The Biequivalence of Locally Cartesian Closed Categories and Martin-Löf Type Theory

Pierre Clairambault, Bath
and
Peter Dybjer, Chalmers

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Categorical logic: key correspondences

Cartesian closed categories	$\simeq$	simply typed λ-calculus
Hyperdoctrines	$\simeq$	first-order logic
Toposes	$\simeq$	Higher-order logic
?	$\simeq$	Martin-Löf type theory

Seely's conjecture

R. Seely (1984), Locally cartesian closed categories and type theory:

> *6.3. THEOREM. The categories **ML** and **LCC** are equivalent.*

- **ML** is the category of "Martin-Löf theories" with types $\prod_{x \in A} B[x], \sum_{x \in A} B[x]$, and $I(a, b)$.
- **LCC** is the category of locally cartesian closed categories.

Extensional type theory

We consider *extensional* intuitionistic type theory of Martin-Löf (1979, 1984), *i.e.* identity types satisfy:

$$\frac{\vdash p : I_A(m,n)}{\vdash m = n : A} \qquad \frac{\vdash p : I_A(m,n)}{\vdash p = r_A(m) : I_A(m,n)}$$

Note that these rules are *not* derivable in intensional type theory (and break decidability of type checking).

Locally cartesian closed categories

A category $\mathcal{C}$ is *locally cartesian closed (lccc)* iff either of the following equivalent conditions hold:

- All slice categories $\mathcal{C}/A$ are cartesian closed (and $\mathcal{C}$ has a terminal object),
- $\mathcal{C}$ has finite limits and the functor $f^* : \mathcal{C}/B \to \mathcal{C}/A$ has a right adjoint Π_f for $f : A \to B$. (The left adjoint Σ_f always exists.)

Seely's **LCC** is the category of lcccs and lccc-structure preserving functors.

Martin-Löf theory and lccc - correspondences

- Contexts are objects of $\mathcal{C}$.
- Types in context Γ are morphisms $A : dom(A) \to \Gamma$ (objects of $\mathcal{C}/\Gamma$), called *display maps*,
- Terms of type A are sections of A, *i.e.* morphisms $a : \Gamma \to dom(A)$ such that $A \circ a = id_\Gamma$.
- Type substitution is pullback:

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "f^* A"'] & \cdot \arrow[d, "A"] \\ \Delta \arrow[r, "f"'] & \Gamma \end{tikzcd}$$

- Extensional identity types are equalizers
- Σ-types are left adjoints $\Sigma_f \dashv f^*$
- Π-types are right adjoints $f^* \dashv \Pi_f$

Substitution up to isomorphism

Flaw in Seely's proof: pullbacks compose only up to isomorphism.

$$A[M/x][N/y] \quad = \quad A[M[N/y]/x]$$

$$\begin{tikzcd}
{} \arrow[r] \arrow[d, "{\llbracket N \rrbracket^*(\llbracket M \rrbracket^*(\llbracket A \rrbracket))}"'] & {} \arrow[r] \arrow[d, "{\llbracket M \rrbracket^*(\llbracket A \rrbracket)}"] & {} \arrow[d, "{\llbracket A \rrbracket}"] & & {} \arrow[r] \arrow[d, "{(\llbracket M \rrbracket \circ \llbracket N \rrbracket)^*(\llbracket A \rrbracket)}"] & {} \arrow[d, "{\llbracket A \rrbracket}"] \\
D \arrow[r, "{\llbracket N \rrbracket}"'] & C \arrow[r, "{\llbracket M \rrbracket}"'] & B & \cong & D \arrow[r, "{\llbracket M \rrbracket \circ \llbracket N \rrbracket}"'] & B
\end{tikzcd}$$

It amounts to the fact that a locally cartesian closed category, as a fibration, is not necessarily split.

Curien and Hofmann

Two proposed solutions for the interpretation of MLTT $\mathcal{T}$ in a LCCC $\mathcal{C}$:

$$\begin{array}{rccc} \text{Curien (93)} & : & \mathcal{T}^* & \rightarrow & \mathcal{C} \\ \text{Hofmann (95)} & : & \mathcal{T} & \rightarrow & \mathcal{B}(\mathcal{C}) \end{array}$$

Where:

- $\mathcal{T}^*$ is an extension of $\mathcal{T}$ with *explicit substitutions*, and special terms for *type isomorphisms*.
- $\mathcal{B}$ is the *Bénabou construction*, which associates to any fibration an equivalent *split* fibration.

