A remark on Jim Laird's games category for general references

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Abstract

In [Lai02], Jim Laird introduces a category of games as a concrete *sequoidal category*, which is a categorical model of general references. We begin by giving a non-negative version of this category. Then we show that there is a (strong monoidal) equivalence of categories between this category of games, and the category $\mathcal{C}$ of Conway games, which can be transported to the corresponding negative situation.

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1 Definitions

1.1 Laird's category of games

We give here the definitions of [Lai02], with some changes of formulation which makes the definitions easier to manipulate for our purpose.

Definition 1. *A game A is a tuple $< M_A, \lambda_A, S_A, \rhd_A, \lhd_A >$ where:*

- *M_A is a set of moves*
- *$\lambda_A : M_A \to \{O, P\}$ is Player/Opponent labelling (by convention, $\overline{O} = P$, $\overline{P} = O$),*
- *$S_A \subseteq M_A$ is a set of Opponent moves – the* starting *moves,*
- *$\rhd_A \subseteq M_A \times M_A$ is a binary relation called* triggering *($m \rhd n$ means m is a trigger for n),*
- *$\lhd_A \subseteq M_A \times M_A$ is a binary relation on moves called* blocking.

Notation. Given a game $A = (M_A, \lambda_A, S_A, \rhd_A, \lhd_A)$, we generalize λ_A, S_A, $\rhd_A$ and $\lhd_A$ to unary predicates on M_A^* to express the compatibility of a sequence with these relations. The abuse of notation is justified by the fact that whether we use λ_A, S_A, $\rhd_A$ and $\lhd_A$ as in their initial definition or as unary relations is always clear from the context.
Let the variables a, b range over moves and s, t range over sequences. Let u be a sequence, then we define compatibility of u with respect to:

- **Initialization.** The first move of u belongs to S_A:

$$S_A(u) \Leftrightarrow (\forall s, a \;\; as \sqsubseteq u \Rightarrow a \in S_A)$$

- **Alternation.** Player and Opponent move alternate in u:

$$\lambda_A(u) \Leftrightarrow (\forall s, a, b \;\; sab \sqsubseteq u \Rightarrow \lambda_A(a) = \overline{\lambda_A(b)})$$

- **Triggering.** Every non-starting move in u is preceded by at least one trigger:

$$\rhd_A(u) \Leftrightarrow (\forall t, b \;\; tb \sqsubseteq u \wedge b \notin S_A \Rightarrow \exists sa \sqsubset {}^1 tb. \;\; a \rhd_A b)$$

- **Blocking.** No move in u is preceded by a move which can block it:

$$\lhd_A(u) \Leftrightarrow (\forall t, b \;\; tb \sqsubseteq u \Rightarrow \forall sa \sqsubseteq t. \;\; \neg(a \lhd_A b))$$

Definition 2. *For any game A, the set L_A of legal sequences is the (automatically prefix closed) set of sequences satisfying compatibility with alternation, triggering and blocking:*

$$L_A = \{u \in M_A^* \mid \lambda_A(u) \;\wedge\; \rhd_A(u) \;\wedge\; \lhd_A(u)\}$$

Remark 1. *In his paper, Laird only considers games where every move is self-blocking, which entails that no legal sequence contains repeated moves. This does not not harm the encoding of Conway games, since we won't use any repeated move, even to encode non-linear Conway games. We'll discuss this later.*

Now, let's define $\bullet$, $\multimap$ and the cartesian product &:

Definition 3. *Given games A, B, form:*

- *$A \bullet B = (M_{A\bullet B}, \lambda_{A\bullet B}, S_{A\bullet B}, \rhd_{A\bullet B}, \lhd_{A\bullet B})$, where:*

[1] $\sqsubseteq$ has been replaced by $\sqsubset$. A typo in Laird's paper? A pretty bad one then, since it allows strategies with a "non-initial" but self-triggering opening Player move, which cause associativity to fail. . .

- $M_{A \bullet B} = M_A + M_B$
- $\lambda_{A \bullet B} = [\lambda_A, \lambda_B]$
- $S_{A \bullet B} = S_A + S_B$
- $\rhd_{A \bullet B} = in_l(\rhd_A) \cup in_r(\rhd_B)$
- $\lhd_{A \bullet B} = in_l(\lhd_A) \cup in_r(\lhd_B)$

In other terms, $A \bullet B$ inherits its labelling, triggering and blocking from A and B, so a legal sequence in $A \bullet B$ consists of interleaved sequences in A and B (not necessary legal, they can fail the alternation property).

- $A \multimap\!\bullet\; B = (M_{A \multimap\bullet B}, \lambda_{A \multimap\bullet B}, S_{A \multimap\bullet B}, \rhd_{A \multimap\bullet B}, \lhd_{A \multimap\bullet B})$, *where:*
  - $M_{A \multimap\bullet B} = M_A + M_B$
  - $\lambda_{A \multimap\bullet B} = [\overline{\lambda_A}, \lambda_B]$
  - $S_{A \multimap\bullet B} = S_B$
  - $\rhd_{A \multimap\bullet B} = in_l(\rhd_A) \cup in_r(\rhd_B) \cup (in_r(S_B) \times in_l(S_A))$
  - $\lhd_{A \multimap\bullet B} = in_l(\lhd_A) \cup in_r(\lhd_B)$

  $A \multimap\bullet B$ is similar, except that polarities in A are interchanged, and the only starting moves are those from B, which become triggers for the starting moves of A.

