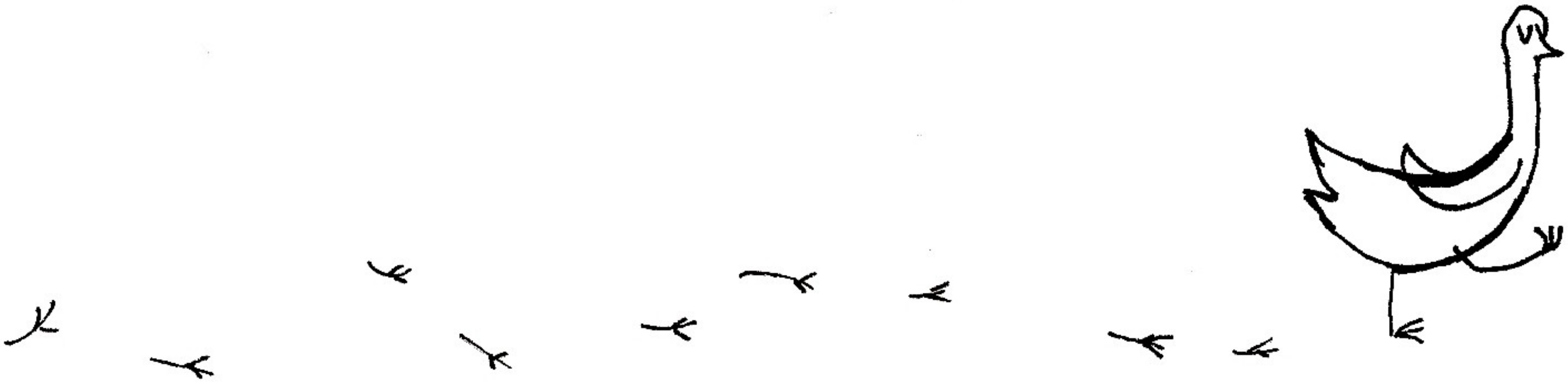
Q-categories as generalized domains

Paweł Waszkiewicz



Ralph Kopperman's advice

MR. MCQUIRE
Ben - I just want to say one word to you - just one word -

BEN
Yes, sir.

MR. MCQUIRE
Are you listening?

BEN
Yes I am.

MR. MCQUIRE
(gravely)
Plastics.

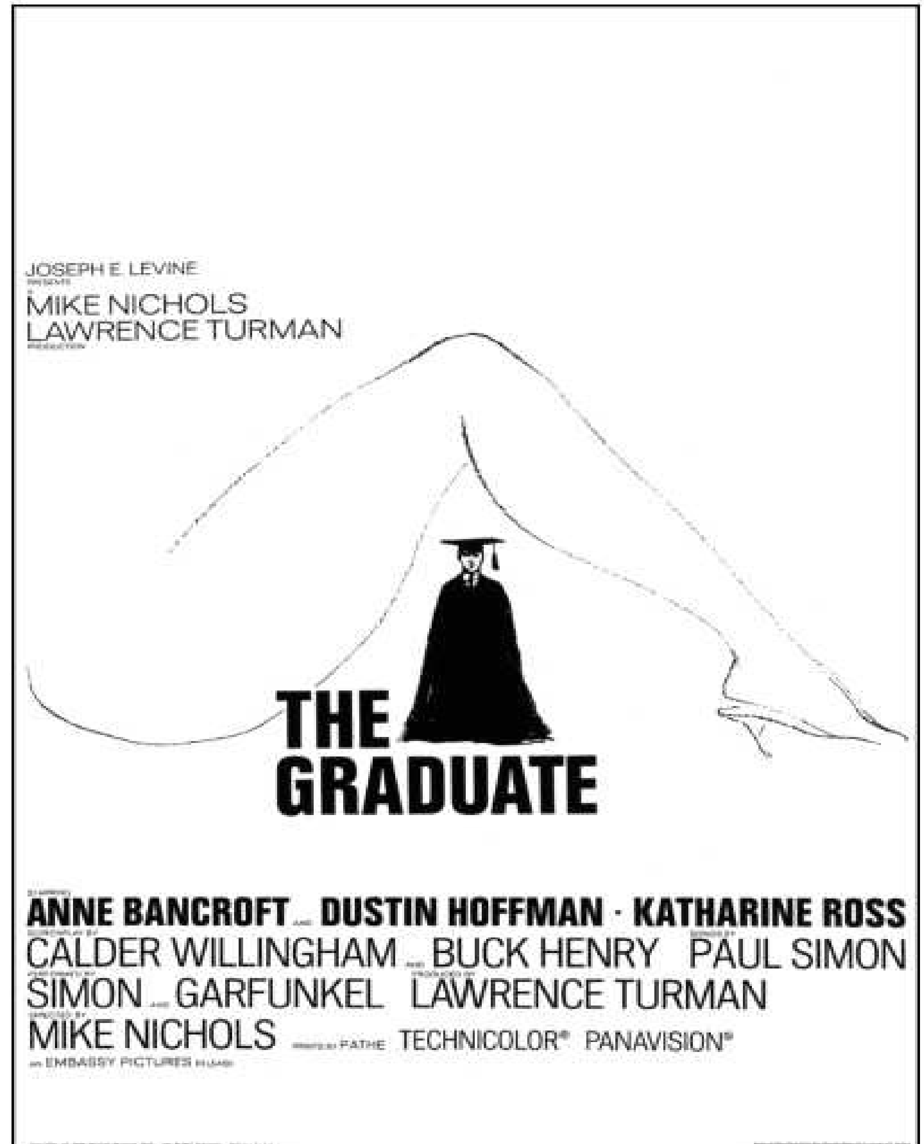
They look at each other for a moment.

