Least and Greatest Fixpoints in Game Semantics

Pierre Clairambault
PPS — CNRS & Université Paris 7
pierre.clairambault@pps.jussieu.fr

1 Introduction

The idea to model logic by game-theoretic tools can be traced back to the work of Lorenzen [19]. The idea is to interpret a formula by a game between two players O and P, O trying to refute the formula and P trying to prove it. The formula A is then valid if P has a *winning strategy* on the interpretation of A. Later, Joyal remarked [18] that it is possible to compose strategies in Conway games [10] in an associative way, thus giving rise to the first category of games and strategies. This, along with parallel developments in Linear Logic and Geometry of Interaction, led to the more recent construction of compositional game models for a large variety of logics [1, 21, 11] and programming languages [17, 3, 20, 2].

On the other hand, games with parity conditions [4] have been used accurately in order to model languages such as the propositional μ-calculus [22]. The idea is to build a μ-bicomplete category of games, *i.e.* a category with finite products, finite coproducts and initial algebras/terminal coalgebras for all functors definable with products, coproducts and parametrized initial/terminal algebras/coalgebras. However, this category of games is a bit unsatisfactory from the point of view of proof theory since it is not *closed*, *i.e.* it does not admit an interpretation of functional types or implication.

2 On previous work

In a previous paper [6], we presented a logic named μLJ which is an extension of both the propositional μ-calculus and of the intuitionistic sequent LJ[15]. Variants of this language have already been considered, see for example [5]. Basically, it consists of the usual rules of LJ plus the following rules to express fixpoints, and the action of formulas with a free type variable as covariant (denoted T) or contravariant (denoted N) endofunctors.

Rules for Fixpoints and Functors

$$\frac{\Gamma \vdash T[\mu X.T/X]}{\Gamma \vdash \mu X.T}\ \mu_r \qquad \frac{T[A/X] \vdash A}{\mu X.T \vdash A}\ \mu_l$$

$$\frac{T[\nu X.T/X] \vdash B}{\nu X.T \vdash B}\ \nu_l \qquad \frac{A \vdash T[A/X]}{A \vdash \nu X.T}\ \nu_r$$

$$\frac{A \vdash B}{T(A) \vdash T(B)}\ [T] \qquad \frac{B \vdash A}{N(A) \vdash N(B)}\ [N]$$

This logic is then equipped with the usual reduction rules of LJ, with the addition of the following rules for fixpoints, and rules for the expansion of functors.

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash T[\mu X.T/X]}}{\Gamma \vdash \mu X.T}\,\mu_r \qquad \dfrac{\dfrac{\pi_2}{T[A/X] \vdash A}}{\mu X.T \vdash A}\,\mu_l}{\Gamma \vdash A}\,Cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash T[\mu X.T/X]} \qquad \dfrac{\dfrac{\dfrac{\pi_2}{T[A/X] \vdash A}}{\mu X.T \vdash A}\,\mu_l}{T[\mu X.T/X] \vdash T[A/X]}\,[T]}{\Gamma \vdash T[A/X]}\,Cut \qquad \dfrac{\pi_2}{\Gamma, T[A/X] \vdash A}}{\Gamma \vdash A}\,Cut$$

Figure 1: Cut reduction for μ

In [6], we show how to build a games model of this logic in the setting of *arena games* [17, 16]. We start from McCusker's model of recursive types in arena games [20] where recursive types are obtained by infinite iteration of the functors, in the spirit of Knaster-Tarski's fixed point theorem. We first revisit his work by replacing this infinite iteration process by loops in arenas. For this purpose, we introduce a general form of functors in game semantics, called *open functors*, which are semantic counterparts of formulas with free type variables. Such functors are in one-to-one correspondence with *open arenas*, *i.e.* arenas with special distinguished moves called *holes*, representing type variables. These arenas admit a *loop construction*, giving rise to a *minimal invariant* [14, 12, 13] for the corresponding functor. Moreover, we show that this loop construction can be enriched by *parity winning conditions*, providing initial algebras and terminal coalgebras for most covariant open functors. Hence, we have built a category of games which is cartesian closed, has (weak) coproducts and initial algebras/terminal coalgebras for all covariant functors definable with the language constructors (including fixpoints and implication) : this is our model for μLJ.

