$\lambda\mu\widetilde{\mu}$,
une interprétation du calcul des séquents classique

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The implicative sequent calculus

Propositions are made only

- of propositional variables
- and of the implication operators.

The implicative sequent calculus (the rules)

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (ax)$$

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$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta}\ (\rightarrow L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta}\ (\rightarrow R)$$

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$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\,(cut)$$

A proof of the Pierce law

$$\vdash ((A \to B) \to A) \to A$$

A proof of the Pierce law

$$\dfrac{(A \to B) \to A \vdash A}{\vdash ((A \to B) \to A) \to A}\;(\to R)$$

A proof of the Pierce law

$$\dfrac{\dfrac{\vdash A \rightarrow B, A \qquad\qquad A \vdash A}{(A \rightarrow B) \rightarrow A \vdash A}\,(\rightarrow L)}{\vdash ((A \rightarrow B) \rightarrow A) \rightarrow A}\,(\rightarrow R)$$

A proof of the Pierce law

Easy

$$\dfrac{\dfrac{\vdash A \to B, A \qquad\qquad A \vdash A}{(A \to B) \to A \vdash A}\,(\to L)}{\vdash ((A \to B) \to A) \to A}\,(\to R)$$

A proof of the Pierce law

$$\dfrac{\dfrac{\vdash A \to B, A \qquad \dfrac{}{A \vdash A}\ (ax)}{(A \to B) \to A \vdash A}\ (\to L)}{\vdash ((A \to B) \to A) \to A}\ (\to R)$$

A proof of the Pierce law

$$
\dfrac{\dfrac{\dfrac{}{A \vdash B, A}\,(ax)}{\vdash A \to B, A}\,(\to R) \qquad \dfrac{}{A \vdash A}\,(ax)}{\dfrac{(A \to B) \to A \vdash A}{\vdash ((A \to B) \to A) \to A}\,(\to R)}\,(\to L)
$$

The active formula 1 / 3

The **active formula** is the formula on the lower part of a rule which is «*split*» by the rule.

For instance in

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta} (\rightarrow L)$$

the active formula is $A \rightarrow B$.

The active formula 2 / 3

It makes sense to track the active formulae and to suppose that A and B become the new active formulae :

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta} (\rightarrow L)$$

Similarly

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta} (\rightarrow R)$$

We have to prove B using the proposition A and to split B if necessary.

The active formula 3 / 3

But our proof of the Pierce law does not fulfill this statement on active formulae.

$$\dfrac{\dfrac{\dfrac{}{A \vdash B, A}\,(ax)}{\vdash A \to B, A}\,(\to R) \qquad \dfrac{}{A \vdash A}\,(ax)}{\dfrac{(A \to B) \to A \vdash A}{\vdash ((A \to B) \to A) \to A}\,(\to R)}\,(\to L)$$

The active formula 3 / 3

But our proof of the Pierce law does not fulfill this statement on active formulae.

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The rules of the implicative sequent calculus with active formulæ

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (L-ax) \qquad \frac{}{\Gamma, A \vdash \Delta, A}\ (R-ax)$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\ (cut)$$

The rules of the implicative sequent calculus with active formulæ

Four requirements :

- One needs to introduce two axioms according to the side of the active formula.
- In (cut) the new introduced proposition becomes the active formula.
- The lower sequent of (cut) has no active formula.
- One needs to introduce a new rule that **activates** a formula and enables a (cut) above that rule.

The rules of the implicative sequent calculus with active formulæ

$$\frac{}{\Gamma, A \vdash \Delta, A}\,(L-ax) \qquad \frac{}{\Gamma, A \vdash \Delta, A}\,(R-ax)$$

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$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\,(cut)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A, \Delta}\,(\mu) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \vdash \Delta}\,(\widetilde{\mu})$$

A new proof of the Pierce law 1 / 2

$$\dfrac{(A \to B) \to A \vdash (A \to B) \to A, A \qquad \dfrac{\mathcal{A}_1 \qquad\qquad \mathcal{A}_2}{(A \to B) \to A, (A \to B) \to A \vdash A}}{\dfrac{\dfrac{(A \to B) \to A \vdash A}{(A \to B) \to A \vdash A}\,(\mu)}{\vdash ((A \to B) \to A) \to A}\,(\to R)}\,(cut)$$

