event structures with polarity $\varnothing \to A$. Is it a strategy in A? Is it a deterministic strategy? Consider now the identity map $\mathrm{id}_A : A \to A$ on an event structure with polarity A. Is it a strategy? Is it a deterministic strategy? [Your answer may depend on A. If so specify how.]

Question 6. (⋆) Say an event structure is *set-like* if its causal dependency relation is the identity relation and all pairs of distinct events are inconsistent. Let A and B be games with underlying event structures which are set-like event structures. In this case, can you see a simpler way to describe deterministic strategies $A \mathrel{\rightarrow\!\!\!\!\!+} B$? What does composition of deterministic strategies between set-like games correspond to? What do strategies in general between set-like games correspond to? What does composition of strategies between set-like games correspond to? [No proofs are required.]

Question 7. (⋆) Let $\sigma : S \to B$ be a strategy in a game B. Let $f : A \to B$ be a total map of event structures with polarity. Prove that $f^*\sigma$, the pullback of σ along f, is a strategy in A.

$$
\begin{tikzcd}
S' \arrow[r] \arrow[d, "f^*\sigma"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "\sigma"] \\
A \arrow[r, "f"'] & B\,.
\end{tikzcd}
$$

[In fact this result also holds when f is partial.]
Deduce that if $\sigma_1 : S_1 \to A$ and $\sigma_2 : S_2 \to A$ are strategies in a game A, then their pullback $\sigma_1 \wedge \sigma_2$ — see the diagram — is also a strategy in A.

$$
\begin{tikzcd}
 & S_1 \wedge S_2 \arrow[dl, "\Pi_1"'] \arrow[dr, "\Pi_2"] \arrow[dd, dotted, "\sigma_1\wedge\sigma_2"] & \\
S_1 \arrow[dr, "\sigma_1"'] & & S_2 \arrow[dl, "\sigma_2"] \\
 & A\,. &
\end{tikzcd}
$$

The strategy $\sigma_1 \wedge \sigma_2$ is a form of conjunction between strategies. Can you describe its behaviour informally in terms of that of σ_1 and σ_2?

2 Cartesian closed categories

2.1 Cartesian structure

For this section, fix a cartesian category $\mathcal{C}$. For each objects $A, B \in \mathcal{C}$, we fix a product $(A \times B, \pi_1^{A,B}, \pi_2^{A,B})$ (the A, B annotations on projections are often omitted when they can be recovered from the context). For $f : X \to A$ and $g : X \to B$, we write $\langle f, g\rangle : X \to A \times B$ for the pairing given by the universal property.

Recall also that the functorial action of $\times$ is then defined as

$$f_A \times f_B = \langle f_A \circ \pi_1, f_B \circ \pi_2\rangle : A_1 \times B_1 \to A_2 \times B_2$$

for $f_A : A_1 \to A_2$ and $f_B : B_1 \to B_2$.

Question 8. Show that:
(a) For all $f : X \to A, g : X \to A$, and $h : Y \to X$, we have the following equation:

$$\langle f, g\rangle \circ h = \langle f \circ h, g \circ h\rangle$$

(b) For all $f : X \to A \times B$, we have the following equation:

$$f = \langle \pi_1 \circ f, \pi_2 \circ f \rangle$$

(c) For all $f_B : A \to B, f_C : A \to C, g_B : B \to B', g_C : C \to C'$, we have the following equation:

$$(g_B \times g_C) \circ \langle f_B, f_C \rangle = \langle g_B \circ f_B, g_C \circ f_C \rangle$$

Question 9. Show that there is a unique $\delta_A : A \to A \times A$ (the **diagonal**) such that $\pi_1 \circ \delta_A = \mathrm{id}_A$ and $\pi_2 \circ \delta_A = \mathrm{id}_A$. Show that for any $f_B : A \to B, f_C : A \to C$, the following diagram commutes:

$$\begin{tikzcd}
A \arrow[d, "\delta_A"'] \arrow[dr, "\langle f_B, f_C \rangle"] & \\
A \times A \arrow[r, "f_B \times f_C"'] & B \times C
\end{tikzcd}$$

Deduce that δ_A is natural in A.

Question 10. Show that the projections yield natural transformations:

$$\pi_1^{-,B} : (-) \times B \to (-) \qquad \pi_2^{A,-} : A \times (-) \to (-)$$

between functors $\mathcal{C} \to \mathcal{C}$.

2.2 Cartesian closed structure

Question 11. Let $\mathcal{C}$ be a cartesian category. Recall that an **exponential** of A to B is a pair (E, ev) such that $E \in \mathcal{C}$ and $\mathsf{ev} \in \mathcal{C}[E \times A, B]$, satisfying the following universal property: for all object C and morphism $f : C \times A \to B$, there is a unique $h : C \to E$ such that the following diagram commutes:

$$\begin{tikzcd}
C \times A \arrow[d, "h \times A"'] \arrow[dr, "f"] & \\
E \times A \arrow[r, "\mathsf{ev}"'] & B
\end{tikzcd}$$

Show that the exponentials of A to B are unique up to isomorphism, *i.e.* if (E, ev) and (E', ev') are two exponentials of A to B, then there is an isomorphism $\phi : E \to E'$ such that the following diagram commutes:

$$\begin{tikzcd}
E \times A \arrow[rr, "\phi \times A"] \arrow[dr, "\mathsf{ev}"'] & & E' \times A \arrow[dl, "\mathsf{ev}'"] \\
& B &
\end{tikzcd}$$

From now on, we assume that $\mathcal{C}$ is cartesian closed, and we fix, for any two objects A, B, an exponential of A to B $(A \Rightarrow B, \mathsf{ev}_{A,B})$. For $f : B \times A \to C$, write $\Lambda(f) : B \to A \Rightarrow C$ the morphism given by universal property.