Whereas this version of μLJ is interesting in its own right, there remains a drawback making it unsatisfactory for a plausible programming language with induction/coinduction, namely, the absence of context in the rules for μ/ν : this restriction would correspond to a programming language where one can only iterate a closed term. We are now interested in the following extended rules:

Extended Rules for Fixpoints and Functors

$$\dfrac{\Gamma \vdash T[\mu X.T/X]}{\Gamma \vdash \mu X.T}\,\mu_r \qquad \dfrac{\Gamma, T[A/X] \vdash A}{\Gamma, \mu X.T \vdash A}\,\mu_l$$

$$\dfrac{\Gamma, T[\nu X.T/X] \vdash B}{\Gamma, \nu X.T \vdash B}\,\nu_l \qquad \dfrac{\Gamma, A \vdash T[A/X]}{\Gamma, A \vdash \nu X.T}\,\nu_r$$

$$\dfrac{\Gamma, A \vdash B}{\Gamma, T(A) \vdash T(B)}\,[T] \qquad \dfrac{\Gamma, B \vdash A}{\Gamma, N(A) \vdash N(B)}\,[N]$$

While it is true that, as claimed in [6], these general rules can be derived from the previous ones, it is unclear and non-trivial whether these derivations are correct from the dynamical point of view. For example, one would have to show that the reduction presented in Figure 2 holds. Unfortunately, due to the complexity of the derivations for the extended rules, the required verifications turn out to be unfeasible, at least by hand. Hence, we instead take these extended rules and reductions as primitive, and investigate possible strengthenings of our games model which could validate them.

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash T[\mu X.T/X]}}{\Gamma \vdash \mu X.T}\,\mu_r \qquad \dfrac{\dfrac{\pi_2}{\Gamma, T[A/X] \vdash A}}{\Gamma, \mu X.T \vdash A}\,\mu_l}{\Gamma \vdash A}\,Cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash T[\mu X.T/X]} \qquad \dfrac{\dfrac{\dfrac{\pi_2}{\Gamma, T[A/X] \vdash A}}{\Gamma, \mu X.T \vdash A}\,\mu_l}{\Gamma, T[\mu X.T/X] \vdash T[A/X]}\,[T]}{\Gamma \vdash T[A/X]}\,Cut \qquad \dfrac{\pi_2}{\Gamma, T[A/X] \vdash A}}{\Gamma \vdash A}\,Cut$$

Figure 2: Extended cut reduction for μ

3 A categorical setting of strong functors

We first investigate what categorical structure is needed in order to interpret the extended rules. Let us suppose given a cartesian closed category $\mathscr{C}$, with (weak) coproducts. In this section, we will be interested in the notion of *strong endofunctors*. These are defined as functors $T : \mathscr{C} \to \mathscr{C}$ equipped with a *strength*, *i.e.* a transformation

$$\theta^T_{\Gamma,A} : \Gamma \times T(A) \to T(\Gamma \times A)$$

natural in Γ and A, and satisfying unarity and associativity constraints. Such strong functors have already been considered in the past, see for example [7, 8, 9]. The difference here is the presence of functional types, which forces us to consider a notion dual to strengths, that we call *contravariant strengths*. A contravariant functor $N : \mathscr{C}^{op} \to \mathscr{C}$ is *strong* if there is a transformation:

$$\rho^N_{\Gamma,A} : \Gamma \times N(\Gamma \times A) \to N(A)$$

which is natural in A, dinatural in Γ and satisfies the following unarity and associativity constraints:

$$\begin{array}{ccc}
1 \times N(A) & \xrightarrow{\pi_2} & N(A) \\
{\scriptstyle 1\times N(\pi_2)}\downarrow & & \parallel \\
1 \times N(1 \times A) & \xrightarrow[\rho^N_{1,A}]{} & N(A)
\end{array}$$