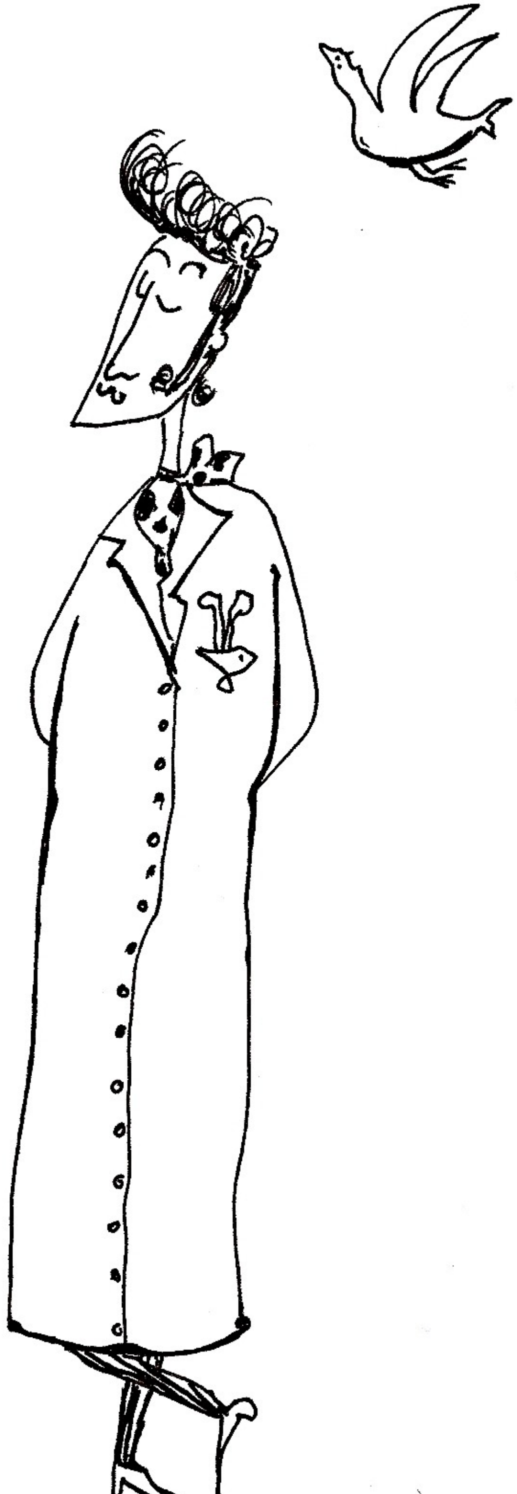
BEN
Exactly how do you mean?

MR. MCQUIRE
There is a great future in plastics. Think about it. Will you think about it?

BEN
Yes, I will.

MR. MCQUIRE
Okay. Enough said. That's a deal.



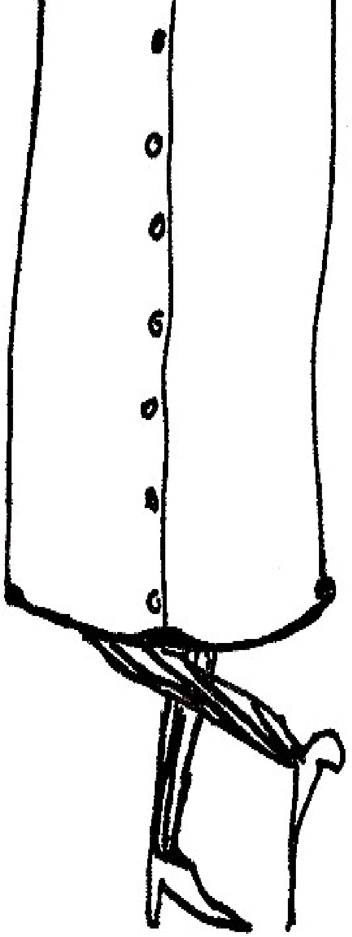


Quasi-uniformities

$\mathcal{U} \subseteq \mathcal{P}(X \times X)$ is a quasi-uniformity if:

- $\forall V \in \mathcal{U} \;\; \Delta \subseteq V$,
- $\forall U \in \mathcal{U} \;\; \exists V \in \mathcal{U} \;\; V \cdot V \subseteq U$.

