Reality and Virtual Reality in Mathematics

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Abstract

This article introduces three of the twentieth century's main philosophies of mathematics and argues that of those three, one describes mathematical reality, the \reality" of the other two being merely virtual.

What are mathematical objects, really? What, for example, is that thing that we call \the number one", or \the set of all positive whole numbers", or \the shortest path between two points on the surface of a sphere"?

Most mathematicians (let alone most people) would ¯nd little interest in such questions, since they are totally preoccupied with the practice of their discipline rather than with questions about its meaning. In this essay I shall outline three[1] of the standard philosophical approaches to the meaning of mathematics and present a case that one of those three represents the reality of mathematics, each of the other two amounting to virtual reality.

The ¯rst approach that I want to mention is known as platonism. The platonist mathematician believes that mathematical objects do exist, in perfect \forms", and that what mathematicians actually work with are, in Plato's vivid metaphor, mere shadows cast by those perfect forms on the wall of the mathematical cave in which our intellects are con¯ned. For the platonist, the number we call \one" is a real object, of which we only work with imperfect representatives (on paper or chalkboard, or in the mind's eye). Likewise, there is a perfect form of the sphere, of which the earth itself is a very imperfect[2] representative; and when we measure the shortest distance between New York and London by following the great circle route along the surface of the earth, we are working with a representative of the shortest distance between (the perfect platonic form of) two points on the \real" sphere.

The platonist believes in truth values;[3] in other words, for the platonist every syntactically correct mathematical statement is either true or false. The task of

[1] There is (at least) a fourth standard approach, known as logicism, in which mathematics is regarded as, or reduced to, the formal, axiomatic theory of logical propositions. This philosophy, advocated especially by Bertrand Russell and A.N. Whitehead [9], is, from the viewpoint of meaning at least, similar to formalism, in that mathematics, an extension of logic, has form without content.

[2] In fact, the earth is an extremely bad representative of the sphere: it is really an oblate spheroid, in which the equatorial diameter is measurably larger than the polar one.

[3] But, as Pilate famously asked, \What is truth?".

the mathematician is systematically to determine the truth value (true or false) of the statements of mathematics. In this view it is as if there were a catalogue, necessarily one with in¯nitely many entries, containing all mathematical statements; whenever the mathematician proves the truth of a statement P, a tick is entered against P in the catalogue; whenever P is shown to be false, it is deleted from the catalogue. The ultimate aim of mathematics | an aim that, in view of the in¯nite number of entries in the catalogue, can never be achieved | is to have a complete catalogue with all false entries deleted and, duly ticked as proved, all the true ones remaining.

While admiring, perhaps wistfully the theological nobility in platonism | I confess to ¯nding the quasi{fundamentalist security of platonism more than super¯cially attractive | I regard it as less than perfectly suited for a mathematical world{view. For it still leaves open the fundamental questions: what, and exactly where, are those perfect forms?

The second of my three philosophies is formalism, which holds mathematics to be the study of axiomatic formal systems without regard to any meaning underlying the axioms or the theorems deduced therefrom. Even if not concerned with the meaning of mathematics, the mathematician surely feels some moral obligation to circumscribe our freedom in the choice of the axiomatic systems within which mathematics is developed. The circumscribing factor is consistency: it must be demonstrable that we cannot derive, as a consequence of our axioms, a contradiction such as \1 = 2 ".

The leading proponent of formalism was David Hilbert (1862{1943), who summarised the epistemology of formalism in a famous aphorism about Euclidean geometry:

> \One must be able to say at all times | instead of points, lines and planes | tables, chairs and beer mugs".

In other words, the interpretation of the axioms, in terms of either geometrical objects or the furnishings of a drinking{establishment, is irrelevant; all that matters is that the axiomatic system be consistent. Hilbert actually made this requirement a bit stronger: for him it was essential that metamathematics | the formal study of axiomatic mathematical systems | employ only techniques that did not themselves require justi¯cation. For example, so{called indirect existence proofs, in which the existence of an object is established by assuming its non{existence and then deriving a contradiction, were not permitted in Hilbert's metamathematics. If the consistency of, for example, axiomatic arithmetic or Euclidean geometry, could be established under such rigorous conditions, then the formalists would have justi¯ed metamathematically their use of such controversial techniques as indirect existence proofs within formal mathematics itself, and therefore overcome the objections of Brouwer (see below) to formalism.