- $A \& B = (M_{A\&B}, \lambda_{A\&B}, S_{A\&B}, \rhd_{A\&B}, \lhd_{A\&B})$, *where:*
  - $M_{A\&B} = M_A + M_B$
  - $\lambda_{A\&B} = [\lambda_A, \lambda_B]$
  - $S_{A\&B} = S_A + S_B$
  - $\rhd_{A\&B} = in_l(\rhd_A) \cup in_r(\rhd_B)$
  - $\lhd_{A\&B} = in_l(\lhd_A) \cup in_r(\lhd_B) \cup (in_l(M_A) \times in_r(M_B)) \cup (in_r(M_B) \times in_l(M_A))$

  In the additive product $A\&B$, starting moves in A are blocked by starting moves from B and vice-versa – thus a legal sequence in $A\&B$ is a sequence from A, or from B.

Now, we define strategies:

Definition 4. *The **strategies** on a game A are the even-prefix closed and evenly-branching sets sets of even-length legal sequences over A.*

Composition of strategies by "parallel composition plus hiding" follows the standard definition.

Definition 5. *From strategies $\sigma : A \multimap\bullet B$ and $\tau : B \multimap\bullet C$, we form $\sigma;\tau : A \multimap\bullet C$:*

$$\sigma;\tau = \{s \in L_{A \multimap\bullet C} | \exists u \in (M_A + M_B + M_C)^* \;\; s = u_{\upharpoonright_{A,C}} \;\wedge\; u_{\upharpoonright_{A,B}} \in \sigma \;\wedge\; u_{\upharpoonright_{B,C}} \in \tau\}$$

Thus we can form a category $\mathscr{L}$ in which the objects are games and the morphisms from A to B are strategies on $A \multimap\bullet B$.

Proposition 1. *$(\mathscr{L}, I, \bullet, \multimap\bullet)$ is a symmetric monoidal closed category*

1.2 Generalization to non-negative Laird-like games

We generalize Laird's definition of games to a compact closed category of non-negative games, in a way that is strictly similar to the relationship between negative Conway games and general Conway games.

Definition 6. *A game A is a tuple $< M_A, \lambda_A, \rhd_A, \lhd_A >$ where:*

- *M_A is a set of moves*
- *$\lambda_A : M_A \to \{O, P\}$ is Player/Opponent labelling (by convention, $\overline{O} = P$, $\overline{P} = O$),*
- *$\rhd_A \subseteq M_A \times M_A$ is a binary relation called* triggering *($m \rhd n$ means m is a trigger for n),*
- *$\lhd_A \subseteq M_A \times M_A$ is a binary relation on moves called* blocking.

Remark 2. *Following the previous construction, we need unary predicates λ_A, $\rhd_A$ and $\lhd_A$ to ease the definitions. They're almost suitable natively for our purpose, but one should however be careful about $\rhd_A$: it has been defined in terms of S_A, which does not exist here. The definition of $\rhd_A$ can be fixed in two independent ways:*

- *Ignore S_A and allow efficiently self-triggering moves:*

$$\rhd_A(u) \Leftrightarrow (\forall t, b \;\; tb \sqsubseteq u \Rightarrow \exists sa \sqsubseteq tb. \;\; a \rhd_A b)$$

- *Keep the previous definition, with S_A defined as self-triggering moves:*

$$S_A = \{x \in M_A \mid x \rhd_A x\}$$

These two formulations are equivalent, thus there's no harm in choosing the first one which is easier to manipulate.

Definition 7. *For any game A, the set L_A of legal sequences is the (automatically prefix closed) set of Opponent-beginning sequences satisfying compatibility with alternation, triggering and blocking. Let $O(u)$ be the unary predicate stating that u begins by an Opponent move, then :*

$$L_A = \{u \in M_A^* \mid \lambda_A(u) \;\wedge\; \rhd_A(u) \;\wedge\; \lhd_A(u) \;\wedge\; O(u)\}$$

Now, let's define $\bullet$, the dual $\_^*$ and $\multimap$:

Definition 8. *Given games A, B, form:*

- *$A \bullet B = (M_{A\bullet B}, \lambda_{A\bullet B}, \rhd_{A\bullet B}, \lhd_{A\bullet B})$, where:*
  - $M_{A\bullet B} = M_A + M_B$
  - $\lambda_{A\bullet B} = [\lambda_A, \lambda_B]$
  - $\rhd_{A\bullet B} = in_l(\rhd_A) \cup in_r(\rhd_B)$
  - $\lhd_{A\bullet B} = in_l(\lhd_A) \cup in_r(\lhd_B)$

  In other terms, $A \bullet B$ inherits its labelling, triggering and blocking from A and B, so a legal sequence in $A \bullet B$ consists of interleaved sequences in A and B (not necessary legal, they can fail the alternation property).