However only the interpretation is investigated, not whether this gives an *equivalence*.

Hofmann's solution: Bénabou's construction

A type A over Γ is no longer a *display map* $A : dom(A) \to \Gamma$, but a *family* $\overrightarrow{A}$ of display maps:

$$\begin{tikzcd} {} \arrow[r] \arrow[d, "\overrightarrow{A}(\delta\gamma)"] & {} \arrow[r] \arrow[d, "\overrightarrow{A}(\delta)"] & {} \arrow[d, "\overrightarrow{A}(id_\Gamma)"] \\ \Omega \arrow[r, "\gamma"'] & \Delta \arrow[r, "\delta"'] & \Gamma \end{tikzcd}$$

Such that all the squares are pullback squares.

Definition

A *functorial family* over Γ is a functor $\overrightarrow{A} : \mathcal{C}/\Gamma \to \mathcal{C}^{\to}$ satisfying some conditions.

Our contribution

Seely's statement

*The categories **LCC** and **ML** are equivalent.*

Theorem

*The 2-categories **LCC** and* $\mathbf{CwF}^{\mathrm{I_{ext}}\Sigma\Pi}_{\mathrm{dem}}$ *are biequivalent*

- Instead of an equivalence, we have a weaker *biequivalence*,
- We use *categories with families* (**CwF**) with extra structure ($\mathrm{I_{ext}}, \Sigma, \Pi$ and dem) as a replacement for syntax.

What is a biequivalence?

When are two objects ”abstractly the same”?

Equality (set) $a = b$

Isomorphism (category) $a \cong b$
$$\begin{tikzcd} a \arrow[r, bend left, "f"] & b \arrow[l, bend left, "g"] \end{tikzcd}$$
$$fg = 1_b \qquad gf = 1_a$$

Equivalence (bicategory) $a \simeq b$
$$\begin{tikzcd} a \arrow[r, bend left, "f"] & b \arrow[l, bend left, "g"] \end{tikzcd}$$
$$fg \cong 1_b \qquad gf \cong 1_a$$

Biequivalence (tricategory) $a \sim b$
$$\begin{tikzcd} a \arrow[r, bend left, "f"] & b \arrow[l, bend left, "g"] \end{tikzcd}$$
$$gf \simeq 1_a \qquad fg \simeq 1_b$$

What is a biequivalence?

To define biequivalent 2-categories, we consider the tricategory of bicategories which components are:

- 0-cells: bicategories
- 1-cells: pseudofunctors between bicategories

$$\begin{tikzcd} FA \arrow[r, bend left=50, "1_{FA}"{name=U}] \arrow[r, bend right=50, "F1_A"'{name=D}] & FA \arrow[from=U, to=D, Rightarrow, "\phi_A"] \end{tikzcd} \qquad \begin{tikzcd} & FB \arrow[dr, bend left, "F(g)"] \arrow[d, phantom, "\Downarrow \phi_{f,g}"] & \\ FA \arrow[ur, bend left, "F(f)"] \arrow[rr, bend right, "F(gf)"'] & & FC \end{tikzcd}$$

- 2-cells: pseudonatural transformation of pseudofunctors

$$\begin{tikzcd} FA \arrow[r, "\eta_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\eta_B"'] \arrow[ur, Rightarrow, "\eta_f"] & GB \end{tikzcd}$$

- 3-cells: modification of pseudonatural transformations

Proving the biequivalence

We need to provide the following data (and check the appropriate properties):

- **LCC**: the 2-category of locally cartesian closed categories, structure-preserving functors, and natural transformations.
- $\mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I_{ext}}\Sigma\Pi}$: the 2-category of cwfs with extra structure.
- $U : \mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I_{ext}}\Sigma\Pi} \to \mathbf{LCC}$ is a forgetful 2-functor.
- $H : \mathbf{LCC} \to \mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I_{ext}}\Sigma\Pi}$ is a pseudofunctor based on the Bénabou-Hofmann construction.
- $\eta : 1 \to HU$ and $\epsilon : HU \to 1$: pseudonatural transformations, which are inverses up to invertible modifications ϕ, ψ.

Categories with families (cwfs)

A *category with family* $(\mathcal{C}, T)$ is the data of:

- $\mathcal{C}$, a *category of contexts*.
- $T : \mathcal{C}^{op} \to \mathbf{Fam}$, a functor where the

  object part $\Gamma \mapsto (\{a \mid \Gamma \vdash a : A\})_{A \in Type(\Gamma)}$

  arrow part $\gamma \mapsto \begin{cases} A \mapsto A[\gamma] \\ a \mapsto a[\gamma] \end{cases}$
- A *terminal object* [] of $\mathcal{C}$ called the *empty context*.
- A *context comprehension* operation which to an object Γ of $\mathcal{C}$ and a type $A \in Type(\Gamma)$ associates a context $\Gamma \cdot A$ satisfying a product-like universal property.

