Q-categories as generalized domains

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Ralph Kopperman's advice

MR. MCQUIRE
Ben - I just want to say one word to you - just one word -

BEN
Yes, sir.

MR. MCQUIRE
Are you listening?

BEN
Yes I am.

MR. MCQUIRE
(gravely)
Plastics.

They look at each other for a moment.

BEN
Exactly how do you mean?

MR. MCQUIRE
There is a great future in plastics. Think about it. Will you think about it?

BEN
Yes, I will.

MR. MCQUIRE
Okay. Enough said. That's a deal.
Quasi-uniformities

\[ U \subseteq \mathcal{P}(X \times X) \text{ is a quasi-uniformity if:} \]

- \( \forall V \in U \quad \Delta \subseteq V, \)
- \( \forall U \in U \quad \exists V \in U \quad V \cdot V \subseteq U. \)
EXAMPLES

EXAMPLE 1. $\mathcal{U} = \{ A \}$ iff $A$ is a preorder.

EXAMPLE 2. In a metric space $X$, for

$$V_\varepsilon = \{(x, y) \mid X(x, y) < \varepsilon\}$$

define

$$\mathcal{U} = \{ V_\varepsilon \mid \varepsilon > 0 \}.$$
Quasi-uniform spaces as domains of computation


Lawvere's famous 1973 paper

Ordered sets and metric spaces are Q-enriched cats

I WILL:

- Introduce
- Introduce
- Introduce

\[ Q = (Q, \leq, \otimes, 1) \]

Q-Rel

Q-Cat
A quantale is something that resembles non-negative real numbers.

\[ Q = (Q, \leq, \otimes, 1) \]

- \( Q \) is a complete lattice
- has addition \( \otimes \)
- addition has unit \( 1 \)
- \( a \otimes \bigvee S = \bigvee \{ a \otimes s \mid s \in S \} \)
The category $\mathcal{Q}$-$\text{Rel}$:

- objects: sets $X, Y, Z, \ldots$

- morphisms: $r : X \rightarrow Y$ is just a function $r : X \times Y \rightarrow \mathcal{Q}$

- composition:
  for $r : X \rightarrow Y$ and $s : Y \rightarrow Z$

  $$(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z).$$

There is a functor $\text{Set} \rightarrow \mathcal{Q}$-$\text{Rel}$ which maps objects identically and interprets a map $f : X \rightarrow Y$ as a $\mathcal{Q}$-relation $f : X \rightarrow Y$:

$$f(x, y) = \begin{cases} 1 & \text{if } f(x) = y, \\ \bot & \text{otherwise}. \end{cases}$$
For example...

- $2$-$\text{Rel}$ is isomorphic to the category of relations

- $[0,1]$-$\text{Rel}$ is isomorphic to the category of fuzzy relations
A $Q$-category

$$X = (X, X(-, -))$$

is a set $X$ with a $Q$-relation $X : X \to X$ satisfying:

$$1_X \leq X \quad \text{(reflexivity)}$$

$$X \cdot X \leq X \quad \text{(transitivity)}.$$ 

A $Q$-functor $f : X \to Y$ must satisfy

$$f \cdot X \leq Y \cdot f.$$ 

$2$-$

Cat$ is the category $\text{Ord}$ of (pre)orders. $[0, \infty]$-$\text{Cat}$ is the category of (pre)metric spaces.

For the trivial quantale: $1$-$\text{Cat} = \text{Set}$. 
The category of $Q$-categories

$\mathbf{Q-Cat}$ is symmetric monoidal closed with tensor product

$$X \otimes Y((x, y), (a, b)) = X(x, a) \otimes Y(y, b)$$

and internal hom:

$$Y^X(f, g) = \bigwedge_{x \in X} Y(f x, g x).$$

The internal hom describes the pointwise order if $Q = 2$, and the non-symmetrized sup-metric if $Q = [0, \infty]$ or $Q = [0, 1]$. 


My research idea is to...

- phrase domain theory in the language of 2-relations and...
- ...change 2-relations to arbitrary Q-relations...
- ...to obtain results for the general case.
- There are a lot of things that can go wrong, but in most cases it works!
Joint work

- (with Dirk Hofmann)
  We introduce continuous Q-cats

- (with Mateusz Kostanek)
  The Bilimit Theorem: every expanding sequence of Q-cats has a bilimit

- Theorem. $Q\text{-}\text{Cat}$ is cartesian closed iff $\otimes = \wedge$
A $Q$-distributor $\varphi : X \rightarrow Y$ is a $Q$-relation $\varphi : X \rightarrow Y$ with

$$\varphi \cdot X = \varphi = Y \cdot \varphi.$$ 

The $Q$-distributor $X : X \rightarrow Y$ plays the role of the identity in the category $Q$-Dist of sets and $Q$-distributors.

For a $Q$-functor $f : X \rightarrow Y$, both

$$f_* = Y \cdot f \quad \text{and} \quad f^* = f^o \cdot Y$$

are $Q$-distributors.
The following are equivalent for $\mathcal{Q}$-relations $\varphi: X \rightarrow Y$ between $\mathcal{Q}$-categories.