- *$A^* = (M_{A^*}, \lambda_{A^*}, \rhd_{A^*}, \lhd_{A^*})$, where*
  - $M_{A^*} = M_A$
  - $\lambda_{A^*} = \overline{\lambda_A}$
  - $\rhd_{A^*} = \rhd_A$
  - $\lhd_{A^*} = \lhd_A$

A^* *is just* A*, where the polarities O/P have been inverted.*

- $A \multimap\!\bullet\ B = A^* \bullet B$

Remark 3. *Note that in this more general setting, the definability of the dual* A^* *for each game* A *allows to define the closure* $A \multimap\!\bullet\ B$ *as* $A^* \bullet B$*. This is why the category of non-negative games will be compact closed.*

Definition 9. *The* ***strategies*** *on a game A are the even-prefix closed and evenly-branching sets sets of even-length legal sequences over A.*

Definition 10. *From strategies* $\sigma : A \multimap\!\bullet\ B$ *and* $\tau : B \multimap\!\bullet\ C$*, we form* $\sigma;\tau : A \multimap\!\bullet\ C$ *:*

$$\sigma;\tau = \{s \in L_{A \multimap\bullet C} | \exists u \in (M_A + M_B + M_C)^* \ \ s = u_{\restriction_{A,C}} \ \wedge \ u_{\restriction_{A,B}} \in \sigma \ \wedge \ u_{\restriction_{B,C}} \in \tau\}$$

Proposition 2. $(\mathcal{L}, I, \bullet, \multimap\!\bullet)$ *is a compact closed category.*

1.3 Definition of Conway Games

We recall here the definition of Conway games. It follows the lines of the definition given by Conway [Con01] to build the field of surreal numbers, and used thereafter by Joyal [Joy] to build a compact closed category of games. A slight difference is that the present formulation do not requires the games to be well-founded.

Definition 11. *A game A is a tuple* $< M_A, \lambda_A, P_A >$ *where:*

- M_A *is a set of* moves,
- $\lambda_A : M_A \to \{O, P\}$ *is Player/Opponent labelling (by convention,* $\overline{O} = P$ *and* $\overline{P} = O$*),*
- $P_A \subset M_A^*$ *is a prefix-closed set of* plays.

Definition 12. *For any game A, the set of legal sequences* L_A *is the set of Opponent-beginning elements of* P_A *which satisfies the* ***alternation*** *condition : Player and Opponent move alternatively.*

$$L_A = \{u \in P_A \mid \lambda_A(u) \ \wedge \ O(u)\}$$

Definition 13. *Given games A and B, form:*

- $A \otimes B = (M_{A\otimes B}, \lambda_{A\otimes B}, P_{A\otimes B})$*, where :*
  - $M_{A\otimes B} = M_A + M_B$
  - $\lambda_{A\otimes B} = [\lambda_A, \lambda_B]$
  - $P_{A\otimes B} = \{s \in (M_A + M_B)^* \ \mid \ s_{\restriction_{M_A}} \in P_A \ \ \wedge \ \ s_{\restriction_{M_B}} \in P_B\}$
- $A^* = (M_{A^*}, \lambda_{A^*}, P_{A^*})$*, where:*
  - $M_{A^*} = M_A$
  - $\lambda_{A^*} = \overline{\lambda_A}$
  - $P_{A^*} = P_A$
- $A \multimap B = A^* \otimes B$

Definition 14. *The* ***strategies*** *on a game A are even-prefix closed, evenly branching sets of even-length elements of* L_A*.*

Definition 15. *From strategies* $\sigma : A \multimap B$ *and* $\tau : B \multimap C$*, we form* $\sigma;\tau : A \multimap C$*:*

$$\sigma;\tau = \{s \in L_{A \multimap\bullet C} | \exists u \in (M_A + M_B + M_C)^* \ \ s = u_{\restriction_{A,C}} \ \wedge \ u_{\restriction_{A,B}} \in \sigma \ \wedge \ u_{\restriction_{B,C}} \in \tau\}$$

Proposition 3. $(\mathcal{C}, I, \otimes, \multimap)$ *is a compact closed category.*

2 Equivalence of $\mathcal{L}$ and C

The equivalence we're going to prove may seem surprising at first glance, but it's really natural when one thinks of the structure of games in a Laird-like category as a way to describe the tree of plays in another way.

2.1 Functor U from $\mathcal{L}$ to C

We call U the functor from $\mathcal{L}$ to C because this functor really *is* a forgetful functor : it associates with each triggering/blocking combination the tree of possible plays generated. It will remain to check to which extent the structural constructs in $\mathcal{L}$ and in $U(\mathcal{L})$ coincide.

Image of objects. Let $A = (M_A, \lambda_A, \rhd_A, \lhd_A)$ be a game of $\mathcal{L}$. Then :

- $M_{U(A)} = M_A$
- $\lambda_{U(A)} = \lambda_A$
- $P_{U(A)} = \{u \in M_A^* \mid \rhd_A(u) \wedge \lhd_A(u)\}$

Image of arrows. Let $\sigma : A \multimap\!\!\bullet\, B$ be a strategy in $\mathcal{L}$. We prove that without any syntactical change, σ is already a strategy in $U(A) \multimap U(B)$. This boils down to the two followings lemmas.

