On Multiset Orderings

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Abstract: In this paper, two well-founded orderings on multisets which extend strictly Dershowitz and Manna ordering are proposed. These orderings do not verify a monotonicity property that Dershowitz and Manna does. That suggests to use monotonicity to provide a new characterization of Dershowitz and Manna ordering. Last section proposes an efficient and correct implementation of that ordering.

Résumé: Dans cette note, on propose deux ordres bien fondés qui étendent l'ordre de Dershowitz et Manna. Ces ordres ne vérifient pas une propriété de monotonicité que l'ordre de Dershowitz et Manna vérifie. Aussi cela suggère d'utiliser la monotonicité comme une nouvelle caractérisation de l'ordre de Dershowitz et Manna. La dernière section propose une implantion efficace et correcte de cet ordre.

1. Introduction

The multiset ordering $\lll$ proposed by Dershowitz and Manna [2] is a main tool of many orderings used to prove the finite termination of programs and also of term rewriting systems [1]. It is thus important to have an efficient implementation of this ordering, and that is the problem we deal with in this paper. As that happens often, deriving an algorithm straight from the mathematical definition provides an inefficient implementation. Therefore a more suitable definition must be found and the equivalence of the former definition and of the new one must be proven, that provides the correctness of the implementation. We attempted to follow that way with multiset ordering. We tried two definitions, both have efficient implementation but both fail to be proven equivalent. In fact they are stronger than Dershowitz and Manna multiset ordering but do not verify a monotonicity property. As an explanation of these facts, we give then a new definition of Dershowitz and Manna multiset ordering based on a main characterization of this ordering: it does not exist a stronger monotonic ordering on multisets. In the last section of this paper we propose a correct and efficient implementation of Dershowitz and Manna ordering.

2. Dershowitz and Manna ordering

Intuitively a multiset on E is an unordered collection of elements of E, with possibly many occurrences of given elements. A multiset can be seen as a mapping $E \rightarrow \mathcal{N}$ where $\mathcal{N}$ is the set of natural numbers. Let $\mathcal{M}(E)$ be the set of all the finite multisets on E, i.e. the multisets M such that their support $\{x \in E \mid M(x) \neq 0\}$ is finite. The empty multiset $\{\ \}$ is the multiset such that $\{\ \}(x) = 0$, for all x in E. A set is a particular case of a multiset such that S(x) is 0 or 1. Usually multisets are denoted by lists $\{x_1, \ldots, x_m\}$ with a straightforward interpretation. If M is a multiset, $x \in M$ means $M(X) > 0$.

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Definition: *Sum of multisets*. The sum of two multisets M and N is the multiset M + N such that $M + N(x) = M(x) + N(x)$.

Remark that if M and N are sets, M + N is a set only if M and N are disjoint and in this case + is the classical disjoint union or direct sum of sets. On another hand, the sum is an associative, commutative operation on $\mathcal{M}(E)$ with neutral element { }. If $M_1, M_2, \ldots, M_p$ is a family of multisets $\sum_{i=1}^{p} M_i$ is the multiset such that $(\sum_{i=1}^{p} M_i)(x) = \sum_{i=1}^{p} M_i(x)$.

Definition: *Inclusion of multisets*. A multiset M is included into a multiset N (written $M \subseteq N$) if and only if $(\forall\, x \in E)\ M(x) \leq N(x)$.

Definition: *Difference of multisets*. If $M \subseteq N$, the difference N – M is defined by $N - M(x) = N(x) - M(x)$.

In this paper, an ordering on a set is a partial or total strict ordering i.e. an irreflexive and transitive relation on E. We use the notation $x \# y$ to mean $\neg(x \prec y$ or $x = y$ or $x \succ y)$. Suppose E is ordered by $\prec$. Dershowitz and Manna ordering $\prec\!\!\prec$ is defined in the following way:

Definition: *Dershowitz and Manna Ordering*.
$M \prec\!\!\prec N$ if there exist two multisets X and Y in $\mathcal{M}(E)$ where

(i) $\{\ \} \neq X \subseteq N$
(ii) $M = (N - X) + Y$
(iii) X dominates Y that means $(\forall\, y \in Y)(\exists x \in X)\ x \succ y$.