$$\begin{array}{ccc}
(B \times A) \times N((A \times B) \times C) & \xrightarrow{\alpha_{B,A,N((A\times B)\times C)}} & B \times (A \times N((A \times B) \times C)) \\
{\scriptstyle s_{A,B}\times N((A\times B)\times C)}\uparrow & & \downarrow{\scriptstyle B\times(A\times N(\alpha^{-1}_{A,B,C}))} \\
(A \times B) \times N((A \times B) \times C) & & B \times (A \times N(A \times (B \times C))) \\
{\scriptstyle \rho^N_{A\times B,C}}\searrow & & \downarrow{\scriptstyle B\times\rho^N_{A,B\times C}} \\
 & & B \times N(B \times C) \\
 & N(C) & \swarrow{\scriptstyle \rho^N_{B,C}}
\end{array}$$

Let us now take any object Γ of $\mathscr{C}$, and consider the comonad $\Gamma \times -$. It gives rise to a co-Kleisli category denoted $\mathscr{C}_\Gamma$, and corresponds to the category of morphisms (terms) in the context Γ. The main interest of strong (co/contra)-variant functors is that they can be extended to $\mathscr{C}_\Gamma$ in the following way:

Proposition 1. *Let $T : \mathscr{C} \to \mathscr{C}$ and $N : \mathscr{C}^{op} \to \mathscr{C}$ be strong functors. If we define:*

- *On objects, $T_\Gamma(A) = T(A)$. On morphisms, if $f : A \to B$ (in $\mathscr{C}_\Gamma$, hence $f : \Gamma \times A \to B$ in $\mathscr{C}$):*

$$T_\Gamma(f) = \Gamma \times T(A) \xrightarrow{\theta^T_{\Gamma,A}} T(\Gamma \times A) \xrightarrow{T(f)} T(B)$$

- *On objects, $N_\Gamma(A) = N(A)$. On morphisms, if $f : B \to A$ (in $\mathscr{C}_\Gamma$),*

$$N_\Gamma(f) = \Gamma \times N(A) \xrightarrow{\Gamma \times N(f)} \Gamma \times N(\Gamma \times B) \xrightarrow{\rho^N_{\Gamma,B}} N(B)$$

Then $T_\Gamma : \mathscr{C}_\Gamma \to \mathscr{C}_\Gamma$ and $N_\Gamma : \mathscr{C}_\Gamma^{op} \to \mathscr{C}_\Gamma$ are well-defined functors.

Moreover, it can be proved that this functor extension operation preserves the existence of initial algebras/terminal coalgebras. More precisely, we can prove the following proposition:

PROPOSITION 2. *Let $T : \mathscr{C} \to \mathscr{C}$ be strong and Γ be an object of $\mathscr{C}$, then we have the following propositions:*

- *If T has an initial algebra in $\mathscr{C}$, then T_Γ has an initial algebra in $\mathscr{C}_\Gamma$.*
- *If T has a terminal coalgebra in $\mathscr{C}$, then T_Γ has a terminal coalgebra in $\mathscr{C}_\Gamma$.*

While this structure already allows to interpret most of the extended rules of μLJ, there are still some gaps. In particular, what guarantees that the behaviour of T is not modified when building T_Γ ? More precisely, the $(-)_\Gamma$ construction has to satisfy a certain number of equations like $(T \times T')_\Gamma = T_\Gamma \times T'_\Gamma$. These equations reduce to properties of strengths, which lead to the following definition.