EXAMPLES



EXAMPLE 1. $\mathcal{U} = \{A\}$ iff A is a preorder.

EXAMPLE 2. In a metric space X, for

$$V_\varepsilon = \{(x, y) \mid X(x, y) < \varepsilon\}$$

define

$$\mathcal{U} = \{V_\varepsilon \mid \varepsilon > 0\}.$$

Quasi-uniform spaces as domains of computation

[Smy88] Smyth, M.B. (1988) Quasi-uniformities: Reconciling Domains and Metric Spaces. *Lecture Notes in Computer Science* **298**, pp. 236–253.

[Smy91] Smyth, M.B. (1991) Totally bounded spaces and compact ordered spaces as domains of computation. In G.M. Reed, A. W. Roscoe, and R. F. Wachter, editors, Topology and Category Theory in Computer Science, pp. 207–229. Clarendon Press.

[Smy94] Smyth, M.B. (1994) Completeness of quasi-uniform and syntopological spaces. *Journal of the London Mathematical Society* **49**, pp. 385–400.

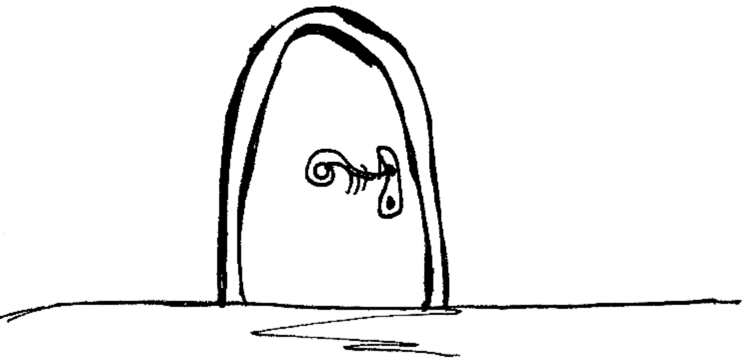
Lawvere's famous 1973 paper

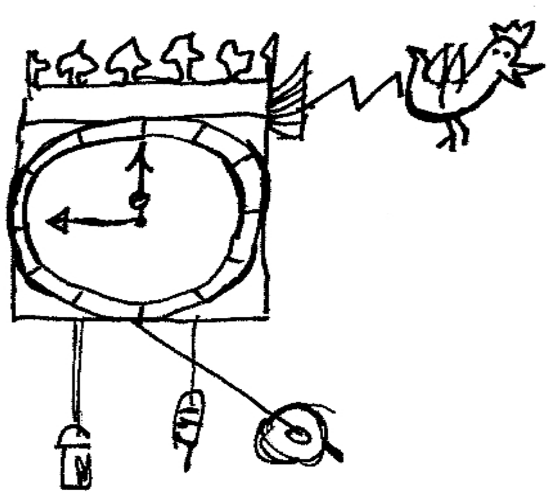
Ordered sets and metric spaces are Q-enriched cats

Lawvere, F.W. (1973) Metric spaces, generalized logic, and closed categories. Rend. Sem. Mat. Fis. Milano 43, pp. 135—166.

I WILL:

- Introduce $\mathcal{Q} = (Q, \leqslant, \otimes, 1)$
- Introduce $\mathcal{Q}$-$\mathbf{Rel}$
- Introduce $\mathcal{Q}$-$\mathbf{Cat}$