Lemma 1. *Let A, B be Laird games, let $u \in (M_A + M_B)^*$, then*

(i) $\rhd_{A\bullet B}(u) \Leftrightarrow \rhd_A(u_{\restriction_A}) \wedge \rhd_B(u_{\restriction_B})$

(ii) $\lhd_{A\bullet B}(u) \Leftrightarrow \lhd_A(u_{\restriction_A}) \wedge \lhd_B(u_{\restriction_B})$

Proof of (i).

$$\begin{array}{rcl}
\rhd_{A\bullet B}(u) & \Leftrightarrow & \forall t, b \;\; tb \sqsubseteq u \Rightarrow \exists sa \sqsubseteq tb \;\; a \rhd_{A\bullet B} b \\
& \Leftrightarrow & \forall t, b \;\; tb \sqsubseteq u \Rightarrow \exists sa \sqsubseteq tb \;\; (a \rhd_A b) \vee (a \rhd_B b) \\
& \Leftrightarrow & \left\{ \wedge \begin{array}{l} \forall t, b \;\; tb \sqsubseteq u_{\restriction_A} \Rightarrow \exists sa \sqsubseteq tb. \;\; a \rhd_A b \\ \forall t, b \;\; tb \sqsubseteq u_{\restriction_B} \Rightarrow \exists sa \sqsubseteq tb. \;\; a \rhd_B b \end{array} \right. \\
& \Leftrightarrow & \rhd_A(u_{\restriction_A}) \wedge \rhd_B(u_{\restriction_B})
\end{array}$$

□

Proof of (ii).

$$\begin{array}{rcl}
\lhd_{A\bullet B}(u) & \Leftrightarrow & \forall t, b \;\; tb \sqsubseteq u \Rightarrow \forall sa \sqsubseteq tb \;\; \neg(a \lhd_{A\bullet B} b)) \\
& \Leftrightarrow & \forall t, b \;\; tb \sqsubseteq u \Rightarrow \forall sa \sqsubseteq tb \;\; \neg((a \lhd_A b) \vee (a \lhd_B b)) \\
& \Leftrightarrow & \forall t, b \;\; tb \sqsubseteq u \Rightarrow \forall sa \sqsubseteq tb \;\; (\neg(a \lhd_A b)) \wedge (\neg(a \lhd_B b)) \\
& \Leftrightarrow & \lhd_A(u_{\restriction_A}) \wedge \lhd_B(u_{\restriction_B})
\end{array}$$

□

Lemma 2. $\forall A, B \in |\mathcal{L}|, \;\; U(A \multimap\!\!\bullet\, B) = U(A) \multimap U(B)$

Proof. Three equalities to check:

$$\begin{aligned}
M_{U(A\multimap\bullet B)} &= M_{A\multimap\bullet B}\\
&= M_A + M_B\\
&= M_{U(A)} + M_{U(B)}\\
&= M_{U(A)\multimap U(B)}
\end{aligned}$$

$$\begin{aligned}
\lambda_{U(A\multimap\bullet B)} &= \lambda_{A\multimap\bullet B}\\
&= [\overline{\lambda_A}, \lambda_B]\\
&= [\overline{\lambda_{U(A)}}, \lambda_{U(B)}]\\
&= \lambda_{U(A)\multimap U(B)}
\end{aligned}$$

$$\begin{aligned}
P_{U(A\multimap\bullet B)} &= \{u \in M^*_{A\multimap\bullet B} \mid \rhd_{A\multimap\bullet B}(u) \ \wedge\ \lhd_{A\multimap\bullet B}(u)\}\\
&= \{u \in M^*_{A^*\bullet B} \mid \rhd_{A^*\bullet B}(u) \ \wedge\ \lhd_{A^*\bullet B}(u)\}\\
&= \{u \in (M_A + M_B)^* \mid \rhd_{A^*}(u_{\restriction A^*}) \ \wedge\ \rhd_B(u_{\restriction B}) \ \wedge\ \lhd_{A^*}(u_{\restriction A^*}) \ \wedge\ \lhd_B(u_{\restriction B})\}\\
&= \{u \in (M_A + M_B)^* \mid \rhd_A(u_{\restriction A}) \ \wedge\ \rhd_B(u_{\restriction B}) \ \wedge\ \lhd_A(u_{\restriction A}) \ \wedge\ \lhd_B(u_{\restriction B})\}\\
&= \{u \in (M_{U(A)} + M_{U(B)})^* \mid \rhd_A(u_{\restriction U(A)}) \ \wedge\ \lhd_A(u_{\restriction U(A)}) \ \wedge\ \rhd_B(u_{\restriction U(B)}) \ \wedge\ \lhd_B(u_{\restriction U(B)})\}\\
&= \{u \in (M_{U(A)} + M_{U(B)})^* \mid u_{\restriction U(A)} \in P_{U(A)} \ \wedge\ u_{\restriction U(B)} \in P_{U(B)}\}\\
&= P_{U(A)\multimap U(B)}
\end{aligned}$$

□

Lemma 3. *The two versions of the game have the same notion of legal play:*

$$\forall A \in |\mathcal{L}|,\ L^{\mathcal{L}}_A = L^{\mathcal{C}}_{U(A)}$$

Proof. Let $A \in |\mathcal{L}|$. Then

$$\begin{aligned}
L^{\mathcal{L}}_A &= \{u \in M^*_A \mid \lambda_A(u) \ \wedge\ \rhd_A(u) \ \wedge\ \lhd_A(u)\}\\
&= \{u \in M^*_A \mid \rhd_A(u) \ \wedge\ \lhd_A(u)\} \cap \{u \in M^*_A \mid \lambda_A(u)\}\\
&= P_{U(A)} \cap \{u \in M^*_A \mid \lambda_A(u)\}\\
&= \{u \in P_{U(A)} \mid \lambda_A(u)\}\\
&= L^{\mathcal{C}}_{U(A)}
\end{aligned}$$

