On-line chain partitioning problems

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Chain partitioning problem

Definition

Chain partitioning of a given partial order P is a set $\mathcal{C}$ of disjoint chains such that $P = \bigcup \mathcal{C}$.

Theorem (Dilworth)

The least number of chains needed to partition a given order P is equal to the width of P (the maximal size of an antichain in P).

On-line chain partitioning problem

Definition

- the partial order is introduced (by Spoiler) point by point at each step
- each new point must be assign (by Algorithm) to the one of chains, decisions of Algorithm cannot be changed in the future

Problem

What is the smallest number $val(w)$ of chains enought to partition each order of width w introduced in on-line way?

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Best bounds for $val(w)$

Theorem (Kierstead 1981)

$$val(w) \leqslant \frac{5^w - 1}{4}$$

Known upper bounds for small cases:

- $val(2) = 5$ Felsner 1997 (with use $val(w) \geqslant 4w - 3$, Kierstead)
- $9 \leqslant val(3) \leqslant 16$ Bosek 2008

Theorem (Szemerédi, published by Kierstead in 1986)

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The main open problem

$$\binom{w+1}{2} \leqslant val(w) \leqslant \frac{5^w - 1}{4}$$

$$val(w) = ?$$

Restriction on Spoiler's moves — the up-growing case

Definition (Felsner)

An on-line order is presented in an up-growing way iff each new point is maximal at a moment of its arrival.

Theorem (Felsner 1997)

$val(w) = \binom{w+1}{2}$ *when Spoiler presents up-growing order.*

In 2007 Agarwal and Garg presented a very simple proof for the upper bound.

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Interval orders

Definition

Partial order P is an interval order if there is a function f which assigns to each $x \in X$ a closed interval $f(x) = [a_x, b_x]$ so that $x < y$ in P iff $b_x < a_y$.

P is an interval order iff it does not contain $\mathbf{2} + \mathbf{2}$.

Example

Interval order together with its interval representation.

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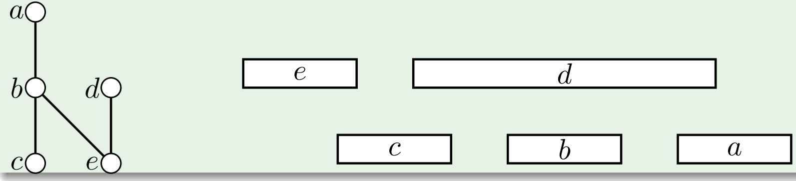

On-line chain partitioning problem for interval orders

Theorem (Kierstead, Trotter 1981; Chrobak, Ślusarek 1988)

The value of on-line chain partitioning problem for interval orders of width w is $3w - 2$.

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Theorem (Bair, Bosek, Micek 2007)

The value of on-line chain partitioning problem for up-growing, interval orders of width w is $2w - 1$.

Interval orders - results of on-line chain partitioning

class	U	R	value	remarks
interval orders			$3w-2$	Kierstead, Trotter 1981
interval orders		$\times$	$3w-2$	Chrobak, Ślusarek 1988
interval orders	$\times$		$2w-1$	Baier, Bosek, Micek 2007
interval orders	$\times$	$\times$	w	Broniek 2005

Semi-orders — definition and examples

Definition

An interval order P is called a semi-order if it has a unit interval representation.

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Semi-order together with its interval representation.

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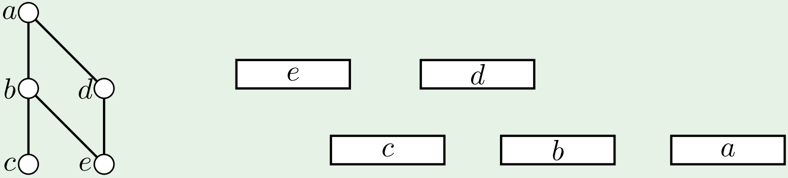
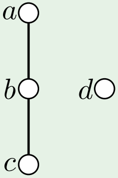
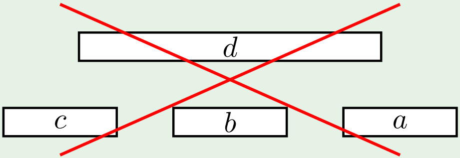


Semi-orders — continued

Example

The following $\mathbf{3}+\mathbf{1}$ interval order is not a semi-order:





Theorem

Semi-orders are exactly these interval orders which do not contain a $\mathbf{3}+\mathbf{1}$ order as a subposet.

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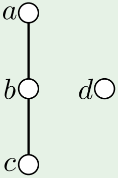

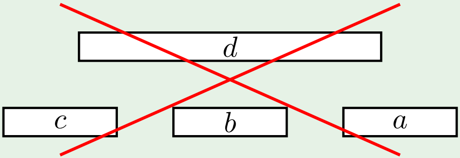

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Theorem (Felsner, Kloch, Matecki, Micek)

The value of on-line chain partitioning problem for up-growing, semi-orders of width w is $\lfloor \frac{1+\sqrt{5}}{2} w \rfloor$.

Semi-orders - results of on-line chain partitioning

class	U	R	value	remarks
semi-orders			$2w-1$	obvious
semi-orders		$\times$	$?$	$\left\lfloor \frac{3}{2}w \right\rfloor \leqslant ? \leqslant 2w-1$
semi-orders	$\times$		$\left\lfloor \frac{1+\sqrt{5}}{2}w \right\rfloor$	Felsner, Kloch, Matecki, Micek
semi-orders	$\times$	$\times$	w	obvious

First-Fit algorithm on $(\mathbf{k}+\mathbf{k})$-free orders.

Theorem (Bosek, Krawczyk, Szczypka 2009)

First-Fit algorithm uses at most $4kw^2$ on $(\mathbf{k}+\mathbf{k})$-free orders of width w.

The case $k = 2$ (interval orders) has been studied for many years and by many authors. The best known bounds are:

- $FF(w) \geqslant 4.4w - c$ by Chrobak and Ślusarek 1988,
- $FF(w) \geqslant 4.99w - c$ by Kierstead and Trotter (resently),
- $FF(w) \leqslant 8w$ by Narayanaswamy and Babu 2008.

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Bosek and Krawczyk reports that $val(w) = O(w^{1+\log w})$.

Thank you.