DEFINITION 1. *A category $\mathscr{C}$ has* strong types *if it is cartesian closed, has (weak, functorial) coproducts, a (weak) initial object, and is equipped with a class $\mathscr{F}$ of functors $T : \mathscr{C}^k \times (\mathscr{C}^{op})^p \to \mathscr{C}$ satisfying the following properties:*

- *$\mathscr{F}$ contains the identity, constant functors and base constructors $-_1 + -_2$, $-_1 \times -_2$ and $-_1 \Rightarrow -_2$;*
- *$\mathscr{F}$ is stable by composition : if $F, G \in \mathscr{F}$ and F, G are composable, then $FG \in \mathscr{F}$;*
- *$\mathscr{F}$ is stable by* contraction *: if $F(-_1, -_2, -_3) : \mathscr{C} \times \mathscr{C} \times \mathscr{D} \to \mathscr{C}$ is in $\mathscr{F}$, then $F(-_1, -_1, -_3) : \mathscr{C} \times \mathscr{D} \to \mathscr{C}$. Same condition with $\mathscr{C}^{op}$ instead of $\mathscr{C}$ at the left.*

Those functors in $\mathscr{F}$ that are also unary (i.e. *$P : \mathscr{C} \to \mathscr{C}$ or $N : \mathscr{C}^{op} \to \mathscr{C}$) are* strong. *The strengths have to satisfy the following conditions:*

- *For both covariant and contravariant strengths, the families $\theta^T_{\Gamma,A}$ and $\rho^N_{\Gamma,A}$ are also* natural in *T/N;*
- *Compatibility with identity:*

$$\theta^-_{\Gamma,A} = id_{\Gamma \times A}$$

- *Compatibility with constant functors:*

$$\theta^B_{\Gamma,A} = \pi_2$$

- *Compatibility with composition:*

$$\begin{array}{lrcl} F \text{ and } G \text{ covariant:} & \theta^{FG}_{\Gamma,A} & = & \theta^{F}_{\Gamma,G(A)}; F(\theta^{G}_{\Gamma,A}) \\ F \text{ covariant and } G \text{ contravariant:} & \rho^{FG}_{\Gamma,A} & = & \theta^{F}_{\Gamma,G(\Gamma\times A)}; F(\rho^{G}_{\Gamma,A}) \\ F \text{ contravariant and } G \text{ covariant:} & \rho^{FG}_{\Gamma,A} & = & \Gamma\times F(\theta^{G}_{\Gamma,A}); \rho^{F}_{\Gamma,G(A)} \\ F \text{ and } G \text{ contravariant:} & \theta^{FG}_{\Gamma,A} & = & \Gamma\times F(\rho^{G}_{\Gamma,A}); \rho^{F}_{\Gamma,G(\Gamma\times A)} \end{array}$$

- *Compatibility with contraction, with P covariant and N contravariant:*

$$\begin{array}{rcl} \theta^{P(-,-)}_{\Gamma,A} & = & \langle \pi_1, \theta^{P(-,A)}_{\Gamma,A}\rangle; \theta^{P(\Gamma\times A,-)}_{\Gamma,A} \\ \rho^{N(-,-)}_{\Gamma,A} & = & \langle \pi_1, \rho^{N(-,\Gamma\times A)}\rangle; \rho^{N(A,-)}_{\Gamma,A} \end{array}$$

- *Compatibility with cartesian closed structure.*

$$\begin{array}{rcl} \rho^{-\Rightarrow C}_{\Gamma,A} & = & \Lambda(\langle\langle \pi_2;\pi_1,\pi_1\rangle, \pi_2;\pi_2\rangle; ev) \\ \theta^{C\Rightarrow -}_{\Gamma,A} & = & \Lambda(\langle \pi_2;\pi_1, \langle \pi_1, \pi_2;\pi_2\rangle; ev\rangle) \end{array}$$

In a category with strong types, all the equations required for the expansion of functor rules hold:

PROPOSITION 3. *For any $P, P' : \mathscr{C} \to \mathscr{C}$, $N, N' : \mathscr{C}^{op} \to \mathscr{C}$, Γ, the following equations hold:*

$$\begin{array}{rcl} (P+P')_\Gamma & = & P_\Gamma + P'_\Gamma \\ (N+N')_\Gamma & = & N_\Gamma + N'_\Gamma \\ (P\times P')_\Gamma & = & P_\Gamma \times P'_\Gamma \\ (N\times N')_\Gamma & = & N_\Gamma \times N'_\Gamma \\ (N\Rightarrow P)_\Gamma & = & N_\Gamma \Rightarrow P_\Gamma \\ (P\Rightarrow N)_\Gamma & = & P_\Gamma \Rightarrow N_\Gamma \end{array}$$