We will refer to this definition by (DM).

The Dershowitz and Manna definition is difficult to use in order to prove that two multisets are not related by an inclusion. Their definition shows only how to reduce a multiset. In [3] Huet and Oppen give another more tractable definition.

Definition: *Huet and Oppen Definition*.

$$M \prec\!\!\prec N \text{ iff } M \neq N \ \&\ [M(y) > N(y) \Rightarrow (\exists x \in E)\ x \succ y \ \&\ M(x) < N(x)]. \qquad \text{(HO)}$$

Lemma 1: Dershowitz and Manna definition is equivalent to Huet and Oppen definition.

<u>Proof:</u> Let us denote by $\prec\!\!\prec_{DM}$ (respectively $\prec\!\!\prec_{HO}$) the ordering associated with Dershowitz and Manna definition (respectively Huet and Oppen definition).

Suppose $M \prec\!\!\prec_{HO} N$ and define X and Y as follows:

$X(x) = \max\{N(x) - M(x), 0\}$
$Y(y) = \max\{M(y) - N(y), 0\}$.

Let us prove (i)

1) $X \subseteq N$ is clear by definition.
2) Because $M \neq N$ there exists z such that $M(z) \neq N(z)$. If $M(z) < N(z)$ then $z \in X$, if $M(z) > N(z)$, by (HO) there exists $x \succ z$ such that $M(x) < N(x)$, that means $x \in X$. In both cases $X \neq \{\ \}$.

(ii) is true by construction. To prove (iii), let $y \in Y$. By hypothesis, there exists x, such that $x \succ y$ and $M(x) < N(x)$, that means $(\exists x \in X) x \succ y$.

Suppose $M \lll_{DM} N$. $M \neq N$ because $X \neq \{\}$. $N(y) > M(y)$ implies $y \in Y$, thus tne second alternative of (HO) follows because X dominates Y. ∎

Another important property of the Dershowitz and Manna ordering is the monotonicity.

Definition: *Monotonicity*. Let $\prec$ be a partial ordering on E and τ a mapping from $E \times E$ into $\mathcal{M}(E) \times \mathcal{M}(E)$. $\tau(\prec)$ is said to be a monotonic extension of $\prec$ iff:

(1) $\tau(\prec)$ is an ordering.

(2) τ is monotonic i.e. $\prec \subseteq < \Rightarrow \tau(\prec) \subseteq \tau(<)$.

Lemma 2: Monotonicity Lemma. The Dershowitz and Manna ordering $\lll$ is a monotonic extension of $\prec$.

Proof: Straightforward using Huet and Oppen definition. ∎

3. Partition based orderings

We define now two multiset orderings, using the same idea of building a partition from a multiset. We say that $\{M_i \mid i = 1,\dots,n\}$ is a partition of a multiset iff $M = \sum_{i=1}^{p} M_i$. Assume now, we are able to compare the M_i using an ordering $\prec$ and thus sort them such that $M_1 \succeq M_2 \succeq \dots \succeq M_p$. It is now easy to define a new ordering $\lll$ to compare the multisets $M = \sum_{i=1}^{p} M_i$ and $N = \sum_{i=1}^{q} N_i$ using lexicographical extension of $\prec$:

$M \prec N$ iff $M_1 M_2 \dots M_p \prec^{lex} N_1 N_2 \dots N_q$.

In practice, we have to define the basic ordering $\prec$ and the way to construct the partition of a given multiset.

3.1 The multiset ordering $\lll_{\mathcal{M}}$

Let us assume that the partition $\tilde{M} = \{M_i \mid i = 1,\dots,p\}$ of the multiset M verifies the following properties:

(1) $x \in M_i \Rightarrow M_i(x) = M(x)$.

(2) $x \in M_i$ and $y \in M_i \Rightarrow$ x and y are incomparable.

(3) $\forall\ i \in [2..p] \quad x \in M_i \Rightarrow (\exists y \in M_{i-1})\ y \succ x$.

In a more intuitive way, the partition is built by first computing the multiset M_1 of all the maximal elements and then recursively computing the partition of $M - M_1$.

Example 1: $M = \{a, a, 2, 2, b, 1, 1\}$ with $a \prec b$, $1 \prec 2$. $M_1 = \{2, 2, b\}$ and $M_2 = \{a, a, 1, 1\}$.