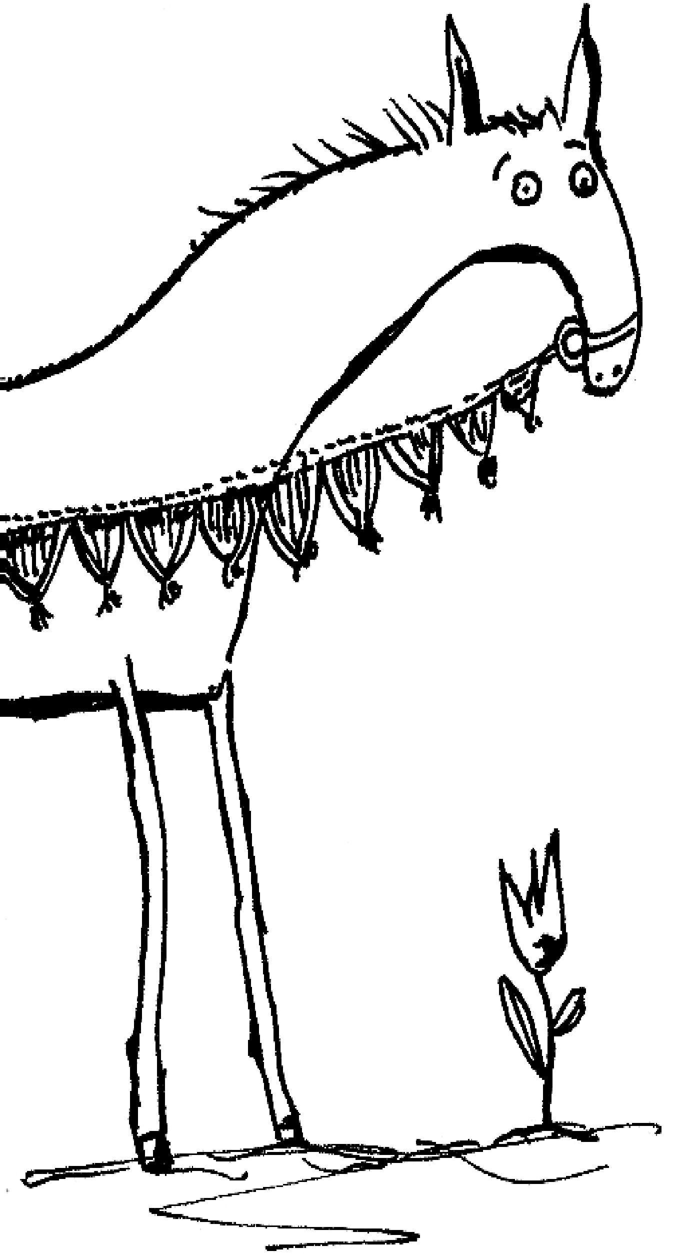


QUANTALES

A quantale is something that resembles non-negative real numbers.

$$\mathcal{Q} = (Q, \leqslant, \otimes, \mathbf{1})$$

- Q is a complete lattice
- has addition $\otimes$
- addition has unit $\mathbf{1}$
- $a \otimes \bigvee S = \bigvee \{a \otimes s \mid s \in S\}$



The category $\mathcal{Q}$-$\mathbf{Rel}$:

- objects: sets $X, Y, Z, \ldots$
- morphisms: $r \colon X \longrightarrow\!\!\!\!\!\!\!+\;\, Y$ is just a function $r \colon X \times Y \to \mathcal{Q}$
- composition: for $r \colon X \longrightarrow\!\!\!\!\!\!\!+\;\, Y$ and $s \colon Y \longrightarrow\!\!\!\!\!\!\!+\;\, Z$

$$(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z).$$

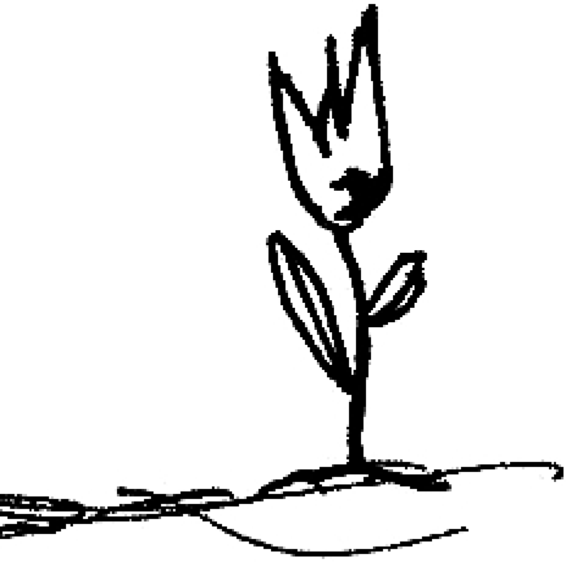
There is a functor $\mathbf{Set} \to \mathcal{Q}$-$\mathbf{Rel}$ which maps objects identically and interprets a map $f : X \to Y$ as a $\mathcal{Q}$-relation $f \colon X \longrightarrow\!\!\!\!\!\!\!+\;\, Y$:

$$f(x, y) = \begin{cases} \mathbf{1} & \text{if } f(x) = y, \\ \bot & \text{otherwise.} \end{cases}$$

For example...

- 2-Rel is isomorphic to the category of relations

- [0,1]-Rel is isomorphic to the category of fuzzy relations



A $\mathcal{Q}$-category

$$X = (X, X(-,-))$$

is a set X with a $\mathcal{Q}$-relation $X\colon X \longrightarrow\!\!\!\!\!\!\mapsto X$ satisfying:

$$1_X \leqslant X \ \ \text{(reflexivity)}$$

$$X \cdot X \leqslant X \ \ \text{(transitivity)}.$$

A $\mathcal{Q}$-functor $f\colon X \to Y$ must satisfy

$$f \cdot X \leqslant Y \cdot f.$$

$\mathbf{2}$-$\mathbf{Cat}$ is the category $\mathbf{Ord}$ of (pre)orders. $[0,\infty]$-$\mathbf{Cat}$ is the category of (pre)metric spaces.