□

The remaining part of the proof is straigthforward : for any $A, B \in |\mathcal{L}|$, we have

$$\begin{aligned}
L^{\mathcal{L}}_{A\multimap\bullet B} &= L^{\mathcal{C}}_{U(A\multimap B)}\\
&= L^{\mathcal{C}}_{U(A)\multimap U(B)}
\end{aligned}$$

And since in both categories strategies have been defined as even-prefix closed, evenly branching sets of even-length legal sequences, the morphisms on $A \multimap\bullet B$ and $U(A) \multimap U(B)$ are syntactically the same.

2.2 U is essentially surjective

To each game $A \in |\mathcal{C}|$, we will associate a game $\tilde{A} \in U(\mathcal{L})$ isomorphic to A, which will prove the equivalence. The trick is to define the moves of $\tilde{A}$ as sequences in A : then we'll be able to describe plays in P_A just by means of triggering and blocking on M^*_A. For M a countable set, let $M^\bullet$ denote the set of non-empty strings on M.

Definition 16.

$$\tilde{F} = \begin{cases} M_A^* \to ([M_A{}^\bullet])^* &], [\notin M_A \\ a_1 \dots a_n \mapsto \prod_{i=1}^{n} [\prod_{k=1}^{i} a_i] \end{cases}$$

which extends naturally to:

$$\tilde{F} = \begin{cases} \mathcal{P}(M_A^*) \to \mathcal{P}(([M_A{}^\bullet])^*) \\ S \mapsto \{\tilde{F}(s) \mid s \in S\} \end{cases}$$

Definition 17. *We define $\tilde{A}$ by:*

- $M_{\tilde{A}} = [M_A{}^\bullet]$
- $\lambda_{\tilde{A}} = ([sa] \mapsto \lambda_A(a))$
- $P_{\tilde{A}} = \tilde{F}(P_A)$

Lemma 4. *Let A be a game in $\mathcal{C}$, then A is isomorphic to $\tilde{A}$.*

Proof. The isomorphism is just a twisted form of the copycat strategy. We first define $\eta_A : A \multimap \tilde{A}$ as all the legal sequences of the form $a[a]b[ab][abc]c\dots$ or more formally:

$$\eta_A = \{s \in L_{A \multimap \tilde{A}} \mid \forall t \sqsubseteq^P s,\ \ t_{\restriction_{\tilde{A}}} = \tilde{F}(t_{\restriction_A})\}$$

Similarly, we define η_A^{-1}:

$$\eta_A^{-1} = \{s \in L_{\tilde{A} \multimap A} \mid \forall t \sqsubseteq^P s,\ \ t_{\restriction_{\tilde{A}}} = \tilde{F}(t_{\restriction_A})\}$$

By construction, we have $\eta_A : A \multimap \tilde{A}$ and $\eta_A^{-1} : \tilde{A} \multimap A$. Now, note that these are very near to the definition of the identity, except that we have the condition that $t_{\restriction_{\tilde{A}}} = \tilde{F}(t_{\restriction_A})$ instead of $t_{\restriction_{LHS}} = t_{\restriction_{RHS}}$. But since $\tilde{F}$ induces a bijection $M_A^* \to \tilde{F}(M_A^*)$, we have a one-to-one correspondence between witnesses for $\eta_A; \eta_A^{-1}$ and witnesses for $id_A; id_A$, thus $\eta_A; \eta_A^{-1} = id_A$. The same proof is valid for $\eta_A^{-1}; \eta_A$, which concludes. □

Lemma 5. *Let A be a game in $\mathcal{C}$, then there is $B \in |\mathcal{L}|$ such that $\tilde{A} = U(B)$.*

Proof. We give the construction of B. Let S_A denote the set $\{a \in M_A \mid a \in P_A\}$ of *opening moves* in A.

- $M_B = M_{\tilde{A}}$
- $\lambda_B = \lambda_{\tilde{A}}$
- $\rhd_B = \{([s], [sa]) \in M_B \times M_B \mid (s, sa) \in P_A \times P_A\} \ \cup\ \{([a], [a]) \mid a \in S_A\}$
- $\lhd_B = \{([s], [t]) \in M_B \times M_B \mid |t| \leq |s|\}$ where $|s|$ denotes the length of s.

