The limit-colimit coincidence in $\mathcal{Q}$-DCPO

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Outline

Comparison of metrics and posets

Ingredients of poset

- A set X.
- Order function $X \times X \to \mathbf{2}$
- Laws.

Ingredients of metric space

- A set X.
- Distance function $X \times X \to \mathbb{R}$
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Quantal

Definition

A $\mathcal{Q} = (\mathrm{Q}, \leqslant, \otimes, \mathbf{1})$ is a quantale if:

- $(Q, \leqslant)$ is a complete lattice.
- Tensor $\otimes \colon Q \times Q \to Q$ is an associative, commutative operation that preserves suprema.
- Unit $\mathbf{1}$ is a neutral element with respect to $\otimes$ and it is simultaneously the top element in Q.

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$\mathcal{Q}$-poset

Definition

A $\mathcal{Q}$-poset is a set X together with a function $X(-,-) : X \times X \to \mathcal{Q}$ satisfying:

- Reflexivity: $X(x,x) = \mathbf{1}$,
- Transitivity: $X(x,y) \otimes X(y,z) \leqslant X(x,z)$
- Antisymmetry: $X(x,y) = \mathbf{1}$ and $X(y,x) = \mathbf{1}$ imply $x = y$.

We will call $X(-,-)$ from now on a “distance on X”

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Examples

Example (Posets)

- $Q := \{0, 1\}$
- $\otimes := \wedge$
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Example (Metric Spaces)

- $Q := [0, \infty]$
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Definition (Dual $\mathcal{Q}$-poset)

$$X^{op} := X$$
$$X^{op}(x, y) := X(y, x)$$

Example

If X is a $\mathcal{Q}$-poset, then X^{op} is also a $\mathcal{Q}$-poset.

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$\mathcal{Q}$-functor

Definition

A $\mathcal{Q}$-functor is a function between $\mathcal{Q}$-posets X and Y that satisfies $X(x, y) \leqslant Y(fx, fy)$.

Example (2-functor)

2-functors are monotone maps between posets.

Example ($[0, \infty]$-functor)

$[0, \infty]$-functors are non expansive maps between metric spaces.

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Interesting examples of $\mathcal{Q}$-functors are those that have X^{op} as their domain

Example

A **2**-functor $X^{op} \to \mathbf{2}$ is a characteristic map of some lower subset of X.

Example (Yoneda)

For every x a function $y_x = X(-, x)$ is a $\mathcal{Q}$-functor $X^{op} \to \mathcal{Q}$.

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Definition ($\mathcal{Q}$-POS)

A $\mathcal{Q}$-**POS** is a category with

- $\mathcal{Q}$-posets as objects.
- $\mathcal{Q}$-functors as morphisms.

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Ideals

Definition (Cauchy net)

A net (c_i) in a $\mathcal{Q}$-poset X is Cauchy if
$\forall_{\varepsilon \prec \mathbf{1}}\ \exists_n\ \forall_{i>j>n}\ \varepsilon \prec X(c_j, c_i)$.

Definition (Ideal)

A $\mathcal{Q}$-distributor ϕ is an ideal if there exists a Cauchy net satisfying:

$$\phi(x) = \bigvee_i \bigwedge_{j>i} X(x, c_j)$$

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Ideals

Example ($\mathbf{2}$-ideal)

$\mathbf{2}$-ideal is a characteristic map of an order ideal.

Example ($[0, \infty]$-ideal)

In complete metric spaces a $[0, \infty]$-ideal is the distance from the limit of a Cauchy net.

Suprema

Definition (Supremum)

An ideal ϕ has a supremum $\mathcal{S}\phi$ if for all x we have:
$X(\mathcal{S}\phi, x) = \widehat{X}(\phi, X(-, x))$

Example (2-supremum)

A **2**-supremum is just a supremum of a set.

Example ($[0, \infty]$-supremum)

A $[0, \infty]$-supremum is the limit of a Cauchy sequence.

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$\mathcal{Q}$-dcpo

Definition ($\mathcal{Q}$-dcpo)

A $\mathcal{Q}$-poset X is a $\mathcal{Q}$-dcpo iff every ideal over X has a supremum.

Continous $\mathcal{Q}$-functor

Definition (Continous $\mathcal{Q}$-functor)

A $\mathcal{Q}$-functor f is continous iff it preservs suprema i.e.

$$f(\mathcal{S}\phi) = \mathcal{S}\underline{f}(\phi)$$

$\mathcal{Q}$-DCPO

Definition ($\mathcal{Q}$-DCPO)

A $\mathcal{Q}$**-DCPO** is a category with

- $\mathcal{Q}$-dcpos as objects.
- Continous $\mathcal{Q}$-functors as morphisms.

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Outline

Bilimit theorem

We will see a generalisation of a theorem that is described as:

> *"One of the fundamental tools of domain theory, which plays its most prominent role in the solution of recursive domain equations."*

Expanding sequence

Definition (Expanding sequence – informal)

An expanding sequence is an step by step growth of a poset with following constrains:

- The order is preserved.
- New points appear above existing ones.

