

The Biequivalence of Locally Cartesian Closed Categories and Martin-Löf Type Theory

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Categorical logic: key correspondences

Cartesian closed categories \simeq simply typed λ -calculus

Hyperdoctrines \simeq first-order logic

Toposes \simeq Higher-order logic

? \simeq Martin-Löf type theory

Seely's conjecture

R. Seely (1984), Locally cartesian closed categories and type theory:

*6.3. THEOREM. The categories **ML** and **LCC** are equivalent.*

- **ML** is the category of "Martin-Löf theories" with types $\prod_{x \in A} B[x]$, $\sum_{x \in A} B[x]$, and $I(a, b)$.
- **LCC** is the category of locally cartesian closed categories.

Extensional type theory

We consider *extensional* intuitionistic type theory of Martin-Löf (1979, 1984), *i.e.* identity types satisfy:

$$\frac{\vdash p : I_A(m, n)}{\vdash m = n : A} \qquad \frac{\vdash p : I_A(m, n)}{\vdash p = r_A(m) : I_A(m, n)}$$

Note that these rules are *not* derivable in intensional type theory (and break decidability of type checking).

Locally cartesian closed categories

A category C is *locally cartesian closed* (lccc) iff either of the following equivalent conditions hold:

- All slice categories C/A are cartesian closed (and C has a terminal object),
- C has finite limits and the functor $f^* : C/B \rightarrow C/A$ has a right adjoint Π_f for $f : A \rightarrow B$. (The left adjoint Σ_f always exists.)

Seely's **LCC** is the category of lcccs and lccc-structure preserving functors.

Martin-Löf theory and lccc - correspondences

- Contexts are objects of C .
- Types in context Γ are morphisms $A : \text{dom}(A) \rightarrow \Gamma$ (objects of C/Γ), called *display maps*,
- Terms of type A are sections of A , *i.e.* morphisms $a : \Gamma \rightarrow \text{dom}(A)$ such that $A \circ a = \text{id}_\Gamma$.
- Type substitution is pullback:

$$\begin{array}{ccc}
 & \longrightarrow & \\
 f^*A \downarrow & & \downarrow A \\
 \Delta & \xrightarrow{f} & \Gamma
 \end{array}$$

- Extensional identity types are equalizers
- Σ -types are left adjoints $\Sigma_f \dashv f^*$
- Π -types are right adjoints $f^* \dashv \Pi_f$

Substitution up to isomorphism

Flaw in Seely's proof: pullbacks compose only up to isomorphism.

$$A[M/x][N/y] = A[M[N/y]/x]$$

The diagram shows two commutative diagrams related by an isomorphism symbol \cong .

The left diagram consists of three objects D , C , and B arranged in a horizontal line. Above them are three objects: $\llbracket N \rrbracket^*(\llbracket M \rrbracket^*(\llbracket A \rrbracket))$, $\llbracket M \rrbracket^*(\llbracket A \rrbracket)$, and $\llbracket A \rrbracket$. Vertical arrows point from each top object to the object below it. Horizontal arrows point from D to C (labeled $\llbracket N \rrbracket$) and from C to B (labeled $\llbracket M \rrbracket$). Top horizontal arrows connect the top objects: from $\llbracket N \rrbracket^*(\llbracket M \rrbracket^*(\llbracket A \rrbracket))$ to $\llbracket M \rrbracket^*(\llbracket A \rrbracket)$, and from $\llbracket M \rrbracket^*(\llbracket A \rrbracket)$ to $\llbracket A \rrbracket$.

The right diagram consists of two objects D and B in a horizontal line. Above them are two objects: $(\llbracket M \rrbracket \circ \llbracket N \rrbracket)^*(\llbracket A \rrbracket)$ and $\llbracket A \rrbracket$. Vertical arrows point from each top object to the object below it. A horizontal arrow points from D to B (labeled $\llbracket M \rrbracket \circ \llbracket N \rrbracket$). A top horizontal arrow connects the two top objects.

It amounts to the fact that a locally cartesian closed category, as a fibration, is not necessarily split.