And, unfolding the definition of U:

$$\begin{aligned} M_{U(B)} &= M_B \\ &= M_{\tilde{A}} \\[2em] \lambda_{U(B)} &= \lambda_B \\ &= \lambda_{\tilde{A}} \\[2em] P_{U(B)} &= \{u \in M_B^* \mid \rhd_B (u) \ \wedge\ \lhd_B (u)\} \end{aligned}$$

Let $u \in P_{U(B)}$, there is $n \in \mathbb{N}$ and $s_1, \dots, s_n \in M_A{}^\bullet$ such that $u = \prod_{i=1}^{n} [s_i]$. We prove by induction on n that each s_{i+1} is an immediate suffix of s_i and that both are in P_A. Suppose $n = 1$, $[s_1]$ then (by compatibility with $\rhd_B$) $[s_1]$ must be self-triggering, which implies that $s_1 \in S_A$. Therefore s_1 is an immediate suffix of ϵ and $s_1, \epsilon \in P_A$.
Now suppose $n > 0$. By compatibility with $\lhd_B$, $[s_n] \notin [S_A]$ since it would have been blocked by the

first move of the play. Therefore by compatibility with $\vartriangleright_B$, there is $i \in \{1 \ldots (n-1)\}$ such that s_n is an immediate suffix of s_i. Then the corresponding prefix has to be s_{n-1} otherwise it would satisfy $|s_n| \leq |s_{n-1}|$ (using induction hypothesis), which would break compatibility with $\vartriangleleft_B$.
Reciprocally, any such sequence is suitable : it satisfies compatibility with triggering and blocking. Thus:

$$\begin{aligned} P_{U(B)} &= \{\prod_{k=1}^{n}[\prod_{i=1}^{k} a_i] \mid \prod_{i=1}^{n} a_i \in P_A\} \\ &= \{\tilde{F}(s) \mid s \in P_A\} \\ &= \tilde{F}(P_A) \\ &= P_{\tilde{A}} \end{aligned}$$

□

Corollary 1. *There is an equivalence of categories between $\mathcal{L}$ and $\mathcal{C}$.*

Proof. Lemma 3 assures that $U : \mathcal{L} \to \mathcal{C}$ defined previously is full and faithful. Moreover, we have proven above that it is *essentially surjective* : any game A in $\mathcal{C}$ is isomorphic to $U(B)$ with $B \in |\mathcal{L}|$. This proves the equivalence, and the proof is constructive since we have an effective construction, for $A \in |\mathcal{C}|$, of the corresponding B. □

2.3 Monoidality of the equivalence

We deduce from the construction above the functor F from $\mathcal{C}$ to $\mathcal{L}$:

Image of objects. Let $A = (M_A, \lambda_A, P_A) \in |\mathcal{C}|$. The image of A by F is just the $B \in |\mathcal{L}|$ such that $F(B) = \tilde{A}$. Let S_A denote the set $\{a \in M_A \mid a \in P_A\}$ of *initial moves* in A, then the definition of $F(A)$ is :

- $M_{F(A)} = M_{\tilde{A}} = [M_A{}^\bullet]$
- $\lambda_{F(A)} = \lambda_{\tilde{A}} = ([sa] \mapsto \lambda_A(a))$
- $\vartriangleright_{F(A)} = \{([s],[sa]) \in M_{F(A)} \times M_{F(A)} \mid (s, sa) \in P_A \times P_A\} \cup \{([a],[a]) \mid a \in S_A\}$
- $\vartriangleleft_{F(A)} = \{([s],[t]) \in M_{F(A)} \times M_{F(A)} \mid |t| \leq |s|\}$ where $|s|$ denotes the length of s.

Image of arrows. Let $A, B \in |\mathcal{C}|$, and $f : A \to B$ in $\mathcal{C}$. We have isomorphisms $\eta_A : A \to UF(A)$ and $\eta_B : B \to UF(B)$, thus we can build $\eta_A^{-1};\sigma;\eta_B : UF(A) \to UF(B)$. Then, U being full and faithful, this induces an unique arrow $U^{-1}(\eta_A^{-1};\sigma;\eta_B) : F(A) \to F(B)$ (which is in fact syntactically equal to $\eta_A^{-1};\sigma;\eta_B$ since U is the identity on arrows), and we define $F(\sigma) = U^{-1}(\eta_A^{-1};\sigma;\eta_B)$.

In practice, it suffices to compute $\eta_A^{-1};\sigma;\eta_B$ since F is the identity on arrows. Therefore, $F(\sigma)$ is easy to describe : take each $s \in \sigma$, and for each move a in s (*ie.* $s = s_1 a s_2$), we do the following : if $a \in M_A$, we replace it by $[s_1{\upharpoonright_A} a]$. If $a \in M_B$, we replace it by $[s_1{\upharpoonright_B} a]$. Notice that this informal description sticks totally to what composition with η_A and η_B^{-1} actually do.

We now describe the behaviour of U and F with respect to the monoidal structure of $\mathcal{L}$ and $\mathcal{C}$.

Lemma 6. *U is monoidal strict : Let A, B be games in $\mathcal{L}$. Then $U(A \bullet B) = U(A) \otimes U(B)$.*

Proof. As usual, we reason equationally:

$$\begin{array}{rcl} M_{U(A\bullet B)} & = & M_{A\bullet B} \\ & = & M_A + M_B \\ & = & M_{U(A)} + M_{U(B)} \\ & = & M_{U(A)\otimes U(B)} \end{array}$$

$$\begin{array}{rcl} \lambda_{U(A\bullet B)} & = & \lambda_{A\bullet B} \\ & = & [\lambda_A, \lambda_B] \\ & = & [\lambda_{U(A)}, \lambda_{U(B)}] \\ & = & \lambda_{U(A)\otimes U(B)} \end{array}$$

$$\begin{array}{rcl} P_{U(A\bullet B)} & = & \{u \in M^*_{A\bullet B} \mid \rhd_{A\bullet B}(u) \ \wedge \ \lhd_{A\bullet B}(u)\} \\ & = & \{u \in M^*_{U(A)\otimes U(B)} \mid \rhd_A(u_{\restriction_A}) \ \wedge \ \rhd_B(u_{\restriction_B}) \ \wedge \ \lhd_A(u_{\restriction_A}) \ \wedge \ \lhd_B(u_{\restriction_B})\} \\ & = & \{u \in M^*_{U(A)\otimes U(B)} \mid u_{\restriction_{U(A)}} \in P_{U(A)} \ \wedge \ u_{\restriction_{U(B)}} \in P_{U(B)}\} \\ & = & P_{U(A)\otimes U(B)} \end{array}$$