Question 12. Show that for all $f : B_2 \times A \to C$, for all $g : B_1 \to B_2$, we have:

$$\Lambda(f) \circ g = \Lambda(f \circ (g \times A))$$

Question 13. Assume we have typed terms:

$$\frac{}{\Gamma, x : A \vdash M : B} \qquad\qquad \frac{}{\Gamma \vdash N : A}$$

Recall from the lecture that:

$$[\![\Gamma \vdash M\,N : B]\!] = \mathsf{ev}_{[\![A]\!],[\![B]\!]} \circ \langle \Lambda([\![M]\!]), [\![N]\!]\rangle : [\![\Gamma]\!] \to [\![B]\!]$$

(note that since we assume here that x is the last variable in the typing context of M, there is no need to reorder the context via a γ isomorphism when computing $[\![\lambda x.\,M]\!]$)

Show that:

$$[\![(\lambda x.\,M)\,N]\!] = [\![M[N/x]]\!]$$

using the substitution lemma seen in the lecture.

2.3 Optional: isomorphisms in cartesian closed categories

Question 14. $(\star)$ Let $\mathcal{C}$ be a category, and A, B be objects.
(a) Show that for all $h : A \to B$, the function

$$\begin{array}{rccc} \varphi_X & : & \mathcal{C}[X,A] & \to & \mathcal{C}[X,B] \\ & & f & \mapsto & h \circ f \end{array}$$

is natural in X.
(b) Show that reciprocally, any natural transformation

$$\varphi \quad : \quad \mathcal{C}[-,A] \quad \to \quad \mathcal{C}[-,B]$$

has the form $\varphi_X(f) = h \circ f$ (for some $h : A \to B$).
(c) Deduce that for any $A, B \in \mathcal{C}$, $A \cong B$ iff the functors $\mathcal{C}[-,A]$ and $\mathcal{C}[-,B]$ are naturally isomorphic.

Question 15. $(\star)$ Show that for any A, B, C, we have the following isomorphisms:

$$\begin{array}{rcl} A \Rightarrow 1 & \cong & 1 \\ 1 \Rightarrow A & \cong & A \\ (A \times B) \Rightarrow C & \cong & A \Rightarrow (B \Rightarrow C) \\ A \Rightarrow (B \times C) & \cong & (A \Rightarrow B) \times (A \Rightarrow C) \end{array}$$

If $\mathcal{V}$ is a set of variables, we consider the following arithmetic expressions on $\mathcal{V}$:

$$e, e' ::= x \mid e \cdot e' \mid e^{e'} \mid 1$$

where $x \in \mathcal{V}$. Given a **valuation**, *i.e.* some $v : \mathcal{V} \to \mathbb{N}$, we define $\langle e \rangle_v \in \mathbb{N}$ by $\langle x \rangle_v = v(x)$, $\langle 1 \rangle_v = 1$, $\langle e \cdot e' \rangle_v = \langle e \rangle_v \times \langle e' \rangle_v$, and $\langle e^{e'} \rangle_v = \langle e \rangle_v^{\langle e' \rangle_v}$. We say that

$$\mathbb{N} \vDash e = e' \quad \Leftrightarrow \quad \forall v : \mathcal{V} \to \mathbb{N},\ \langle e \rangle_v = \langle e' \rangle_v$$

We recall the following theorem:

Theorem 1 (Martin, 1972). For e, e' arithmetic expressions as above, we have $\mathbb{N} \vDash e = e'$ iff e and e' are convertible using the following "high school algebra" equations:

$$\begin{array}{rclcrcl} 1 \cdot x & = & x & & 1^x & = & 1 \\ x^1 & = & x & & x \cdot y & = & y \cdot x \\ (x \cdot y) \cdot z & = & x \cdot (y \cdot z) & & x^{y \cdot z} & = & (x^y)^z \\ (x \cdot y)^z & = & x^z \cdot y^z & & & & \end{array}$$

But we may also, given a valuation $\rho : \mathcal{V} \to \mathcal{C}_0$, interpret arithmetic expressions as objects in a cartesian closed category $\mathcal{C}$, with $[\![1]\!]_\rho = 1$, $[\![x]\!]_\rho = \rho(x)$, $[\![e \cdot e']\!]_\rho = [\![e]\!]_\rho \times [\![e']\!]_\rho$, $[\![e^{e'}]\!]_\rho = [\![e']\!]_\rho \Rightarrow [\![e]\!]_\rho$.

Question 16. $(\star\star)$ Using Martin's theorem, show that for all arithmetic expressions e, e', $\mathbb{N} \vDash e = e'$ iff for all cartesian closed category $\mathcal{C}$, for all valuation $\rho : \mathcal{V} \to \mathcal{C}_0$, we have $[\![e]\!]_\rho \cong [\![e']\!]_\rho$.