Cwfs can be presented as a generalised algebraic theory, and be seen as a variable-free syntax for Martin-Löf type theory with explicit substitutions.

Cwf-morphisms

Definition

A *strict cwf-morphism* from $(\mathcal{C}, T)$ to $(\mathcal{D}, T')$ is a pair:

- $F : \mathcal{C} \to \mathcal{D}$ is a functor;
- $\sigma : T \to T'F$ is a natural transformation.

It follows that $\sigma_\Gamma(A)[F\delta] = \sigma_\Delta(A[\delta])$

Definition

A *pseudo cwf-morphism* from $(\mathcal{C}, T)$ to $(\mathcal{D}, T')$ is a pair:

- $F : \mathcal{C} \to \mathcal{D}$ is a functor;
- $\sigma_\Gamma : T\Gamma \to T'F\Gamma$ is a family of **Fam**-morphism.

Such that $\sigma_\Gamma(A)[F\delta] \cong \sigma_\Delta(A[\delta])$, with coherence conditions.

The 2-category $\mathbf{CwF}^{\mathrm{I_{ext}}\Sigma\Pi}_{\mathrm{dem}}$

Definition

There is a 2-category $\mathbf{CwF}^{\mathrm{I_{ext}}\Sigma\Pi}_{\mathrm{dem}}$ with:

- 0-cells are cwfs supporting $\Sigma, \Pi, \mathrm{I_{ext}}$ and dem,
- 1-cells are pseudo cwf-morphisms which also preserve $\Sigma, \Pi, \mathrm{I_{ext}}$ and dem up to isomorphism,
- 2-cells are pseudo cwf-transformations.

The remaining components

$$
\begin{tikzcd}
\textbf{LCC} & & \textbf{CwF}^{\mathrm{I_{ext}}\Sigma\Pi}_{\mathrm{dem}} \\
 & & (\mathcal{C}, T) \arrow[lld, mapsto, "U"'] \arrow[dd, bend right, "\eta_{(\mathcal{C},T)}"'] \\
\mathcal{C} \arrow[rrd, mapsto, "H"'] & & \\
 & & (\mathcal{C}, T_{\mathcal{C}}) \arrow[uu, bend right, "\epsilon_{(\mathcal{C},T)}"']
\end{tikzcd}
$$

- $U(\mathcal{C}, T) = \mathcal{C}$ is a *forgetful 2-functor*,
- $H(\mathcal{C}) = (\mathcal{C}, T_{\mathcal{C}})$ generalises the Bénabou-Hofmann construction to a *pseudofunctor*.

The remaining components

LCC $\quad\quad$ $\mathbf{CwF}^{\mathrm{I_{ext}}\Sigma\Pi}_{\mathrm{dem}}$

$$
\begin{tikzcd}
 & (\mathcal{C}, T) \arrow[dl, mapsto, "U"'] \arrow[dd, bend right, "\eta_{(\mathcal{C},T)}"'] \\
\mathcal{C} \arrow[dr, mapsto, "H"'] & \\
 & (\mathcal{C}, T_{\mathcal{C}}) \arrow[uu, bend right, "\epsilon_{(\mathcal{C},T)}"']
\end{tikzcd}
$$

- $\eta_{(\mathcal{C},T)} = (Id, \sigma)$ where $\sigma(A)$ associates to any substitution δ the display map of $A[\delta]$,
- $\epsilon_{(\mathcal{C},T)} = (Id, \tau)$ where $\tau(\overrightarrow{A}) = \Sigma_{y:\Delta} I_\Gamma(\overrightarrow{A}(id)(y), x)(x : \Gamma)$ builds a syntactic representative type for $\overrightarrow{A}$.

Conclusion

Theorem

*The 2-categories **LCC** and $\mathbf{CwF}^{\mathrm{I_{ext}}\Sigma\Pi}_{\mathrm{dem}}$ are biequivalent.*

Removing everything related to Π in the proof yields the following result:

Theorem

*The 2-categories **FL** and $\mathbf{CwF}^{\mathrm{I_{ext}}\Sigma}_{\mathrm{dem}}$ are biequivalent.*