□

Proposition 4. *F is strong monoidal : Let A, B be games in $\mathcal{C}$. Then $F(A \otimes B) \simeq F(A) \bullet F(B)$.*

Proof. First, note that the equivalence of categories provides us with a natural isomorphism $\epsilon : I \to FU$ in $\mathcal{L}$. This isomorphism can be computed (pointwise) directly by the following way. Consider a game A of $\mathcal{L}$. We have the isomorphism $\eta_{U(A)} : U(A) \to UFU(A)$. Since U is full and faithful there is an isomorphism $U^{-1}(\eta_{U(A)}) : A \to FU(A)$ such that $U(U^{-1}(\eta_{U(A)})) = \eta_{U(A)}$. Since U is the identity on arrows, $U^{-1}(\eta_{U(A)})$ and $\eta_{U(A)}$ are in fact syntactically equal. Thus we define $\epsilon_A = \eta_{U(A)} : A \to FU(A)$. Finally we get the necessary isomorphism by composition of known isomorphisms.

$$F(A \otimes B) \xrightarrow{F(\eta_A \otimes \eta_B)} F(UF(A) \otimes UF(B)) = FU(F(A) \bullet F(B)) \xrightarrow{\epsilon^{-1}_{F(A)\bullet F(B)}} F(A) \bullet F(B)$$

□

Remark 4. *F is not strict monoidal.*

$$\begin{array}{rcl} M_{F(A\otimes B)} & = & [M_{A\otimes B}{}^\bullet] \\ & = & [(M_A + M_B)^\bullet] \\ & \neq & [M_A{}^\bullet] + [M_B{}^\bullet] \\ & = & M_{F(A)\bullet F(B)} \end{array}$$

3 The negative case

We briefly summarize the adjunction situation between the global category of games and the negative one, which is really the same in both the Conway and the Laird situation. Then, we use the particular properties of these parallel adjunctions to transport the above equivalence of categories to the negative case.

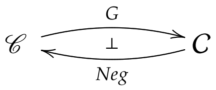
3.1 Adjunction global ↔ negative

We begin by describing the situation for Conway games. As usual, $\mathcal{C}$ will denote the category of Conway games and $\mathscr{C}$ the category of negative Conway games.

Proposition 5. *We define a functor $Neg : \mathcal{C} \to \mathscr{C}$ by taking the negative part of each game; the image of a morphism $\sigma : A \to B$ of $\mathcal{C}$ is obtained by selecting all plays where Opponent starts in B. We define a forgetful functor $G : \mathscr{C} \to \mathcal{C}$ which associates to each negative game the same game in $\mathcal{C}$. Then the closure of $\mathscr{C}$ is inherited from the closure of $\mathcal{C}$: for each games A, B of $\mathscr{C}$,*

$$\begin{aligned} A \multimap B &= Neg(G(A)^* \otimes G(B)) \\ &= Neg(G(A) \multimap G(B)) \end{aligned}$$

As an consequence, there is an adjunction:



Proof. Let's prove the equality. First notice that $A \multimap B$ and $Neg(G(A)^* \otimes G(B))$ have the same set of moves and polarity functions. Let $O(s)$ denote here the fact that s starts with an opponent move, then :

$$\begin{aligned} s \in P_{A \multimap B} &\Leftrightarrow s_{\upharpoonright_A} \in P_A \wedge s_{\upharpoonright_B} \in P_B \wedge O(s) \\ &\Leftrightarrow s_{\upharpoonright_{G(A)}} \in P_{G(A)} \wedge s_{\upharpoonright_{G(B)}} \in P_{G(B)} \wedge O(s) \\ &\Leftrightarrow s \in P_{G(A)^* \otimes G(B)} \wedge O(s) \\ &\Leftrightarrow s \in P_{Neg(G(A)^* \otimes G(B))} \end{aligned}$$

which ends the first part of the proposition.

We show now that there is a natural isomorphism $\phi : Hom_{\mathcal{C}}(G\_, \_) \to Hom_{\mathscr{C}}(\_, Neg\_)$. Let A be a game in $\mathscr{C}$, and B a game in $\mathcal{C}$, and $\sigma : G(A) \to B$. As a strategy, σ acts on the game $G(A)^* \otimes B$. Since $G(A)$ is negative, the first move in σ must be an opponent move in B, which appears as well in $G(Neg(B))$. Therefore, we can view σ as a strategy on $G(A)^* \otimes G(Neg(B))$: this is a negative game and therefore equal to $Neg(G(A)^* \otimes G(Neg(B)))$. We write $\phi_{A,B}(\sigma)$ for σ viewed as a strategy on $Neg(G(A)^* \otimes G(Neg(B)))$, which is equal to $A \multimap Neg(B)$ by the first part of the proposition. Naturality comes for free since ϕ makes no change to its argument.