For the trivial quantale: $\mathbf{1}\text{-}\mathbf{Cat} = \mathbf{Set}$.

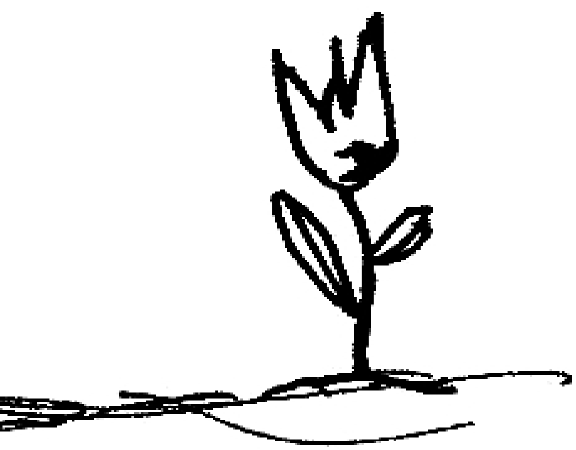
The category of Q-categories

$\mathcal{Q}$-**Cat** is symmetric monoidal closed with tensor product

$$X \otimes Y((x,y),(a,b)) = X(x,a) \otimes Y(y,b)$$

and internal hom:

$$Y^X(f,g) = \bigwedge_{x \in X} Y(fx,gx).$$



The internal hom describes the pointwise order if $\mathcal{Q} = \mathbf{2}$, and the non-symmetrized $\sup$-metric if $\mathcal{Q} = [0,\infty]$ or $\mathcal{Q} = [0,1]$.

The Amsterdam Group

[AR89] America, P. and Rutten, J.J.M.M. (1989) Solving reflexive domain equations in a category of complete metric spaces, *Journal of Computer and System Sciences* **39**(3), pp. 343–375.

[BvBR96] Bonsangue, M. M., van Breugel, F. and Rutten, J. J. M. M. (1996) Alexandroff and Scott Topologies for Generalized Metric Spaces, Annals of the New York Academy of Sciences, pp. 49–68.

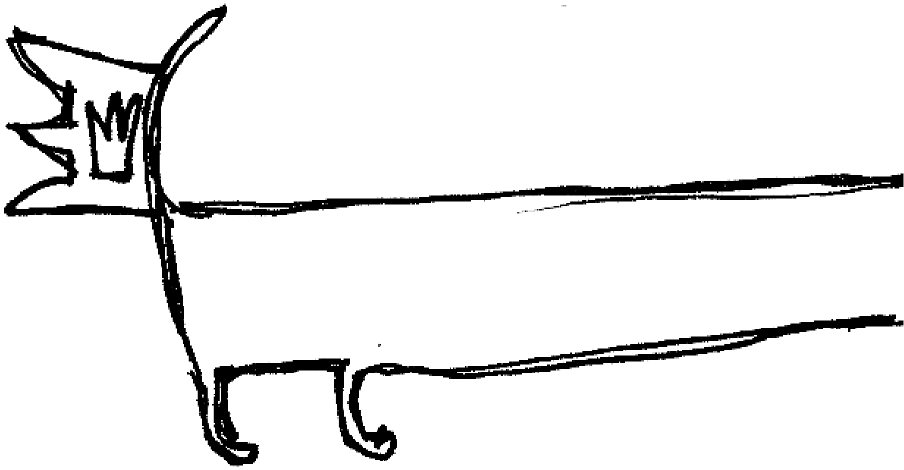
[BvBR98] Bonsangue, M.M., van Breugel, F. and Rutten, J.J.M.M. (1998) Generalized Metric Spaces: Completion, Topology, and Powerdomains via the Yoneda Embedding, *Theoretical Computer Science* **193**(1-2), pp. 1–51.

[Rut96] Rutten, J.J.M.M. (1996) Elements of generalized ultrametric domain theory, *Theoretical Computer Science* **170**, pp. 349–381.