Let $\prec$ be the following basic ordering on multisets.

$M \prec_{\mathcal{M}} N$ iff $M \neq N$ and $(\forall x \in M)\ M(x) < N(x)$ or $(\exists y \in N)\ y \succ x$.

If M and N contain only incomparable elements, this definition provides an ordering equivalent to Dershowitz and Manna ordering.

Definition: Let M and N be multisets. We say that $M \lll_{\mathcal{M}} N$ iff $\tilde{M} \prec^{lex}_{\mathcal{M}} \tilde{N}$.

Example 2: If $a \prec b$ and $M = \{1, a, b\}$ then $M_1 = \{1, b\}$, $M_2 = \{a\}$. If $N = \{1,1,b\}$ then $N_1 = \{1,1,b\}$ and $M \prec\prec_{\mathcal{Mb}} N$.

It is easy to see that $\prec\prec_{\mathcal{Mb}}$ is an ordering because lexicographical extension preserves the orderings. Let us show now that this ordering is more powerful than Dershowitz and Manna one.

Lemma 3: $M \prec\prec N$ implies $M \prec\prec_{\mathcal{Mb}} N$.

Proof: Suppose $M \prec\prec N$ and $\tilde{M} = M_1 \ldots M_p$ and $\tilde{N} = N_1 \ldots N_q$ and prove $M \prec\prec_{\mathcal{Mb}} N$. The proof is by induction on $p+q$. The result is obvious if $M = \{\ \}$. Assume now that $p \neq 0$ and $q \neq 0$. Three cases have to be distinguished.

- $M_1 \prec_{\mathcal{Mb}} N_1$ the result is straightforward.
- $M_1 = N_1$. Thus $M_2 \ldots M_p \prec\prec_{\mathcal{Mb}} N_2 \ldots N_q$ and the result follows by the induction hypothesis.
- $M_1 \succ_{\mathcal{Mb}} N_1$ or M_1 and N_1 are incomparable. In both cases that implies there must exist an $x \in M_1$ such that $M_1(x) > N_1(x)$ and $(\forall y \in N_1)\ x \not\prec y$. This contradicts the hypothesis $M \prec\prec N$. ∎

The contrary is surely not true. Let us use the last example. We see that $M(a) = 1 > N(a) = 0$. However the only element in N greater than a, that is b, has one occurrence in both M and N. Thus M and N cannot be compared using Dershowitz and Manna ordering. On the other hand, the multiset ordering $\prec\prec_{\mathcal{Mb}}$ has a serious drawback, which forbid its use in some usual cases, where incremental orderings are required: it is not monotonic. Let us go back to Example 2 and assume now $1 \prec a \prec b$. That increase the basic ordering by adding the new pair $1 \prec a$. We now get:

$M_1 = \{b\}$, $M_2 = \{a\}$, $M_3 = \{1\}$

$N_1 = \{b\}$, $N_2 = \{1,1\}$.

Thus $N \prec\prec_{\mathcal{Mb}} M$!!

3.2 The multiset ordering $\prec\prec_{\mathcal{S}}$

An entirely different way to build a partition from a set is to require that different occurrences of a same element x belong to different multisets of the partition, for example to assume that all the multisets of the partitions are sets. Thus the partition $\bar{M} = \{S_i \mid i = 1 \ldots p\}$ must verify the following properties:

(1) S_i is a set, that is $S_i(x) \leq 1$.

(2) $x \in S_i$ and $y \in S_i$ implies that x and y are incomparable.

(3) $\forall i \in [2..p]\ x \in S_i$ implies $(\exists y \in S_{i-1})\ y \succeq x$.

The only difference with the previous partition definition is in condition (1). In the same way, the partition is built by first computing the set S_1 of maximal elements and then recursively computing the partition of $M - S_1$.

Example 3: $M = \{a,a,2,2,b,1,1\}$ with $a \prec b$ and $1 \prec 2$ and $S_1 = \{2,b\}$, $S_2 = \{2,a\}$, $S_3 = \{1,a\}$, $S_4 = \{1\}$.

