Magistère d'Informatique et Modélisation
Réécriture

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Rewriting - TD n°5

Higman's theorem and simplification orders

Exercise 1 *a constructive proof of Higman's theorem*

1. The **words** (denoted by $x, y, \ldots$ of a two letters alphabet are introduced by :

$$\frac{}{0 : \mathsf{Bin}} \quad \frac{x : \mathsf{Bin}}{S_0(x) : \mathsf{Bin}} \quad \frac{x : \mathsf{Bin}}{S_1(x) : \mathsf{Bin}}$$

The **finite sequence** of words (denoted by $A, B, \ldots$) is defined by :

$$\frac{}{\mathsf{Nil} : \mathsf{Seq}} \quad \frac{A : \mathsf{Seq}\ x : \mathsf{Bin}}{Ax : \mathsf{Seq}}$$

We introduce a relation $x \leq y$ between words, a relation $\mathsf{L}(A, y)$ between finite sequences and words, and two predicates $\mathsf{Good}(A)$ and $\mathsf{Bar}(A)$:

$$\frac{}{0 \leq x} \quad \frac{x \leq y}{x \leq S_0(y)} \quad \frac{x \leq y}{x \leq S_1(y)} \quad \frac{x \leq y}{S_0(x) \leq S_0(y)} \quad \frac{x \leq y}{S_1(x) \leq S_1(y)}$$

$$\frac{x \leq y}{\mathsf{L}(Ax, y)} \quad \frac{\mathsf{L}(A, y)}{\mathsf{L}(Ax, y)}$$

$$\frac{\mathsf{L}(A, x)}{\mathsf{Good}(Ax)} \quad \frac{\mathsf{Good}(A)}{\mathsf{Good}(Ax)}$$

$$\frac{\mathsf{Good}(A)}{\mathsf{Bar}(A)} \quad \frac{(x : \mathsf{Bin})\mathsf{Bar}(Ax)}{\mathsf{Bar}(A)}$$

Formulate Higman's Theorem in this setting. (Higman's theorem : For any infinite sequence $x_1, x_2, \ldots$ of words from A^*, there exists $p < q$ such that $x_p \leq x_q$.)

2. We introduce four binary relations R_0, R_1, T_0, and T_1 between finite sequences of words :

$$\frac{}{\mathsf{R}_0(\mathsf{Nil}, \mathsf{Nil})} \quad \frac{\mathsf{R}_0(A, C)}{\mathsf{R}_0(Ax, CS_0(x))}$$

$$\frac{}{\mathsf{R}_1(\mathsf{Nil},\mathsf{Nil})} \quad \frac{\mathsf{R}_1(A,C)}{\mathsf{R}_1(Ax, CS_1(x))}$$

Prove the following statements :

(a) For all finite sequences A of words, $\mathsf{Bar}(A0)$

(b) The following implications hold :
- If $\mathsf{L}(A,x)$, then $\mathsf{L}(A,S_0(x))$ and $\mathsf{L}(A,S_1(x))$
- If $\mathsf{R}_0(A,C)$ and $\mathsf{Good}(A)$, then $\mathsf{Good}(C)$
- If $\mathsf{T}_1(A,C)$ and $\mathsf{Good}(A)$, then $\mathsf{Good}(C)$
- If $\mathsf{R}_1(B,C)$ and $\mathsf{Good}(B)$, then $\mathsf{Good}(C)$
- If $\mathsf{T}_0(B,C)$ and $\mathsf{Good}(B)$, then $\mathsf{Good}(C)$
- If $\mathsf{R}_0(A,C)$, then $\mathsf{T}_0(Ax, CS_0(x))$
- If $\mathsf{R}_1(B,C)$, then $\mathsf{T}_1(Bx, CS_1(x))$

(c) If $\mathsf{T}_0(A,C)$, $\mathsf{T}_1(B,C)$, $\mathsf{Bar}(A)$, and $\mathsf{Bar}(B)$, then $\mathsf{Bar}(C)$

(d) If $\mathsf{R}_0(A,C)$ and $\mathsf{Bar}(A)$, then $\mathsf{Bar}(C)$

(e) If $\mathsf{R}_1(B,C)$ and $\mathsf{Bar}(B)$, then $\mathsf{Bar}(C)$

(f) For all x $\mathsf{Bar}([x])$.

3. Prove Higman's Theorem ($\mathsf{Bar}(\mathsf{Nil})$ expreses that any "growing" sequence of binary words $x_1 \to x_1x_2 \to x_1x_2x_3 \ldots$ is eventually good.)

Exercise 2

lexicographical path order

Discuss the termination of the following term rewriting systems :

$$\begin{array}{rcl} (x*y)*z & \longrightarrow & x*(y*z) \\ f(x*y) & \longrightarrow & f(x)*f(y) \end{array}$$

$$\begin{array}{rcl} (x*y)*z & \longrightarrow & x*(y*z) \\ f(x)*f(y) & \longrightarrow & f(x*y) \\ f(x*y)*z & \longrightarrow & f(x)*(f(y)*z) \end{array}$$

$$\begin{array}{rcl} (x*y)*z & \longrightarrow & x*(y*z) \\ f(x)*f(y) & \longrightarrow & f(x*y) \\ f(x)*(f(y)*z) & \longrightarrow & f(x*y)*z \end{array}$$