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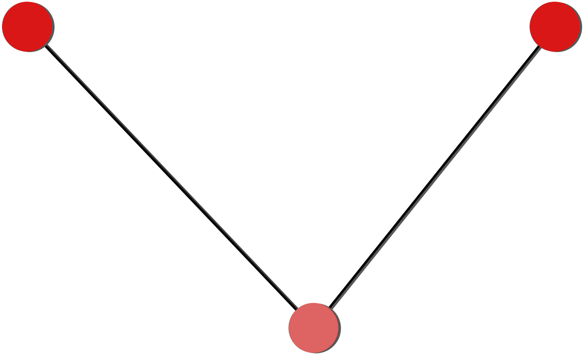

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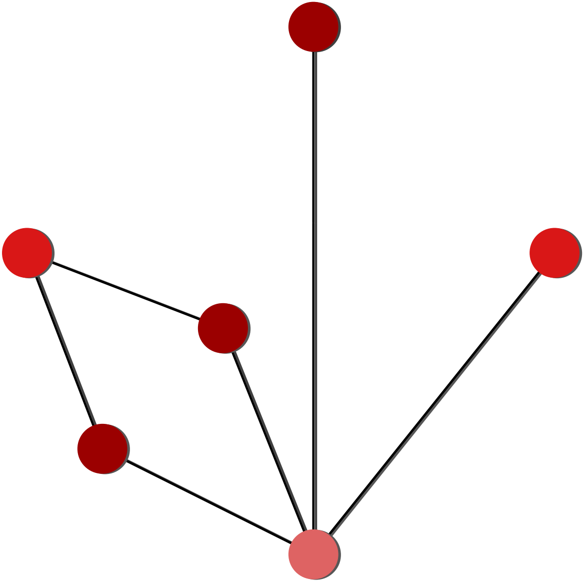

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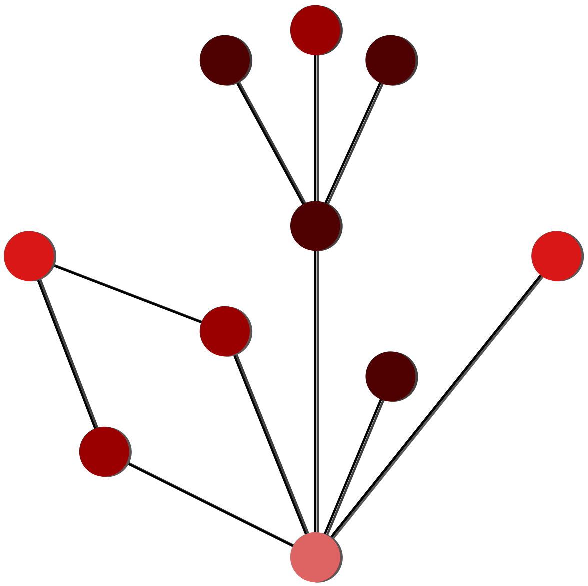

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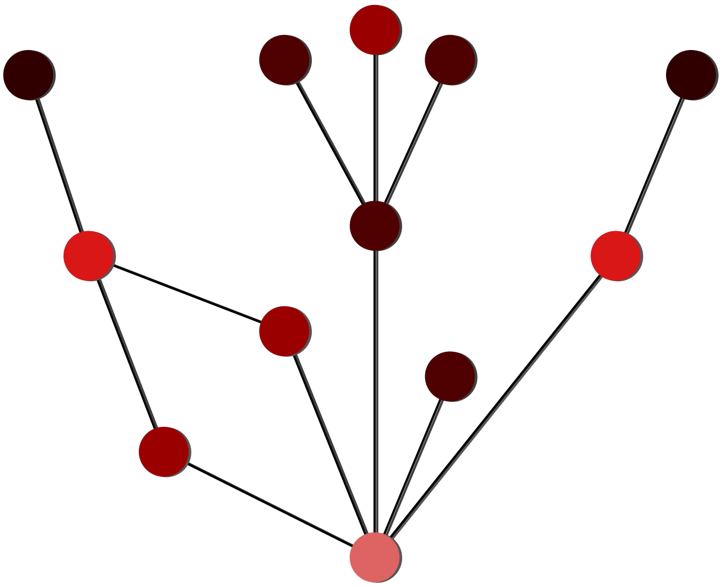

Bilimit of an expanding sequence

Definition (Bilimit – informal)

A bilimit of a given sequence is the smallest dcpo that contains (as induced subgraph) every element of the sequence.