Let $\prec_{\mathcal{S}}$ be the following ordering on sets:

$S \prec_{\mathcal{S}} T$ iff $S \neq T$ and $(\forall x \in S)(\exists y \in T)\ y \succeq x$.

In the following, $\bar{N} = \{T_i \mid i = 1..q\}$ will be the partition of N. If sets are considered as a particular case of multisets, $\prec_{\mathcal{S}}$ is $\prec\prec$ on the sets of incomparable elements.

Definition: Let M and N be multisets. We say that $M \prec\!\!\prec_{\mathcal{S}} N$ iff $\bar{M} \prec^{lex}_{\mathcal{S}} \bar{N}$.

Example 4: Suppose a and b incomparable. If $N = \{a,b\}$ then $T_1 = \{a,b\}$, if $M = \{b,b\}$ then $S_1 = \{b\}$, $S_2 = \{b\}$ and $M \prec\!\!\prec_{\mathcal{S}} N$.

Once more, it is easy to see that $\prec\!\!\prec_{\mathcal{S}}$ is an ordering. Let us now show that this new ordering is more powerful that $\prec\!\!\prec$.

Lemma 4: $M \prec\!\!\prec N$ implies $M \prec\!\!\prec_{\mathcal{S}} N$.

Proof: by induction on p and q. The result is straightforward if $M = \{\ \}$. Else, we have to distinguish three cases.

- $S_1 \prec_{\mathcal{S}} T_1$, the result is true.
- $S_1 = T_1$, by induction hypothesis.
- $S_1 \succ_{\mathcal{S}} T_1$ or S_1 and T_1 are incomparable. That implies that there must exist an $x \in S_1$ such that $(\forall\ y \in T_1)$, $\neg y \succeq x$. This contradicts the hypothesis $M \prec\!\!\prec N$. ∎

Once more, the converse is false, as it is proved by using the last example: $M = \{b,b\}$ and $N = \{a,b\}$.

Suppose now $a \prec b$ with same M and N as in example 4. We get $S_1 = \{b\}$, $M_2 = \{b\}$ and $T_1 = \{b\}$, $T_2 = \{a\}$. Thus $N \prec\!\!\prec_{\mathcal{S}} M$. Thus $\prec\!\!\prec_{\mathcal{S}}$ is not a monotonic ordering.

Now, let try to compare $\prec\!\!\prec_{\mathcal{S}}$ and $\prec\!\!\prec_{\mathcal{M}}$, using two examples with $a \prec b$:

- $M = \{1,a,b\}$, $N = \{1,1,b\}$, $M \prec\!\!\prec_{\mathcal{M}} N$ and $\neg\ M \prec\!\!\prec_{\mathcal{S}} N$.
- $M = \{b,b\}$, $N = \{1,b\}$, $M \prec\!\!\prec_{\mathcal{S}} N$ and $\neg\ M \prec\!\!\prec_{\mathcal{M}} N$.

Thus $\prec\!\!\prec_{\mathcal{S}}$ and $\prec\!\!\prec_{\mathcal{M}}$ do not compare.

3.3 Well foundedness

We may state the following theorem.

Theorem 1: If $\prec$ is well-founded on E, then $\prec\!\!\prec$, $\prec\!\!\prec_{\mathcal{M}}$ and $\prec\!\!\prec_{\mathcal{S}}$ are well-founded on $\mathcal{M}(E)$.

Proof: [2] contains a proof of well-foundedness of $\prec\!\!\prec$. Remark now that a proof of well-foundedness of $\prec\!\!\prec_{\mathcal{S}}$ or $\prec\!\!\prec_{\mathcal{M}}$ is also a proof of well-foundedness of $\prec\!\!\prec$ by Lemma 3 and 4. A proof of well-foundedness of $\prec\!\!\prec_{\mathcal{S}}$ and $\prec\!\!\prec_{\mathcal{M}}$ is easy to obtain by proving that $\prec_{\mathcal{S}}$ and $\prec_{\mathcal{M}}$ are well-founded. This can be done by using König Lemma as in [2]. On another hand, it is also possible to remark that $\prec_{\mathcal{S}}$ and $\prec_{\mathcal{M}}$ are particular cases of $\prec\!\!\prec$. ∎

4. A property of maximality of Dershowitz and Manna Ordering

In the previous section, we exhibited two non monotonic orderings containing the Dershowitz and Manna ordering. Therefore a question arises naturally: Does exist monotonic orderings on $\mathcal{M}(E)$ which contain Dershowitz and Manna ordering? The answer is negative and provides a new important characterization of Dershowitz and Manna ordering. Let us first prove an important lemma.