A new proof of the Pierce law 2 / 2

where

$$\dfrac{\mathcal{A}_1 \qquad\qquad \mathcal{A}_2}{(A \to B) \to A, (A \to B) \to A \vdash A}$$

$$=$$

$$\dfrac{\dfrac{\dfrac{\dfrac{(A \to B) \to A, A \vdash A, B, A \qquad (A \to B) \to A, A, A \vdash A, B}{(A \to B) \to A, A \vdash B, A}\,(cut)}{(A \to B) \to A, A \vdash B, A}\,(\mu)}{(A \to B) \to A \vdash A \to B, A}\,(\to R) \qquad (A \to B) \to A, A \vdash A}{(A \to B) \to A, (A \to B) \to A \vdash A}\,(\to L)$$

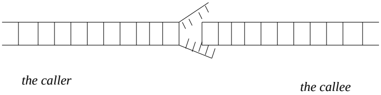
Plan

The model of computation : Herbelin's calculus

The link between the sequent calculus and Herbelin's calculus

A model of computation

The model of computation relies on **capsules**.



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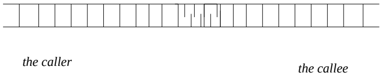

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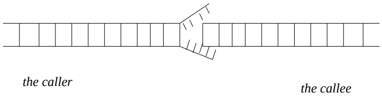

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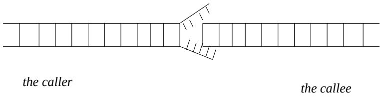

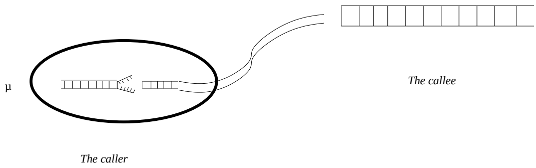

A model of computation : GEMINI

The model of computation relies on **capsules** $\langle r \parallel e \rangle$
that contain two constituents :

- a **caller** r
- and a **callee** e .

with the syntax

$$\begin{array}{lcl} c & ::= & \langle r \parallel e \rangle \\ r & ::= & x \mid \lambda x.r \mid \mu\alpha.c \\ e & ::= & \alpha \mid r \bullet e \mid \widetilde{\mu}x.c \end{array}$$

Callers

A **caller** is

- either a variable x,
- or a λ-abstraction $\lambda x.r$ which expects a value to take the place of x in r,
- or a μ-abstraction $\mu\alpha.c$ which expects a callee to take the place of α in c producing a new capsule.

Note : **values** and **callers** are the same.

Callees

A **callee** is basically a list of values, more precisely it is

- either a variable α,
- or a pair $r \bullet e$ of a value (caller) r and a callee e,
- or an $\widetilde{\mu}$-abstraction $\widetilde{\mu}x.c$

The reductions

$$
\begin{array}{llcl}
(\lambda) & \langle \lambda x \cdot r \parallel r' \bullet e \rangle & \longrightarrow & \langle r[x \leftarrow r'] \parallel e \rangle \\
(\mu) & \langle \mu\alpha \cdot c \parallel e \rangle & \longrightarrow & c[\alpha \leftarrow e] \\
(\widetilde{\mu}) & \langle r \parallel \widetilde{\mu} x \cdot c \rangle & \longrightarrow & c[x \leftarrow r]
\end{array}
$$

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$$

The system is ambiguous !

$\langle \mu\alpha.c \parallel \widetilde{\mu}x.c' \rangle$ has two possible reductions at the top.
$c[\alpha \leftarrow \widetilde{\mu}x.c'], \quad c[x \leftarrow \mu\alpha \cdot c]$ is a **critical pair**.

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This ambiguity is **inherent to proofs** in classical logic

The reductions

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\end{array}
$$

Can we type capsules, callers and callees ?

- to prove that *nothing wrong can happen*, i.e., *capsules reduces always* to capsules,
- to guarantee **termination**, i.e., *a typed capsule always reduces to a normal form* whatever strategy we adopt.

Plan

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The type judgments

Thanks to colors, I will consider three types of judgments
They can be seen as annotations of sequent calculus judgments ;

Judgments for capsules

In $c : x_1 : A_1, \ldots, x_p : A_p \vdash \alpha_1 : B_1, \ldots, \alpha_q : B_q$
(or in short $c : \Gamma \vdash \Delta$),
one says that

- c takes the x_i as arguments with type A_i
- c takes a continuation α_j with type B_j.