Example of a bilimit of an expanding sequence

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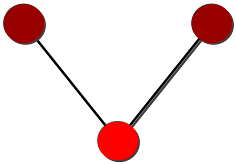

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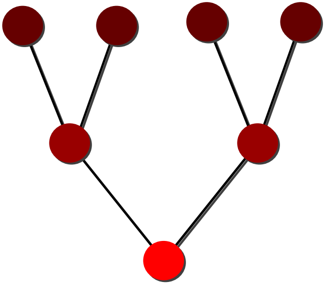

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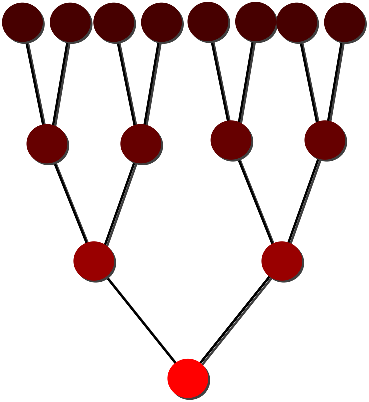

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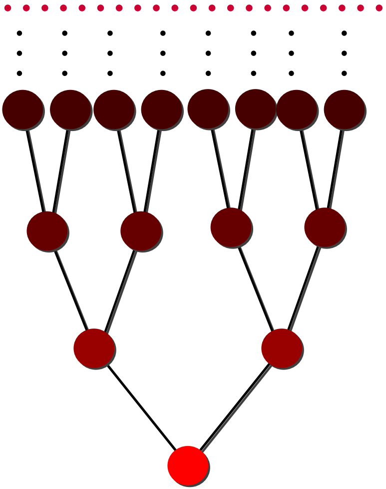

Bilimit in **DCPO**

Theorem

Each expanding sequence in the category ***DCPO*** *has a bilimit.*

Outline

Expanding sequence in $\mathcal{Q}$-DCPO

Definition

Expanding sequence in $\mathcal{Q}$**-DCPO**

$$D_0 \underset{p_{01}}{\overset{e_{10}}{\rightleftarrows}} D_1 \underset{p_{12}}{\overset{e_{21}}{\rightleftarrows}} D_2 \underset{p_{23}}{\overset{e_{32}}{\rightleftarrows}} D_3 \rightleftarrows \cdots$$

1. $e_{kn} \circ e_{nm} = e_{km}$,
2. $p_{mn} \circ p_{nk} = p_{mk}$,
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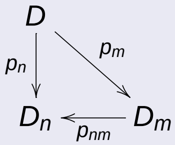
Limit – Colimit

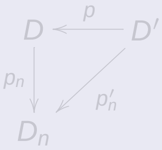
Definition (Limit)

A $\mathcal{Q}$-dcpo D is a limit of an expanding sequence iff

1. There exist $\mathcal{Q}$-functors (p_n) such that following diagram commutes:



2. Furthermore any other candidate D' together with family of $\mathcal{Q}$-functors (p'_n), that satisfies previous condition, factorises through (p_n)



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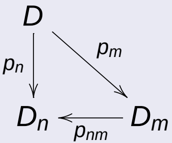

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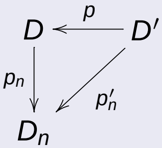

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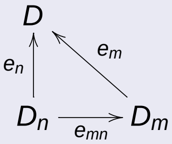
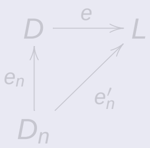


2. Furthermore any other candidate D' together with family of $\mathcal{Q}$-functors (e'_n), so that there exists $\mathcal{Q}$-functor e such that following diagram commutes.



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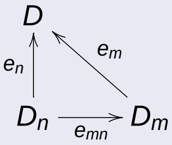
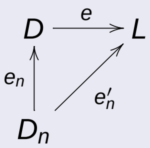


2. Furthermore any other candidate D' together with family of $\mathcal{Q}$-functors (e'_n), so that there exists $\mathcal{Q}$-functor e such that following diagram commutes.



Product

Theorem

*The category $\mathcal{Q}$-**DCPO** is closed under the categorical product. The distance in $\prod_{i \in I} X_i$ is given by:*

$$\prod_{i \in I} X_i((x_j)_{j \in I}, (y_j)_{j \in I}) := \bigwedge_{i \in I} X_i(x_i, y_i)$$

Bilimit

Our candidate

$$D := \left\{ (x_0, x_1, \ldots) \in \prod_{i \in \mathbb{N}} D_i \;\middle|\; p_{ii+1}(x_{i+1}) = x_i \right\}.$$

Lemma

- D is a $\mathcal{Q}$-dcpo.
- Functors e_n and p_n exist and commute as they should.
- Functors e_n and p_n are continuous.
- D is the limit and the colimit.

Bilimit

Our candidate

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Theorem

*The category $\mathcal{Q}$-**DCPO** has bilimits of expanding sequences.*

Corollary

The category of complete quasimetric spaces and nonexpansive maps have bilimits of ω-diagrams.

Bilimit

Theorem

*The category $\mathcal{Q}$-**DCPO** has bilimits of expanding sequences.*

Corollary

The category of complete quasimetric spaces and nonexpansive maps have bilimits of ω-diagrams.

Outline

Summary

- The theorem is true for any quantale.
- For quasimetrics existence of bilimits is a new result.
- Unfortunately in the current shape this theorem trivialises in case of symmetric distances.

THANK YOU.