Lemma 5: Let $\prec$ be a partial ordering on E and M and N be two multisets on E such that $N \preceq\ \preceq_{HO} M$ that is $\neg(N = M$ or $N \prec\!\!\prec_{HO} M)$. Then there exist a partial ordering $<$ on E such that $< \supseteq \prec$ (that is $x \succ y \Rightarrow x > y$) and $M \ll_{HO} N$.

Proof: By induction on the set $D = \{(x,y) \in M \times N \mid x \# y$ i.e. $\neg(x \prec y$ or $x = y$ or $x \succ y\}$.

Basic case: Let $D = \emptyset$. Then $N \preceq\ \preceq_{HO} M \Rightarrow M \prec\!\!\prec_{HO} N$.

General case: Let D be not empty. Then either $M \prec\!\!\prec_{HO} N$ and the result is proved with $< = \prec$, or $M \preceq\ \preceq_{HO} N$ and there must exist a pair (x,y) such that

(1) $M(x) > N(x)$ and $(\forall z \in E)\ x \prec z \Rightarrow M(z) \geq N(z)$.

(2) $M(y) < N(y)$ and $(\forall z \in E)\ y \prec z \Rightarrow M(z) \leq N(z)$.

It follows from (1) and (2) that $x \# y$ and thus $(x,y) \in D$. Let now $<$ the transitive closure of the relation union of $\prec$ and the pair (x,y). $<$ is clearly an ordering containing strictly $\prec$. Therefore $\ll_{HO}$ is an ordering containing strictly $\prec\!\!\prec_{HO}$. As $x \prec y$, either $M \ll_{HO} N$ and the result is true or $M \#_{HO} N$ and the result follows from the induction hypothesis used with a new D, the cardinal of which is strictly less than the previous one. ∎

Theorem 2 *Maximality*: Let $\prec$ a partial ordering on E and $\tau(\prec)$ a monotonic extension of $\prec$ such that $\prec\!\!\prec \subseteq \tau(\prec)$. Then $\prec\!\!\prec$ and $\tau(\prec)$ are the same ordering.

Proof: Assume first $\prec$ is total. Then $\prec\!\!\prec$ is total on $\mathcal{M}(E)$ and must coincide with $\tau(\prec)$. Assume now that $\prec$ is partial on E. Then $\prec\!\!\prec$ is partial on $\mathcal{M}(E)$. Let us suppose that $\tau(\prec) \supsetneq \prec\!\!\prec$. That implies there must exist two multisets M and N such that $M\ \tau(\prec)\ N$ and $M \# N$. Using Lemma 5 there exists $< \supseteq \prec$ such that $N \ll M$, which implies $N\ \tau(<)\ M$ by hypothesis. Using now the monotonicity of the multiset extension τ and the hypothesis $M\ \tau(\prec)\ N$, we get $M\ \tau(<)\ N$, which is a contradiction. ∎

Notice that this main property of Dershowitz and Manna ordering can be used to give a simple proof of equivalence of (HO) and (DM). In the following, we use this technique to give and prove a new definition of $\prec\!\!\prec$. If $\prec$ is a total ordering on E, $\prec^{lex}$ is a total ordering on the ordered lists on E which provides a simple definition of Dershowitz and Manna ordering in that particular case: let $list(M) = \{x_1, x_2, \ldots, x_n\}$ with $i > j \Rightarrow x_i \succeq x_j$ be the sorted list representation of the multiset M. Then $M \prec\!\!\prec N$ iff $list(M) \prec^{lex} list(N)$.

Let us now define a new multiset ordering $\prec\!\!\prec_*$ in the following way:

Definition: Given a partial ordering $\prec$ on E, let $M \prec\!\!\prec_* N$ iff for all $<$ which is a total ordering containing $\prec$, $list(M) <^{lex} list(N)$.

It is now quite easy to prove that this new ordering is exactly Dershowitz and Manna ordering as an application of Theorem 2.