Judgments for callers

$$x_1 : A_1, \ldots, x_p : A_p \vdash r : A, \alpha_1 : B_1, \ldots, \alpha_q : B_q$$

or in short $\Gamma \vdash r : A, \Delta$,

or $\Gamma \vdash \boxed{\text{r: A}}, \Delta$, when one does not have color.

Judgments for callees

$x_1 : A_1, \ldots, x_p : A_p, e : A \vdash \alpha_1 : B_1, \ldots, \alpha_q : B_q$

or in short $\Gamma, e : A \vdash \Delta$.

or $\Gamma, \boxed{\text{e: A}} \vdash \Delta$, when one does not have color.

The type system $\mathbf{G}^{\rightarrow}$

$$\frac{}{\Gamma, \alpha : A \vdash \alpha : A, \Delta}\ (L-ax) \qquad \frac{}{\Gamma, x : A \vdash x : A, \Delta}\ (R-ax)$$

$$\frac{\Gamma \vdash r : A, \Delta \qquad \Gamma, e : B \vdash \Delta}{\Gamma, r \bullet e : A \rightarrow B \vdash \Delta}\ (\rightarrow L) \qquad \frac{\Gamma, x : A \vdash r : B, \Delta}{\Gamma \vdash \lambda x.r : A \rightarrow B, \Delta}\ (\rightarrow R)$$

$$\frac{\Gamma \vdash r : A, \Delta \qquad \Gamma, e : A \vdash \Delta}{\langle r \parallel e \rangle : (\Gamma \vdash \Delta)}\ (cut)$$

$$\frac{c : (\Gamma \vdash \beta : B, \Delta)}{\Gamma \vdash \mu\beta.c : B, \Delta}\ (\mu) \qquad \frac{c : (\Gamma, x : A \vdash \Delta)}{\Gamma, \widetilde{\mu}x.c : A \vdash \Delta}\ (\widetilde{\mu})$$

$$\frac{}{\Gamma, A \vdash \Delta, A}\,(L-ax) \qquad \frac{}{\Gamma, A \vdash \Delta, A}\,(R-ax)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\,(\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\,(\to R)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\,(cut) \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash B, \Delta}\,(\mu)$$

$$\frac{}{\Gamma, \alpha : A \vdash \alpha : A, \Delta}\,(L-ax) \qquad \frac{}{\Gamma, x : A \vdash x : A, \Delta}\,(R-ax)$$

$$\frac{\Gamma \vdash r : A, \Delta \qquad \Gamma, e : B \vdash \Delta}{\Gamma, r \bullet e : A \to B \vdash \Delta}\,(\to L) \qquad \frac{\Gamma, x : A \vdash r : B, \Delta}{\Gamma \vdash \lambda x.r : A \to B, \Delta}\,(\to R)$$

$$\frac{\Gamma \vdash r : A, \Delta \qquad \Gamma, e : A \vdash \Delta}{\langle r \parallel e \rangle : (\Gamma \vdash \Delta)}\,(cut) \qquad \frac{c : (\Gamma \vdash \beta : B, \Delta)}{\Gamma \vdash \mu\beta.c : B, \Delta}\,(\mu)$$

Curry-Howard correspondence

One gets a Curry-Howard correspondence, namely

- **terms** are **proofs**,
- **types** are **propositions**,
- **term reductions** are **proof simplifications** (normalization).

Pierce law again 1 / 2

Let T be $(A \to B) \to A$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{x : T, y : A \vdash y : A, \beta : B, \alpha : A \qquad x : T, y : A, \alpha : A \vdash \alpha : A, \beta : B}{\langle y \parallel \alpha \rangle : (x : T, y : A \vdash \beta : B, \alpha : A)}\,(cut)}{x : T, y : A \vdash \mu\beta.\langle y \parallel \alpha \rangle : B, \alpha : A}\,(\mu)}{x : T \vdash \lambda y.\mu\beta.\langle y \parallel \alpha \rangle : A \to B, \alpha : A}\,(\to R) \qquad x : T, \alpha : A \vdash \alpha : A}{x : T, (\lambda y.\mu\beta.\langle y \parallel \alpha \rangle) \bullet \alpha : T \vdash \alpha : A}\,(\to L)
$$

Pierce law again 1 / 2

Let T be $(A \rightarrow B) \rightarrow A$.