- $\varphi: X \rightarrow Y$ is a $\mathcal{Q}$-distributor.
- $\varphi: X^{\text{op}} \otimes Y \rightarrow \mathcal{Q}$ is a $\mathcal{Q}$-functor.

**EXAMPLE.** For $\mathcal{Q} = 2$, a $\mathcal{Q}$-distributor from $X$ to $1$ corresponds to a monotone map from $X^{\text{op}}$ to $2$, that is, to a lower set in $X$. 
Let \( \hat{X} \) denote the \( Q \)-category of all \( Q \)-distributors \( X \rightarrow 1 \) (i.e. all \( Q \)-functors \( X^{\text{op}} \rightarrow Q \)):

\[
\hat{X}(\phi, \psi) = \bigwedge_{x \in X} Q(\phi x, \psi x).
\]

\( X \) embeds into \( \hat{X} \) via the Yoneda \( Q \)-functor:

\[
X \ni x \mapsto yx := x^* \in \hat{X}
\]

**DEFINITION.** A \( Q \)-category \( X \) is **Cocomplete** if \( y : X \rightarrow \hat{X} \) has a left adjoint \( S : \hat{X} \rightarrow X : S \dashv y \).
We consider any subcategory $J$ of $Q$-$\mathbf{Dist}$ such that for all $Q$-functors $f$:

$$f^* \in J$$

and

$$(X \xrightarrow{\varphi} Y \xrightarrow{y^*} 1) \in J \Rightarrow (X \xrightarrow{\varphi} Y) \in J.$$ 

Define

$$J(X) = \{\varphi : X \rightarrow 1 \mid \varphi \in J\}$$

and

$$J_S(X) = \{\varphi \in J(X) \mid \varphi \text{ has a supremum}\}$$

A $Q$-category $X$ is

**J-COCOMPLETE**

iff

$$S : J_S(X) \rightarrow X \quad \dashv \quad y : X \rightarrow J_S(X).$$
EXAMPLE 1. Choose \( J = \mathcal{Q} - \text{Dist.} \).
\( J \)-cocomplete means cocomplete.

EXAMPLE 2. For \( \mathcal{Q} = \mathbb{2} \), choose

\[
J = \{ \phi : X \rightarrow 1 \mid \phi \text{ is an ideal} \}.
\]

\( J \)-cocomplete means directed-complete.

EXAMPLE 3. For \( \mathcal{Q} = [0, \infty] \), choose

\[
J = \{ \inf_i \sup_{j \geq i} X(-, x_j) \mid (x_i) \text{ is Cauchy} \}
\]

\( J \)-cocomplete means metrically complete.

ddetcetera
A $\mathcal{Q}$-functor $f : X \rightarrow Y$ is Scott-continuous if for all $\varphi \in J_S(X)$,

$$f(S\varphi) = S(\varphi \cdot f^*).$$

PROPOSITION. Let $Y$ be a $J$-cocomplete category. Then any $\mathcal{Q}$-functor $f : X \rightarrow Y$ uniquely extends to a Scott-continuous $\mathcal{Q}$-functor $F : J(X) \rightarrow Y$.

THEOREM. The inclusion functor

$$\text{J-Cocont} \rightarrow \mathcal{Q}\text{-Cat}$$

has a left adjoint which sends a $\mathcal{Q}$-category $X$ to $J(X)$ and a $\mathcal{Q}$-functor $f$ to $(-) \cdot f^*$. 
A fixpoint theorem

THEOREM. Let $X$ be $J$-cocomplete, where
$J = \{ \forall_i \wedge_{j \geq i} X(-, x_j) \mid (x_i) \text{ is Cauchy}\}$. Let $f : X \to X$ be a $Q$-functor. If

$$\phi \cdot f^* = \phi$$

for some $Q$-distributor $\phi : X \to 1$ such that $\phi \in J$, then $f$ has a fixed point.

PROOF: Define $x \leq y$ iff $1 = X(x, y)$. By $J$-cocompleteness, $S\phi$ exists. Thus:

$$S\phi = S(\phi \cdot f^*) \leq f(S\phi) \leq f^2(S\phi) \leq \ldots$$

However, by the choice of $J$, $(X, \leq)$ is directed-complete, and $f$ is monotone wrt $\leq$, so $f$ has a fixpoint by the usual argument using transfinite induction.
Suppose $X$ is a complete (quasi-)metric space. Let $f: X \to X$ be a contraction. Let $x \in X$. Then:

$$\phi = \bigvee_{i} \bigwedge_{j \geq i} X(-, f^j x)$$

is an element of $J$, and satisfies:

$$\phi \cdot f^* = \phi.$$ 

Moreover,

$$f(S\phi) = S(\phi \cdot f^*) = S\phi$$

is the fixed point. It is unique, since $f$ is a contraction.

**Corollary.** A contraction on a complete metric space has a unique fixed point.
THE END