Curien and Hofmann

Two proposed solutions for the interpretation of MLTT \mathcal{T} in a LCCC \mathcal{C} :

$$\begin{array}{lcl} \text{Curien (93)} & : & \mathcal{T}^* \rightarrow \mathcal{C} \\ \text{Hofmann (95)} & : & \mathcal{T} \rightarrow \mathcal{B}(\mathcal{C}) \end{array}$$

Where:

- \mathcal{T}^* is an extension of \mathcal{T} with *explicit substitutions*, and special terms for *type isomorphisms*.
- \mathcal{B} is the *Bénabou construction*, which associates to any fibration an equivalent *split* fibration.

However only the interpretation is investigated, not whether this gives an *equivalence*.

Hofmann's solution: Bénabou's construction

A type A over Γ is no longer a *display map* $A : \text{dom}(A) \rightarrow \Gamma$,
but a *family* \vec{A} of display maps:

$$\begin{array}{ccccc}
 & \longrightarrow & & \longrightarrow & \\
 \downarrow \vec{A}(\delta\gamma) & & \downarrow \vec{A}(\delta) & & \downarrow \vec{A}(id_\Gamma) \\
 \Omega & \xrightarrow{\gamma} & \Delta & \xrightarrow{\delta} & \Gamma
 \end{array}$$

Such that all the squares are pullback squares.

Definition

A *functorial family* over Γ is a functor $\vec{A} : C/\Gamma \rightarrow C^\rightarrow$ satisfying some conditions.

Our contribution

Seely's statement

The categories **LCC** and **ML** are equivalent.

Theorem

The 2-categories **LCC** and $\mathbf{CwF}_{\text{dem}}^{\text{I}_{\text{ext}}\Sigma\Pi}$ are biequivalent

- Instead of an equivalence, we have a weaker *biequivalence*,
- We use *categories with families* (**CwF**) with extra structure (I_{ext} , Σ , Π and dem) as a replacement for syntax.

What is a biequivalence?

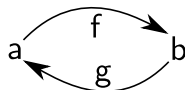
When are two objects "abstractly the same"?

Equality
(set)

$$a = b$$

Isomorphism
(category)

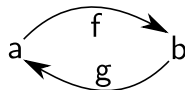
$$a \cong b$$



$$\begin{aligned} fg &= 1_b \\ gf &= 1_a \end{aligned}$$

Equivalence
(bicategory)

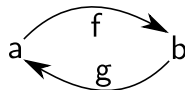
$$a \simeq b$$



$$\begin{aligned} fg &\cong 1_b \\ gf &\cong 1_a \end{aligned}$$

Biequivalence
(tricategory)

$$a \sim b$$

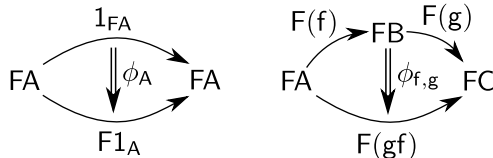


$$\begin{aligned} gf &\simeq 1_a \\ fg &\simeq 1_b \end{aligned}$$

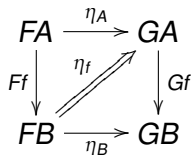
What is a biequivalence?

To define biequivalent 2-categories, we consider the tricategory of bicategories which components are:

- 0-cells: bicategories
- 1-cells: pseudofunctors between bicategories



- 2-cells: pseudonatural transformation of pseudofunctors



- 3-cells: modification of pseudonatural transformations

Proving the biequivalence

We need to provide the following data (and check the appropriate properties):

- **LCC**: the 2-category of locally cartesian closed categories, structure-preserving functors, and natural transformations.
- $\mathbf{CwF}_{\text{dem}}^{\text{Iext}\Sigma\Pi}$: the 2-category of cwfs with extra structure.
- $U : \mathbf{CwF}_{\text{dem}}^{\text{Iext}\Sigma\Pi} \rightarrow \mathbf{LCC}$ is a forgetful 2-functor.
- $H : \mathbf{LCC} \rightarrow \mathbf{CwF}_{\text{dem}}^{\text{Iext}\Sigma\Pi}$ is a pseudofunctor based on the Bénabou-Hofmann construction.
- $\eta : 1 \rightarrow HU$ and $\epsilon : HU \rightarrow 1$: pseudonatural transformations, which are inverses up to invertible modifications ϕ, ψ .