□

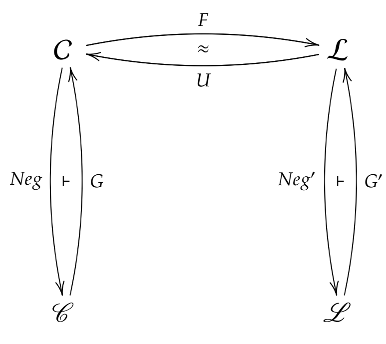
Remark 5. *The situation is exactly the same for Laird-like games : notice that the proof above is only about the* opening *of games, and the situation is the same on the Laird-like side. Thus, we will speak in the sequel of the corresponding adjunction $G' \dashv Neg'$ in Laird's setting.*

3.2 Equivalence of categories between $\mathscr{L}$ and $\mathscr{C}$

As usual, $\mathscr{L}$ is the category of negative Laird-like games and $\mathscr{C}$ is the category of negative Conway games. The above adjunctions allows to transport easily the equivalence of $\mathcal{L}$ and $\mathcal{C}$ to the negative situation.

Proposition 6. *There is a strong monoidal equivalence of categories between $\mathscr{L}$ and $\mathscr{C}$.*

Proof. This diagram shows a brief overview of the situation:



As expected, the functors required for the equivalence will be $Neg \circ U \circ G' : \mathscr{L} \rightarrow \mathscr{C}$ and $Neg' \circ F \circ G : \mathscr{C} \rightarrow \mathscr{L}$, and the required natural isomorphism falls out of the general case :

$$\begin{aligned} Neg \circ U \circ G' \circ Neg' \circ F \circ G &= Neg \circ U \circ F \circ G \quad (G' \circ Neg'_{\restriction \mathscr{L}} = I) \\ &\simeq Neg \circ G \quad \text{(By the equivalence property)} \\ &= I \end{aligned}$$

This isomorphism is natural by composition of equalities and a natural iso, and the construction of the other isomorphism is similar. □

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A Definition of negative Conway games

We recall here the usual definition of negative Conway games.

Definition 18. *A game A is a tuple $< M_A, \lambda_A, P_A >$ where:*

- *M_A is a set of* moves*,*
- *$\lambda_A : M_A \to \{O, P\}$ is Player/Opponent labelling (by convention, $\overline{O} = P$ and $\overline{P} = O$),*
- *$P_A \subset M_A^*$ is a prefix-closed set of* plays *such that the first move of each play belongs to Opponent (*negativity*).*

Definition 19. *For any game A, the set of legal sequences L_A is the set of elements of P_A which satisfies the **alternation** condition : Player and Opponent move alternatively.*

$$L_A = \{u \in P_A \mid \lambda_A(u)\}$$

Definition 20. *Given games A and B, form:*

- *$A \otimes B = (M_{A\otimes B}, \lambda_{A\otimes B}, P_{A\otimes B})$, where :*
  - $M_{A\otimes B} = M_A + M_B$
  - $\lambda_{A\otimes B} = [\lambda_A, \lambda_B]$
  - $P_{A\otimes B} = \{s \in (M_A + M_B)^* \mid s_{\restriction M_A} \in P_A \;\wedge\; s_{\restriction M_B} \in P_B\}$
- *$A \multimap B = (M_{A\multimap B}, \lambda_{A\multimap B}, P_{A\multimap B})$, where:*
  - $M_{A\multimap B} = M_A + M_B$
  - $\lambda_{A\multimap B} = [\overline{\lambda_A}, \lambda_B]$
  - $P_{A\multimap B} = \{s \in (M_A + M_B)^* \mid s_{\restriction M_A} \in P_A \wedge s_{\restriction M_B} \in P_B \wedge \forall x \in (M_A + M_B), \forall u \in (M_A + M_B)^*, x.u \sqsubseteq s \Rightarrow x \in M_B\}$

  We require here that the first move in $A \multimap B$ is in M_B. Note that by negativity of A and B, it is equivalent to requiring that this first move belongs to Opponent.
- *$A\&B = (M_{A\&B}, \lambda_{A\&B}, P_{A\&B})$, where:*
  - $M_{A\&B} = M_A + M_B$
  - $\lambda_{A\&B} = [\lambda_A, \lambda_B]$
  - $P_{A\&B} = \{s \in (M_A + M_B)^* \mid s \in P_A \;\vee\; s \in P_B\}$

Definition 21. *The **strategies** on a game A are even-prefix closed, evenly branching sets of even-length elements of L_A.*

Definition 22. *From strategies $\sigma : A \multimap B$ and $\tau : B \multimap C$, we form $\sigma;\tau : A \multimap C$:*

$$\sigma;\tau = \{s \in L_{A\multimap C} | \exists u \in (M_A + M_B + M_C)^* \;\; s = u_{\restriction A,C} \;\wedge\; u_{\restriction A,B} \in \sigma \;\wedge\; u_{\restriction B,C} \in \tau\}$$

Proposition 7. *$(\mathscr{C}, I, \otimes, \multimap)$ is a symmetric monoidal closed category.*