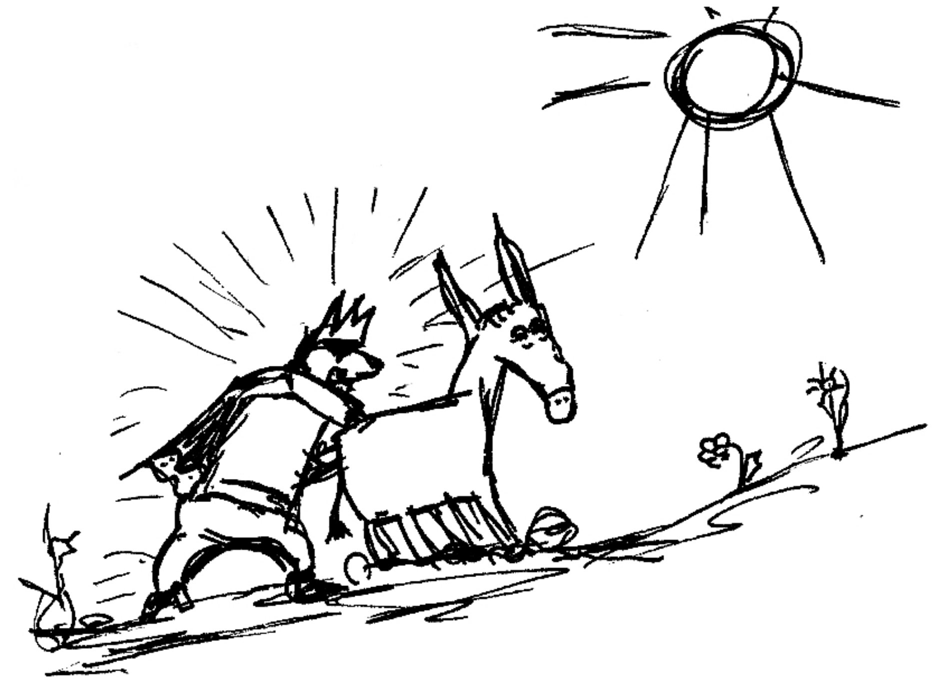
[Rut98] Rutten, J.J.M.M. (1998) Weighted colimits and formal balls in generalized metric spaces, *Topology and its Applications* **89**, pp. 179–202.

My research idea is to...

- phrase domain theory in the language of 2 -relations and...
- ...change 2-relations to arbitrary Q -relations...
- ...to obtain results for the general case.
- There are a lot of things that can go wrong, but in most cases it works!



Joint work



- (with Dirk Hofmann)

We introduce

continuous Q-cats

- (with Mateusz Kostanek)

The Bilimit Theorem: every expanding sequence

of Q-cats has a bilimit

- Theorem. $\mathcal{Q}$-**Cat** is cartesian closed iff $\otimes = \wedge$

A $\mathcal{Q}$-distributor $\varphi\colon X \mathrel{-\!\!\!\circ\!\!\!\rightarrow} Y$ is a $\mathcal{Q}$-relation $\varphi\colon X \mathrel{-\!\!\!+\!\!\!\rightarrow} Y$ with

$$\varphi \cdot X = \varphi = Y \cdot \varphi.$$

The $\mathcal{Q}$-distributor $X\colon X \mathrel{-\!\!\!\circ\!\!\!\rightarrow} X$ plays the role of the identity in the category $\mathcal{Q}$-$\mathbf{Dist}$ of sets and $\mathcal{Q}$-distributors.

For a $\mathcal{Q}$-functor $f\colon X \to Y$, both

$$f_* = Y \cdot f \quad \text{and} \quad f^* = f^\circ \cdot Y$$

are $\mathcal{Q}$-distributors.

The following are equivalent for $\mathcal{Q}$-relations $\varphi\colon X \longrightarrow\!\!\!\!\!\mapsto Y$ between $\mathcal{Q}$-categories.

- $\varphi\colon X \multimap Y$ is a $\mathcal{Q}$-distributor.
- $\varphi\colon X^{\mathrm{op}} \otimes Y \to \mathcal{Q}$ is a $\mathcal{Q}$-functor.

EXAMPLE. For $\mathcal{Q} = \mathbf{2}$, a $\mathcal{Q}$-distributor from X to 1 corresponds to a monotone map from X^{op} to $\mathbf{2}$, that is, to a lower set in X.

Let $\widehat{X}$ denote the $\mathcal{Q}$-category of all $\mathcal{Q}$-distributors $X \multimap 1$ (i.e. all $\mathcal{Q}$-functors $X^{op} \to \mathcal{Q}$):

$$\widehat{X}(\phi, \psi) = \bigwedge_{x \in X} \mathcal{Q}(\phi x, \psi x).$$

X embeds into $\widehat{X}$ via the Yoneda $\mathcal{Q}$-functor:

$$X \ni x \mapsto \mathsf{y}x := x^* \in \widehat{X}$$

DEFINITION. A $\mathcal{Q}$-category X is COCOMPLETE iff $\mathsf{y}\colon X \to \widehat{X}$ has a left adjoint $\mathcal{S}\colon \widehat{X} \to X$:

$$\mathcal{S} \dashv \mathsf{y}.$$

Relative Cocompleteness

We consider any subcategory J of $\mathcal{Q}$-$\mathbf{Dist}$ such that for all $\mathcal{Q}$-functors f:

$$f^* \in J$$

and

$$(X \overset{\varphi}{-\!\!\circ\!\!\!\to} Y \overset{y^*}{-\!\!\circ\!\!\!\to} 1) \in J \Rightarrow (X \overset{\varphi}{-\!\!\circ\!\!\!\to} Y) \in J.$$