Had Hilbert's ingenious metamathematical aim been realised, the power of mathematics might indeed have come close to that of his earlier claim:

> \... to the mathematical understanding there are no bounds ... in mathematics there is no Ignorabimus [we shall not know]; rather we

can always answer meaningful questions ... our reason does not possess any secret art but proceeds by quite definite and statable rules which are the guarantee of the absolute objectivity of its judgement." (address to 1928 International Congress of Mathematicians, in [7])

Alas for Hilbert, in 1931, Kurt Gödel (1906–78) proved two results, the second a consequence of the first, that destroyed the formalists' hopes, at least as originally expressed, for ever. Gödel's first theorem stated that if any formal axiomatic theory T powerful enough to cover elementary arithmetic is consistent, then there is a statement S of arithmetic that cannot be either proved or disproved within the formal theory T. Since one of the axioms of the logic underlying Hilbert's formalism is that for every proposition P, either P or its negation, not P, is true, Gödel's first theorem shows that any sufficiently powerful formal system contains true statements that cannot be proved. Gödel's second theorem follows from this, and states[4] that the consistency of any sufficiently powerful formal system of mathematics cannot be demonstrated within that system itself; you need to step outside the system — that is, enlarge it — in order to establish its consistency.

Now, the formalist could quite reasonably accept the implication of Gödel's theorem that consistency cannot be demonstrated formally, and then take as an act of faith the consistency of formal systems such as those used in the past century to develop mathematics with a breath–taking range of successful applications in the physical world. But this would still leave formalist mathematics as an activity ultimately devoid of meaning, if remarkably effective as an intellectual tool. In Russell's famous words, mathematics would be "the subject in which we never know what we are talking about nor whether what we are saying is true".

Let me now turn to the third of our philosophies of mathematics — intuitionism. Some of the underlying ideas of intuitionism can be traced back to the German algebraist Leopold Kronecker (1823–1891), who wished to base all of analysis on the natural numbers $0, 1, 2, \ldots$ and to eliminate all need for, or reference to, irrational numbers such as π; in Kronecker's own (translated) words,

"God made the integers; all else is the work of Man"

and (to Lindemann, who had proved that π has the important property known as transcendentality),

"Of what use is your beautiful investigation regarding π? Why study such problems, since irrational numbers are non–existent?"

However, intuitionism really begins with the foundational work of the Dutch mathematician L.E.J. Brouwer (1881–1966), who, in his doctoral thesis [2] in 1907, began a lifetime of publication largely devoted to following through his belief that

[4] What Gödel proved was roughly this: for a sufficiently powerful formal system T, the statement "T is consistent" can be encoded as a statement of arithmetic and hence one of T; but the encoded statement cannot be proved within the system T itself.

> Mathematics is a free creation of the human mind.

Actually, Brouwer's philosophy ranged beyond mathematics. For him, our fundamental intuition was that of the passage of time from one instant to the next:

> \Mathematics arises when the subject of twoness, which results from the passage of time, is abstracted from all special occurrences. The remaining empty form of the common content of all these twonesses becomes the original intuition of mathematics and repeated unlimitedly creates new mathematical subjects."

By repeating this process, the human mind creates successively the positive integers $1, 2, 3, \ldots$ For Brouwer, mathematics is intrinsic to the human intellect, preceding language, logic and experience.

