Quantitative comparison of classical and intuitionistic logics*

Full propositional system

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1. Tautologies
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   - Intuitionistic logic
2. Previous results
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   - Tautologies and *implication*
3. Theorem and sketch of the proof
   - Definitions
   - Theorem
   - Simple tautologies
   - Proof’s sketch
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Tautologies

Formulas

- k variables: $x_1, \ldots, x_k$ and the constant *false*: $\perp$

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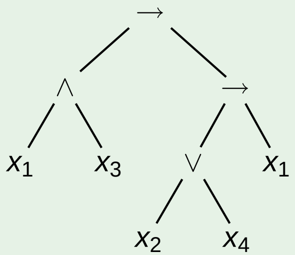

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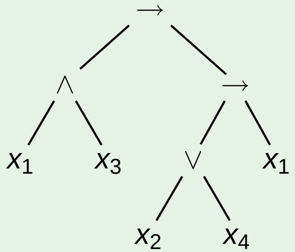
- 

- classical tautology: formula which computes always *true*

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$\Rightarrow$ this proof is not intuitionistically valid

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$\Rightarrow$ the information ($\phi \vee \neg\phi$) is not interesting.

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When k tends to infinity,
the probability that a random tautology is intuitionistic
is equal to $\frac{5}{8}$.

Previous results

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  - Improvement of the bounds

    [Chauvin, Flajolet, Gardy, Gittenberger. *And / Or trees revisited*, 2005]

And/Or trees: Tautologies

- Lower bound of tautologies
  - $\frac{1}{16k} \leqslant \pi_k(\text{true})$
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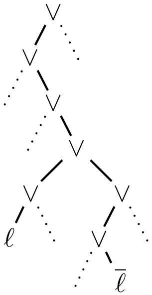
- Almost all tautologies are simple (when $k \to \infty$)
  - there exist 2 paths containing only $\vee$ from the root to ℓ and $\overline{\ell}$



  - [Woods, 2005; Kozik, 2008]

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  - $\lim_{n\to\infty} \frac{Cl_1(n)}{\mathcal{F}_1(n)} = \frac{1}{2} + \frac{\sqrt{5}}{10} \approx 0.7236\ldots$
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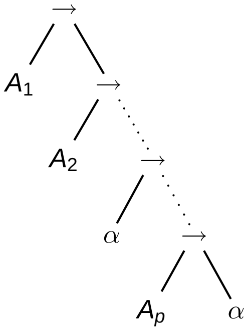
- For two variables
  - $\lim_{n\to\infty} \frac{Int_2(n)}{Cl_2(n)} \approx 0.97\ldots$
  - [Kostrzycka. *On the density of implicational parts of intuitionistic and classical logics*, 2003]

Tautologies and *implication*

- For k variables
  - Almost all tautologies are simple (when $k \to \infty$)



  - $\lim_{k\to\infty}\left(\lim_{n\to\infty}\frac{Int_k(n)}{Cl_k(n)}\right) = 1$
  - [Gardy, Fournier, G., Zaionc. *Classical and intuitionistic logics are asymptotically identical*, 2007]

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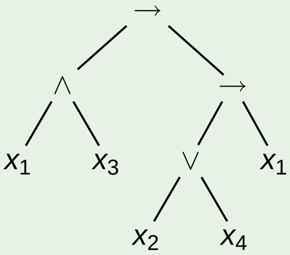
- Size of a formula = number of occurrences of variables or of $\bot$

Example

Formula : $(x_1 \wedge x_3) \rightarrow (x_2 \vee x_4) \rightarrow x_1)$



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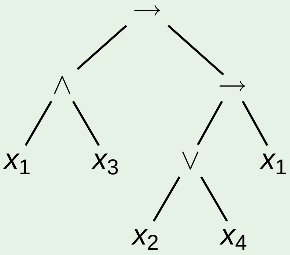

Size: 5

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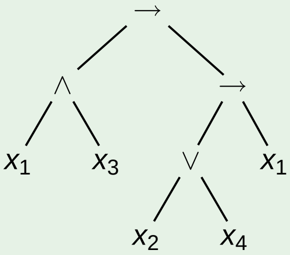

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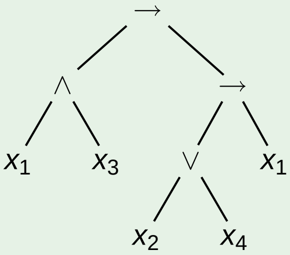

Tautology:

- if $x_1 = 1$ then the formula is *true*
- if $x_1 = 0$ then the formula is still *true*

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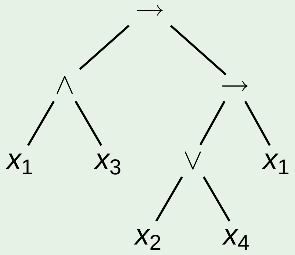

Tautology:

- it is a classical tautology
- it is a intuitionistic tautology (natural deduction, Heyting algebra)

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- Peirce's law

$$((x_1 \to x_2) \to x_1) \to x_1$$

Asymptotic density

Asymptotic density of tautologies

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We conjecture the existence of the limit.

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Definitions:

$$d_k^- = \liminf_{n\to\infty} \frac{Int_k(n)}{Cl_k(n)},$$

$$d_k^+ = \limsup_{n\to\infty} \frac{Int_k(n)}{Cl_k(n)}.$$

Results

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Interpretation:

Asymptotically, about $5/8$ of classical tautologies are intuitionistic.

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Positive path

Path from the root to a node,

- which does not contain $\wedge$-node,
- which does never go in the left subtree of a $\rightarrow$-node.

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Path containing a positive node h, labelled by $\rightarrow$,

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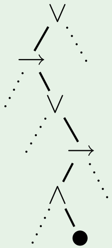
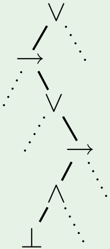


Simple tautologies

Family $\mathcal{S}_\perp$

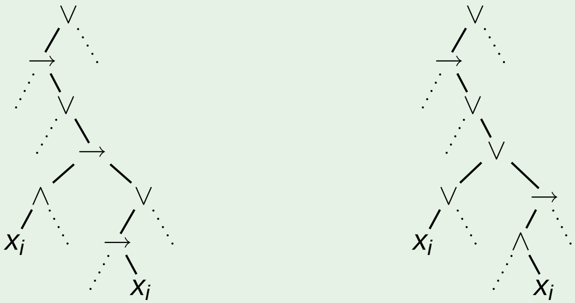
Expressions containing a negative leaf, labelled by $\perp$.



Simple tautologies

Family $\mathcal{S}_R$

Expressions containing a positive leaf and a negative leaf, both labelled by the same variable x_i.



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$x_1 \vee (x_1 \to x_2)$ is not intuitionistic.

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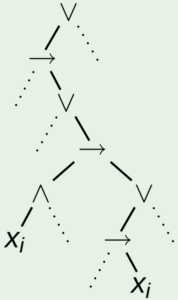

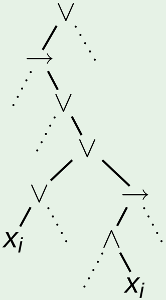

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  - we get conditions on some leaves
  - then we prove that this family has a negligible density.

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- if $\vee$ is one of the connective, then
  - $\{\to,\vee\}$: the limit is equal to $3/13$
  - $\{\to,\vee\}$ et $\bot$: the limit is equal to $2/7$
  - $\{\to,\vee,\wedge\}$: the limit is equal to $5/11$
  - $\{\to,\vee,\wedge\}$ et $\bot$: the limit is equal to $5/8$

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