Define

$$J(X) = \{\varphi \colon X {-\!\!\circ\!\!\!\to} 1 \mid \phi \in J\}$$

and

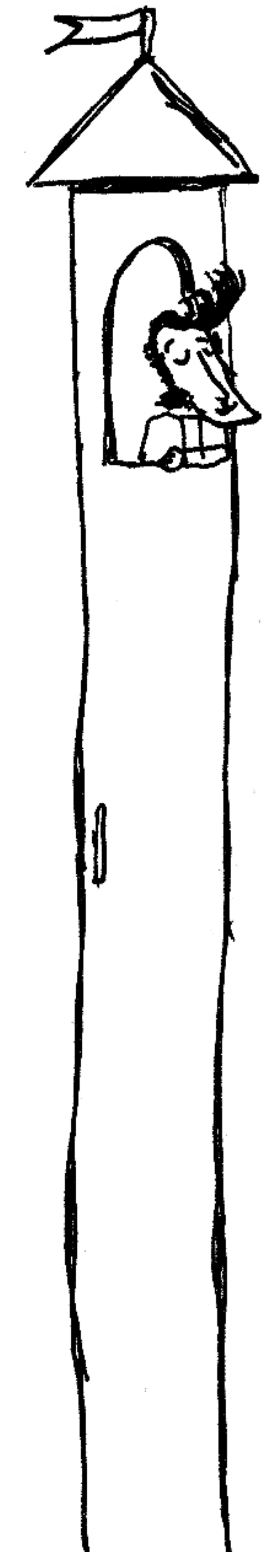
$$J_S(X) = \{\varphi \in J(X) \mid \varphi \,\text{has a supremum}\}$$

A $\mathcal{Q}$-category X is

J-COCOMPLETE

iff

$$\mathcal{S} \colon J_S(X) \to X \;\dashv\; \mathsf{y} \colon X \to J_S(X).$$



EXAMPLE 1. Choose $J = \mathcal{Q}\text{-}\mathbf{Dist}$.

J-cocomplete means cocomplete.

EXAMPLE 2. For $\mathcal{Q} = \mathbf{2}$, choose

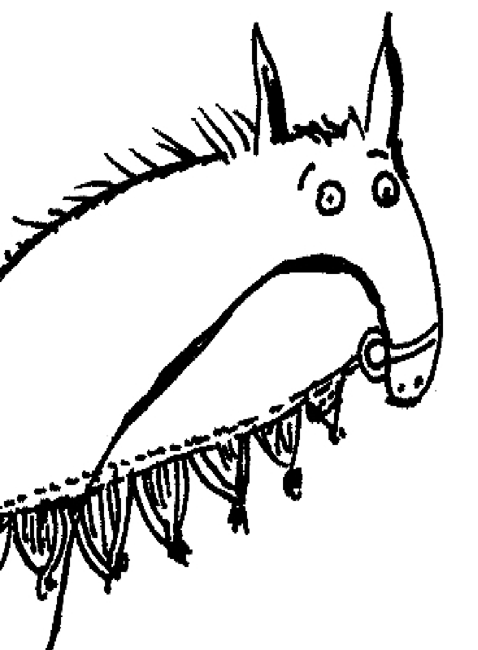
$$J = \{\phi\colon X \mathrel{\multimap} 1 \mid \phi \text{ is an ideal}\}.$$

J-cocomplete means directed-complete.

EXAMPLE 3. For $\mathcal{Q} = [0, \infty]$, choose

$$J = \{\inf_i \sup_{j \geqslant i} X(-, x_j) \mid (x_i) \text{ is Cauchy}\}$$

J-cocomplete means metrically complete.



etcetera

A $\mathcal{Q}$-functor $f\colon X \to Y$ is *Scott-continuous* if for all $\varphi \in J_S(X)$,

$$f(\mathcal{S}\varphi) = \mathcal{S}(\varphi \cdot f^*).$$

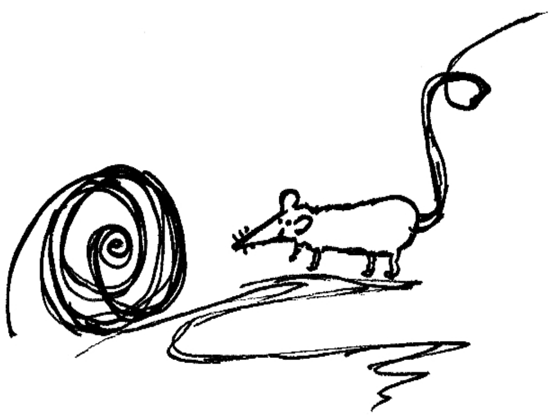
PROPOSITION. Let Y be a J-cocomplete category. Then any $\mathcal{Q}$-functor $f\colon X \to Y$ uniquely extends to a Scott-continuous $\mathcal{Q}$-functor $F\colon J(X) \to Y$.