Now, if one believes that mathematical objects are created in the individual human mind, then it is natural to adopt a \constructive" view of existence, in which to establish the existence of a certain mathematical object x, one must show, at least in principle, how x is created. It is not enough to prove the existence of the mental creation x by assuming its non{existence and then deriving a contradiction:

> \[indirect existence proofs] inform the world that a treasure exists without disclosing its location." ([12])

It follows that, for the intuitionist, the rules of traditional, or as it is usually known classical, logic become seriously suspect. For example, the classical law of excluded middle, or law of excluded third, states that

> For any proposition P, either P holds or not P holds.

What happens if we apply this law to a proposition P that asserts the existence of a certain mathematical object x? We then have that either x exists or x does not exist; but the existence of x, for the intuitionist, means that x can be mentally constructed; so this application of the law of excluded middle tells us that either we can construct x or there is no (mental) construction of x. It is not hard to become convinced that, under that interpretation of existence, the law of excluded middle is unjusti¯able.

For example, let us de¯ne a sequence $a_1, a_2, a_3, \ldots$ of integers as follows. If $2n + 2$ can be written as a sum of two primes numbers[5], set $a_n = 0$; if $2n + 2$ cannot be written as a sum of two prime numbers, set $a_n = 1$. Note that the terms a_n can easily be computed (at least in principle | when n is large, it may be very time{consuming to check whether $2n + 2$ can, or cannot, be written as a sum of two prime numbers). Now consider the proposition

> P : There exists n such that $a_n = 1$.

[5] Recall that the prime numbers $2, 3, 5, 7, 11, 13, 17, \ldots$ are those integers ¸ 2 whose only divisors are themselves and 1.

According to Brouwer, to prove P we must show how to ¯nd (construct) a positive integer n that cannot be written as a sum of two prime numbers; whereas to prove not P we must demonstrate that, and therefore how, each of the in¯nitely many positive integers can be written as a sum of two prime numbers. In the ¯rst case we will have produced an explicit counterexample to Goldbach's Conjecture,

> Every even integer greater than 2 is a sum of two primes.

In the second case we will have proved Goldbach's Conjecture. Since that conjecture has remained neither proved nor disproved[6] since ¯rst stated in 1742, it seems extremely unlikely that, under Brouwer's philosophy of mathematics, we could resolve it by a simple constructive application of the law of excluded middle.

By careful analyses like the foregoing one, Brouwer showed that classical logic was \untrustworthy" for the intuitionist.

> \The belief in the universal validity of the principle of the excluded third in mathematics is considered by the intuitionists as a phenomenon in the history of civilization of the same kind as the former belief in the rationality of π, or in the rotation of the ¯rmament about the earth. The intuitionist tries to explain the long duration of the reign of this dogma by two facts: ¯rstly that within an arbitrarily given domain of mathematical entities the non{contradictority of the principle for a single assertion is easily recognized; secondly that in studying an extensive group of simple everyday phenomena of the exterior world, careful application of the whole of classical logic was never found to lead to error."([10])

This point was perhaps more clearly put by Hermann Weyl, at one stage a follower of Brouwer:

> \According to [Brouwer's] view and reading of history, classical logic was abstracted from the mathematics of ¯nite sets and their subsets. ... Forgetful of this limited origin, one afterwards mistook that logic for something above and prior to all mathematics, and ¯nally applied it, without justi¯cation, to the mathematics of in¯nite sets. This is the Fall and original sin of set theory■." [12]

Believing that logic was both subservient and posterior to mathematics, Brouwer did not attempt to formalise the logic underlying his intuitionistic mathematics. In 1930 his doctoral student Arend Heyting (1898-1980) published the ¯rst set of formal axioms for that intuitionistic logic, which has subsequently become an object of considerable interest within mathematical logic

[6] A prize of one million US dollars has recently been o®ered for the ¯rst correct resolution of Goldbach's Conjecture; see http://www.thetimes.co.uk/news/pages/tim/2000/03/16/timfeafea02004.html

and theoretical computer science. In essence, that logic captures formally the Brouwer–Heyting–Kolmogorov (BHK) interpretation of intuitionistic practice, of which the following are three examples:

- In order to prove a logical disjunction P or Q, we must either produce a proof of P or else produce a proof of Q. (Classically, it is enough to demonstrate that it is impossible that both P and Q be false.)
- In order to prove that there exists an object x with the property P, we must (i) construct a certain object x and (ii) demonstrate that that object x has the property P.
- In order to prove that p implies q, we must produce (i) an algorithm that, applied to any proof of p, converts that proof to one of q, and (ii) a proof that this conversion algorithm actually works.