Categories with families (cwfs)

A *category with family* (C, T) is the data of:

- C , a *category of contexts*.
- $T : C^{op} \rightarrow \mathbf{Fam}$, a functor where the
 - object part** $\Gamma \mapsto (\{a \mid \Gamma \vdash a : A\})_{A \in \text{Type}(\Gamma)}$
 - arrow part** $\gamma \mapsto \begin{cases} A \mapsto A[\gamma] \\ a \mapsto a[\gamma] \end{cases}$
- A *terminal object* $[]$ of C called the *empty context*.
- A *context comprehension* operation which to an object Γ of C and a type $A \in \text{Type}(\Gamma)$ associates a context $\Gamma \cdot A$ satisfying a product-like universal property.

Cwfs can be presented as a generalised algebraic theory, and be seen as a variable-free syntax for Martin-Löf type theory with explicit substitutions.

Cwf-morphisms

Definition

A *strict cwf-morphism* from (C, T) to (\mathcal{D}, T') is a pair:

- $F : C \rightarrow \mathcal{D}$ is a functor;
- $\sigma : T \rightarrow T'F$ is a natural transformation.

It follows that $\sigma_{\Gamma}(A)[F\delta] = \sigma_{\Delta}(A[\delta])$

Definition

A *pseudo cwf-morphism* from (C, T) to (\mathcal{D}, T') is a pair:

- $F : C \rightarrow \mathcal{D}$ is a functor;
- $\sigma_{\Gamma} : T\Gamma \rightarrow T'F\Gamma$ is a family of **Fam**-morphism.

Such that $\sigma_{\Gamma}(A)[F\delta] \cong \sigma_{\Delta}(A[\delta])$, with coherence conditions.

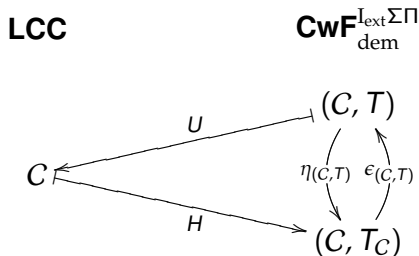
The 2-category $\mathbf{CwF}_{\text{dem}}^{\text{I}_{\text{ext}}\Sigma\Pi}$

Definition

There is a 2-category $\mathbf{CwF}_{\text{dem}}^{\text{I}_{\text{ext}}\Sigma\Pi}$ with:

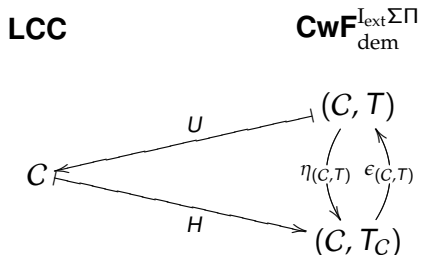
- 0-cells are cwfs supporting $\Sigma, \Pi, \text{I}_{\text{ext}}$ and dem ,
- 1-cells are pseudo cwf-morphisms which also preserve $\Sigma, \Pi, \text{I}_{\text{ext}}$ and dem up to isomorphism,
- 2-cells are pseudo cwf-transformations.

The remaining components



- $U(C, T) = C$ is a *forgetful 2-functor*,
- $H(C) = (C, T_C)$ generalises the Bénabou-Hofmann construction to a *pseudofunctor*.

The remaining components



- $\eta_{(C,T)} = (Id, \sigma)$ where $\sigma(A)$ associates to any substitution δ the display map of $A[\delta]$,
- $\epsilon_{(C,T)} = (Id, \tau)$ where $\tau(\vec{A}) = \Sigma_{y:\Delta} I_{\Gamma}(\vec{A}(id)(y), x)(x : \Gamma)$ builds a syntactic representative type for \vec{A} .

Conclusion

Theorem

The 2-categories **LCC** and $\mathbf{CwF}_{\text{dem}}^{\text{Iext}\Sigma\Pi}$ are biequivalent.

Removing everything related to Π in the proof yields the following result:

Theorem

The 2-categories **FL** and $\mathbf{CwF}_{\text{dem}}^{\text{Iext}\Sigma}$ are biequivalent.