Moreover, if $T : \mathscr{C}\times\mathscr{D} \to \mathscr{C}$ in $\mathscr{F}$ has a parametrized initial algebra or a parametrized terminal coalgebra T^μ/T^ν :

$$\begin{array}{rcl} (T^\mu)_\Gamma & = & (T_\Gamma)^\mu \\ (T^\nu)_\Gamma & = & (T_\Gamma)^\nu \end{array}$$

4 μ-closed categories

Now that we have all the necessary background to interpret the extended rules for functors and their expansion, we can turn to the rules for fixpoints. We define *μ-closed categories* by analogy with the definition of μ-bicomplete categories [23], as those categories with strong types where the canonical interpretation of μLJ formulas as strong functors is total.

DEFINITION 2. *Let $\mathscr{C}$ be a category with strong types. We define a partial interpretation of μLJ formulas as strong functors in the class $\mathscr{F}$ as follows:*

- $[\![0]\!] = X \mapsto 0$ *(the constant functor on the (weak) initial object);*
- $[\![1]\!] = X \mapsto 1$ *(the constant functor on the terminal object);*

- $[\![X]\!] = X \mapsto X$ *(the identity functor);*
- $[\![S \Rightarrow T]\!] = [\![S]\!] \Rightarrow [\![T]\!]$*;*
- $[\![S + T]\!] = [\![S]\!] + [\![T]\!]$*;*
- $[\![S \times T]\!] = [\![S]\!] \times [\![T]\!]$
- $[\![\mu X.T]\!]$ *is, if defined, the parametrized initial algebra of* $[\![T]\!](X, \overrightarrow{Y})$*;*
- $[\![\nu X.T]\!]$ *is, if defined, the parametrized terminal coalgebra of* $[\![T]\!](X, \overrightarrow{Y})$.

In both cases, the notation $\overrightarrow{Y}$ *expresses the fact that, since* T *can have other free type variables,* $[\![T]\!]$ *can admit other arguments than* X.

DEFINITION 3. *Let* $\mathscr{C}$ *a category with strong types.* $\mathscr{C}$ *is* μ-closed *if the interpretation function* $[\![-]\!]$ *is total on* $\mathscr{F}$.

THEOREM 1. *Any* μ*-closed category is a sound model for* μLJ *with the extended rules set.*

5 The Games model

Let $\mathscr{I}$ denote the usual category of arenas and innocent strategies (see for example [20]), and $\mathscr{G}$ denote the category of games for fixpoints introduced in [6]. The extension of $\mathscr{I}$ and $\mathscr{G}$ to take the structure presented here into account goes rather smoothly, in several steps.

PROPOSITION 4. $\mathscr{I}$ *has strong types, with* open functors *as the needed class of functors.*

Now, this strong types structure extends naturally to $\mathscr{G}$ by the addition of parity winning conditions. But we already know that in $\mathscr{G}$, open functors definable by the base constructors have initial algebras and terminal coalgebras, hence we get the following theorem.

THEOREM 2. $\mathscr{G}$ *is* μ*-closed.*

6 Conclusion

This work presents a categorical setting in which it is possible to deal with rules for fixpoints and functors under a given context Γ. This is important, since any plausible programming language with induction and/or coinduction will allow iteration of a functional with free variables. We also show how the games model of [6] can be adapted to this extended setting, thus giving a model of the extended rules of μLJ, with games and winning total strategies.

An interesting open question is whether the definition of μ-closed category is actually redundant, *i.e.* whether the derivations of the extended rules in μLJ which are known to exist behave as needed, from the dynamical point of view. More precisely, it is possible to define strength candidates for all functors built out of the base constructors, but proving that these strength candidates satisfy the required equations turned out to be unfeasible by hand, most notably for the case of strengths for functors already generated by fixpoints (necessary for the modelling of *interleaving* inductive/coinductive types, *i.e.* embedded fixpoints within the definition of fixpoints). We note however that such verifications could be at least partially automatized, thus leading to a rather strong theorem allowing to interpret the extended rules of μLJ in any model of the basic rules.