Brouwer believed that language had the same posterior status relative to mathematics as did logic. For him, mathematics was essentially a language–less mental activity, and language came into action later, when one tried to describe, and communicate to others, one's mathematical creations.[7]

Brouwer's abrasive personality and unflinching advocacy of intuitionism led to a bitter dispute between him and Hilbert, and hence between the intuitionists and the formalists, in the years following World War I. At least part of Hilbert's restricting the methods of metamathematics, in his pursuit of a proof of the consistency of his formal mathematics, originated in the need to demonstrate, once and for all, that the full gamut of classical techniques, such as indirect existence proofs, could be justified beyond all doubt. For Hilbert, the law of excluded middle was an essential tool of analysis:

> "Forbidding a mathematician to make use of the principle of excluded middle is like forbidding an astronomer his telescope or a boxer the use of his fists." [8]

Hilbert and his followers believed that intuitionistic mathematics would forever be skeletal, with none of the flesh that classical techniques could provide; and until the mid–1960s this view appeared to reflect reality. However, all was changed in 1967, when Errett Bishop (1928–83), already famous for his work in classical analysis, published a monograph [1] gathering the fruits of an astonishingly fertile two years in which he had single-handedly developed a vast amount of mathematics, in parallel with the classical theories, using only techniques based on intuitionistic logic.[8] In doing this, Bishop demolished the biggest barrier to belief in an intuitionistic or quasi–intuitionistic view of mathematics: the

[7] This raises philosophical problems with intuitionism which I have neither the competence nor the space to discuss here, problems such as that of the reliability of the language–based communication about one individual's mathematical (mental) creations to another. For more on such questions see [6, 11].

[8] It would not be quite correct to say that Bishop's mathematics was intuitionistic in the fullest Brouwerian sense: Bishop did not use certain principles that Brouwer added to those of his logic.

perception that serious, hard mathematics was virtually impossible to develop intuitionistically.

Let me brie°y summarise the three philosophies of mathematics that we have discussed above. First, there is the platonist view that mathematical objects have a meaningful reality, and that each mathematical statement has an associated truth value; the reality of an object consists in its perfect form, whose representatives are the day{to{day material of mathematical activity. Secondly, there is the formalist view, in which mathematics is a carefully crafted, but ultimately meaningless, game played according to rules that, ideally, can be shown never to lead to a formal contradiction. Finally, there is intuitionism, which is one form of constructivism, a term covering those philosophies in which mathematical objects are seen as mental creations (constructions) and which, in consequence, hold intuitionistic logic as the ideal for mathematical practice.

Which, if any, of these philosophies matches most closely the reality of mathematics | which is not necessarily the current reality of mathematical practice, but the reality of mathematics itself?

Whatever the formalist may claim (see, for example, [3, 4]), most mathematicians that I know seem to sense that what they do is meaningful:

> [The mathematician] \does not believe that mathematics consists in drawing brilliant conclusions from arbitrary axioms, of juggling concepts devoid of pragmatic content, of playing a meaningless game." ([1], p. viii)

Of course, it may be that mathematicians are (as many people do with life as a whole) taking a pragmatic, sanity{preserving attitude that allows them to pretend that there is meaning in what they do, even if at heart they believe that all is ultimately devoid of any absolute signi¯cance.