$$\dfrac{\dfrac{\dfrac{\dfrac{x : T, y : A \vdash y : A, \beta : B, \alpha : A \qquad x : T, y : A, \alpha : A \vdash \alpha : A, \beta : B}{\langle y \parallel \alpha \rangle : (x : T, y : A \vdash \beta : B, \alpha : A)}\,(cut)}{x : T, y : A \vdash \mu\beta.\langle y \parallel \alpha \rangle : B, \alpha : A}\,(\mu)}{x : T \vdash \lambda y.\mu\beta.\langle y \parallel \alpha \rangle : A \rightarrow B, \alpha : A}\,(\rightarrow R) \qquad x : T, \alpha : A \vdash \alpha : A}{x : T, (\lambda y.\mu\beta.\langle y \parallel \alpha \rangle) \bullet \alpha : T \vdash \alpha : A}\,(\rightarrow L)$$

is called $\mathcal{A}$ in the following screens.

Pierce law again 2 / 2

The tree for typing Pierce law is

$$\dfrac{\dfrac{\dfrac{x : T \vdash x : T, \alpha : A \qquad \mathcal{A}}{\langle x \parallel (\lambda y.\mu\beta.\langle y \parallel \alpha\rangle) \bullet \alpha\rangle : (x : T \vdash \alpha : A)}\,(cut)}{x : T \vdash \mu\alpha.\langle x \parallel (\lambda y.\mu\beta.\langle y \parallel \alpha\rangle) \bullet \alpha\rangle : A,}\,(\mu)}{\vdash \lambda x.\mu\alpha.\langle x \parallel (\lambda y.\mu\beta.\langle y \parallel \alpha\rangle) \bullet \alpha\rangle : ((A \to B) \to A) \to A,}\,(\to L)$$

Pierce law again 2 / 2

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$$

The term with type the Pierce law is

$$
\lambda x.\mu\alpha.\langle x \parallel (\lambda y.\mu\beta.\langle y \parallel \alpha\rangle) \bullet \alpha\rangle.
$$

Reductions as simplifications of proofs 1 / 2

Reductions are simplifications (normalizations) of proofs
Let us look at

$$(\lambda) \quad \langle \lambda x \cdot r \parallel r' \bullet e \rangle \quad \longrightarrow \quad \langle r[x \leftarrow r'] \parallel e \rangle$$

Reductions as simplifications of proofs 2 / 2

It corresponds to

$$\dfrac{\dfrac{\dfrac{\mathcal{D}}{\Gamma, x : A \vdash r : B, \Delta}}{\Gamma \vdash \lambda x . r : A \to B, \Delta}\,(\to R) \qquad \dfrac{\Gamma \vdash r' : A, \Delta \quad \Gamma, e : B \vdash \Delta}{\Gamma, r' \bullet e : A \to B \vdash \Delta}\,(\to L)}{\langle \lambda x \cdot r \parallel r' \bullet e \rangle : \Gamma \vdash \Delta}\,(cut)$$

and

$$\dfrac{\dfrac{\mathcal{D}[x \leftarrow r']}{\Gamma, \vdash r[x \leftarrow r'] : B, \Delta} \quad \Gamma, e : B \vdash \Delta}{\langle r[x \leftarrow r'] \parallel e \rangle : \Gamma \vdash \Delta}\,(cut)$$

Reductions as simplifications of proofs 2 / 2

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$$\dfrac{\dfrac{\dfrac{\mathcal{D}}{\Gamma, x : A \vdash r : B, \Delta}}{\Gamma \vdash \lambda x.r : A \to B, \Delta}(\to R) \qquad \dfrac{\Gamma \vdash r' : A, \Delta \quad \Gamma, e : B \vdash \Delta}{\Gamma, r' \bullet e : A \to B \vdash \Delta}(\to L)}{\langle \lambda x \cdot r \parallel r' \bullet e \rangle : \Gamma \vdash \Delta}(cut)$$

and

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash r' : A, \Delta}{} \quad \dfrac{\Gamma \vdash r' : A, \Delta}{}}{\mathcal{D}}}{\Gamma, \vdash r[x \leftarrow r'] : B, \Delta} \qquad \Gamma, e : B \vdash \Delta$$

$$\dfrac{\Gamma, \vdash r[x \leftarrow r'] : B, \Delta \qquad \Gamma, e : B \vdash \Delta}{\langle r[x \leftarrow r'] \parallel e \rangle : \Gamma \vdash \Delta}(cut)$$

Termination or strong normalization

If c is typable in $G^{\rightarrow}$, then c does not start a non terminating reduction.