References

[1] S. Abramsky and R. Jagadeesan. Games and full completeness for multiplicative linear logic. *J. Symb. Log.*, 59(2):543–574, 1994.

[2] Samson Abramsky, Kohei Honda, and Guy McCusker. A fully abstract game semantics for general references. In *LICS*, pages 334–344, 1998.

[3] Samson Abramsky, Radha Jagadeesan, and Pasquale Malacaria. Full Abstraction for PCF. *Inf. Comput.*, 163(2):409–470, 2000.

[4] A. Arnold and D. Niwinski. Rudiments of μ-calculus, 2001.

[5] David Baelde and Dale Miller. Least and greatest fixed points in linear logic. In Nachum Dershowitz and Andrei Voronkov, editors, *LPAR*, volume 4790 of *Lecture Notes in Computer Science*, pages 92–106. Springer, 2007.

[6] Pierre Clairambault. Least and greatest fixpoints in game semantics. In Luca de Alfaro, editor, *FOSSACS*, volume 5504 of *Lecture Notes in Computer Science*, pages 16–31. Springer, 2009.

[7] J. Robin B. Cockett and Dwight Spencer. Strong Categorical Datatypes II: A Term Logic for Categorical Programming. *Theor. Comput. Sci.*, 139(1&2):69–113, 1995.

[8] J.R.B. Cockett and D. Spencer. Strong categorical datatypes I. In *Category theory 1991: proceedings of an International Summer Category Theory Meeting, held June 23-30, 1991*, page 141. Canadian Mathematical Society, 1992.

[9] Robin Cockett and Tom Fukushima. About charity. Technical report, 1992.

[10] J.H. Conway. *On Numbers and Games*. AK Peters, Ltd., 2001.

[11] J. De Lataillade. Second-order type isomorphisms through game semantics. *Ann. Pure Appl. Logic*, 151(2-3):115–150, 2008.

[12] P. Freyd. Algebraically complete categories. In *Proc. 1990 Como Category Theory Conference*, volume 1488, pages 95–104. Springer, 1990.

[13] P. Freyd. Remarks on algebraically compact categories. In *Applications of Categories in Computer Science. Proceedings of the LMS Symposium, Durham*, volume 177, pages 95–106, 1991.

[14] Peter J. Freyd. Recursive Types Reduced to Inductive Types. In *LICS*, pages 498–507. IEEE Computer Society, 1990.

[15] J.Y. Girard, Y. Lafont, and P. Taylor. *Proofs and types*. Cambridge University Press New York, 1989.

[16] R. Harmer. Innocent game semantics. *Lecture notes*, 2004.

[17] J.M.E. Hyland and C.H.L. Ong. On Full Abstraction for PCF: I, II, and III. *Inf. Comput.*, 163(2):285–408, 2000.

[18] A. Joyal. Remarques sur la théorie des jeux à deux personnes. *Gaz. Sc. Math. Qu.*, 1977.

[19] P. Lorenzen. Logik und Agon. *Atti Congr. Internat. di Filosofia*, 1960.

[20] G. McCusker. Games and Full Abstraction for FPC. *Inf. Comput.*, 160(1-2):1–61, 2000.

[21] Paul-André Melliès. Asynchronous Games 4: A Fully Complete Model of Propositional Linear Logic. In *LICS*, pages 386–395. IEEE Computer Society, 2005.

[22] Luigi Santocanale. A Calculus of Circular Proofs and Its Categorical Semantics. In Mogens Nielsen and Uffe Engberg, editors, *FoSSaCS*, volume 2303 of *Lecture Notes in Computer Science*, pages 357–371. Springer, 2002.

[23] Luigi Santocanale. μ-bicomplete categories and parity games. *ITA*, 36(2):195–227, 2002.