THEOREM. The inclusion functor

$$\textbf{J-Cocont} \to \mathcal{Q}\textbf{-Cat}$$

has a left adjoint which sends a $\mathcal{Q}$-category X to $J(X)$ and a $\mathcal{Q}$-functor f to $(-) \cdot f^*$.

A fixpoint theorem



THEOREM. Let X be J-cocomplete, where $J = \{\bigvee_i \bigwedge_{j \geqslant i} X(-, x_j) \mid (x_i) \text{ is Cauchy}\}$. Let $f\colon X \to X$ be a $\mathcal{Q}$-functor. If

$$\phi \cdot f^* = \phi$$

for some $\mathcal{Q}$-distributor $\phi\colon X \mathrel{-\!\!\circ\!\!\rightarrow} 1$ such that $\phi \in J$, then f has a fixed point.

PROOF: Define $x \leqslant y$ iff $\mathbf{1} = X(x, y)$. By J-cocompleteness, $\mathcal{S}\phi$ exists. Thus:

$$\mathcal{S}\phi = \mathcal{S}(\phi \cdot f^*) \leqslant f(\mathcal{S}\phi) \leqslant f^2(\mathcal{S}\phi) \leqslant \ldots$$

However, by the choice of J, $(X, \leqslant)$ is directed-complete, and f is monotone wrt $\leqslant$, so f has a fixpoint by the usual argument using transfinite induction.

Banach fixpoint theorem

Suppose X is a complete (quasi-)metric space. Let $f : X \to X$ be a contraction. Let $x \in X$. Then:

$$\phi = \bigvee_i \bigwedge_{j \geqslant i} X(-, f^j x)$$

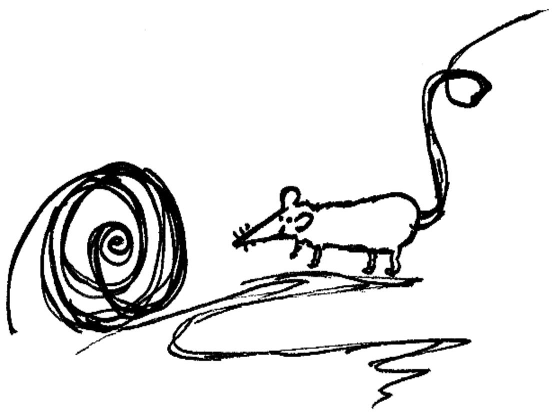
is an element of J, and satisfies:

$$\phi \cdot f^* = \phi.$$

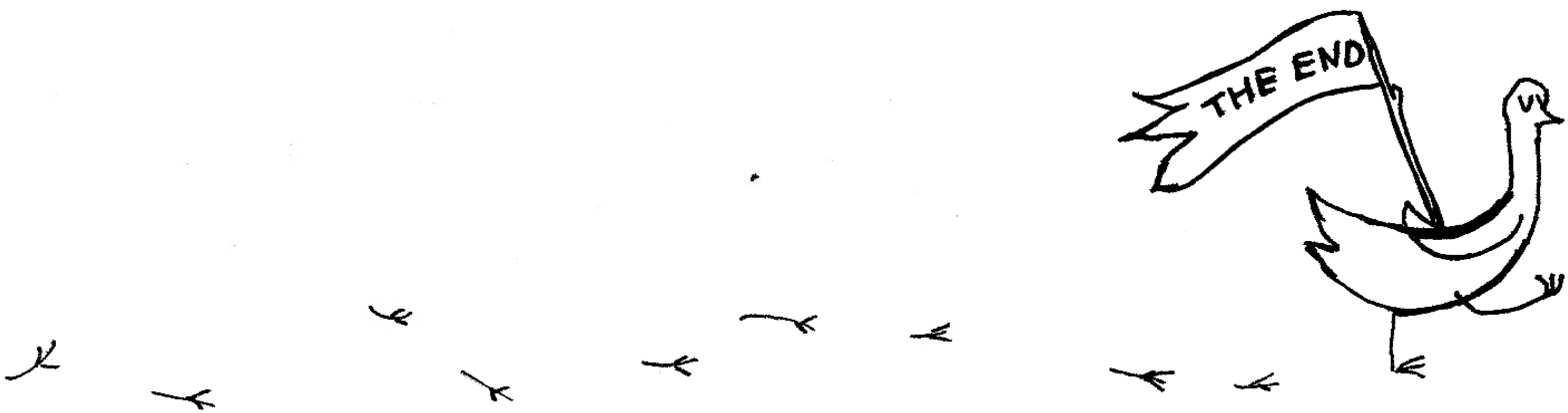
Moreover,

$$f(\mathcal{S}\phi) = \mathcal{S}(\phi \cdot f^*) = \mathcal{S}\phi$$

is the fixed point. It is unique, since f is a contraction.



Corollary. A contraction on a complete metric space has a unique fixed point.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