For those of us who believe that mathematics has a reality of its own, of the three philosophies outlined above, only platonism and constructivism could be tenable. Part of the appeal of platonism is its sense that everyday mathematics is an intimation of a quasi{divine mathematical perfection of relations between platonic forms; thus the mathematician gains a sense of being like an artist, trying to represent on a mathematical canvas the ultimately unrepresentable perfection of creation. On the other hand, by permitting the use of \idealistic" methods, such as deducing the existence of an object by deriving a contradiction from the assumption of its non{existence, platonism leads to theorems whose practical content is nugatory:

> \It appears than that there are certain mathematical statements that are merely evocative, which make assertions without empirical validity. There are also mathematical statements of immediate empirical validity, which state that certain performable operations will produce certain observable results, for instance, the theorem that every positive integer is the sum of four squares." ([1], ibid).

Bishop's use of the word \evocative" here strikes me as sound. An indirect proof that our galaxy contains black holes may be informative to a certain degree, but a direct proof of the existence of galactic black holes would be much more so, since it would enable us to pinpoint where they actually lie in relation to the earth.

In my view, a constructivist philosophy of mathematics gets closer to the heart of mathematical reality than any other.[9] I would suggest that mathematical objects are, indeed, mental constructions, and that to clarify their inter{ relations we must eventually use intuitionistic logic, although we may use the idealistic techniques of classical logic to provide initial information and guidelines for subsequent intuitionistic arguments. It must be emphasised, however, that in saying this, I am

> \not contending that idealistic mathematics is worthless from the constructive point of view. This would be as silly as contending that unrigorous mathematics is worthless from the classical point of view. Every theorem proved with idealistic methods presents a challenge: to¯nd a constructive version, and to give it a constructive proof." ([1], p. x)

Thus we may regard mathematics performed solely with classical logic as describing mathematics in virtual reality.[10] Sometimes the virtually real can be shown to be fully real, as when one replaces an indirect existence proof by an intuitionistic one; at other times, closer examination of the statement about mathematical virtual reality will show that it re°ects an aspect of \reality" that is genuinely virtual, in that it cannot be described using intuitionistic logic. In the latter case, the statement will remain evocative, the virtual being merely chimerical, for ever.

One might well ask:

> \If mathematical objects are mental creations, why are those creations there in the¯rst place? On what, if anything, are our primary mathematical intuitions based?"

My suggestion is that our primary mathematical intuitions, such as those of one{ ness and the passage from one{ness to two{ness, are abstractions from properties of the natural world; that the understanding, or at least mental assimilation, of those properties gave species a substantial evolutionary advantage; and that in the course of evolution, the human brain subsequently developed the ability to build on those primary mathematical intuitions, to produce mathematics with a structure and life of its own, not necessarily tied to the natural world whence the primary intuitions arose, but nevertheless, as the physicist Eugen Wigner

[9] Michael Dummett goes as far as to say, \Of the various attempts made ... to create over{ all philosophies of mathematics providing, simultaneously, solutions to all the fundamental philosophical problems concerning mathematics, only the intuitionist system originated by Brouwer survives today as a viable theory to which, as a whole, anyone could now declare himself an adherent" ([6], Introductory Remarks).

[10] Indeed, one arch{formalist has written that \...it seems to us today that mathematics and reality are almost completely independent, and their contacts more mysterious than ever" [5].

remarked [13], with an \unreasonable e®ectiveness" as a tool for describing and predicting phenomena in that world.

Although constructivist mathematics has had few adherents since Brouwer's initial onslaught against the formalists, the rise of the computer in the last quarter of the twentieth century has raised mathematicians' consciousness of computational, or constructive, issues. It has certainly highlighted a meaningful distinction between proving the existence of something and actually computing (approximations to) it. Nevertheless, very few mathematicians are aware of the power of intuitionistic logic, the sole use of which automatically eliminates non{ computational arguments from mathematics. (Every proof in Bishop's book [1] not only embodies algorithms for the computation of the objects it refers to, but is in itself a veri¯cation that those algorithms meet their speci¯cation | that is, do the job they are supposed to do.) Maybe the next century will, under the increasing in°uence of the computer, bring a greater appreciation of the reality of (constructive) mathematics, evoked by, but lying deeper than, the virtual reality | beautiful and seductive though it may be | of the platonist/formalist.

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