Lemma 6: $\prec\!\!\prec_*$ is a monotonic ordering.

Proof: It follows obviously from the definition. ∎

Lemma 7: $\prec\!\!\prec \subseteq \prec\!\!\prec_*$.

Proof: Suppose $M \prec\prec N$ and $\neg(M \prec\prec_* N)$. There exists a total ordering $<$ such that $< \supseteq \prec$ and $\neg(list(N) <^{lex} list(M))$. Thus there exists y such that $M(y) > N(y)$ and $z > y \Rightarrow N(z) = M(z)$ which implies $z \succ y \Rightarrow N(z) = M(z)$. That implies by (HO) $\neg(M \prec\prec N)$, which is a contradiction. ∎

Theorem 3: $\prec\prec_*$ and $\prec\prec$ are the same multiset ordering.

Proof: It follows from Theorem2, Lemma 6 and Lemma 7.

5. An efficient implementation of Dershowitz and Manna ordering

It is easy to derive an implementation of Dershowitz and Manna ordering from Huet and Oppen definition but it is not efficient because it supposes to try a comparison for each pair of items. Moreover it leads to an algorithm which does not work symmetrically on the data. In this section we propose an implementation based on the following idea. Given a pair of two multisets M_k and N_k, we build a new pair M_{k+1} and N_{k+1} such that at least one of the two multisets decreases in number of elements and the value of the comparison $comp(M_k, N_k)$ does not change which means $comp(M_k, N_k) = comp(M_{k+1}, N_{k+1})$. The process is repeated until it is possible to decide easily. To decrease M_k and N_k one can choose a pair (a,b) in $M_k \times N_k$ and do the following:

1. If $a = b$ and $M_k(a) = N_k(b)$, a is removed from M_k and N_k.
2. If $a \prec b$ or $a = b$ and $M_k(a) < N_k(b)$ then we would like to remove a from M_k without changing the value of $comp(M_k, N_k)$. That is possible if we know that a is maximal in M_k in which case $a \prec b$ or $a = b$ and $M(a) < N(b)$ implies $M_k \prec\prec N_k$ or M_k and N_k are incomparable (denoted by $M_k \# N_k$). Thus $comp(M_k, N_k)$ is not changed by removing a from M_k. It must be noted that in that case, one may remove with a all the elements in M_k which are less than a.

Thus instead of choosing any elements a and b in M_k and N_k, we have to choose maximal elements in M_k and N_k. Thus after removing a from M_k, for example, we have to compute the set of maximal elements of M_{k+1}. This is not difficult. Let us first define the function *succ* as

$(x \in succ(x) \Rightarrow x \succ y) \;\&\; (x \succ y \Rightarrow [\exists\, z \in succ(x)]\; z \succeq y)$,

and then

In case (1), $M_{k+1} = M_k - \{a\}$, then

$$Maximal(M_{k+1}) = Maximal(M_k) - \{a\} + \{x \mid x \in succ(a) \;\&\; x \notin succ(a') \text{ for } a \neq a'\}.$$

In case (2) $M_{k+1} = M_k - \{x \mid x \in succ^*(a)\}$ where $succ^*(a) = \sum_{i=1}^{n} succ^{i}(a)$, then

$$Maximal(M_{k+1}) = Maximal(M_k) - \{a\}.$$

The above discussion suggests to represent a multiset M as a directed acyclic graph representing the relation *succ*. Each node contains a triple: the value of the element, the number of occurrences nb(x,M) of the element x in the multiset M i.e. M(x) in the previous notation, the number of antecedents nb_ant(x,M) of x in M i.e. $card\{y \mid succ(y) = x\}$. If this last number is 0, x belongs to Maximal(M).

In fact the arrows deduced by transitivity are not necessary and the algorithm is more efficient if they are not present. For example if the definition of *succ* is the following minimal one:

$y \in succ(x) \equiv x \succ y \;\&\; \neg\, (\exists z \in M)\; y \succ z \succ x.$

For example if $a \succ c$, $a \succ d$, $b \succ d$, $c \succ e$, $c \succ f$, $d \succ f$ and $M = \{a,a,b,b,b,c,c,d,e,f,f\}$, a representation of M can be:

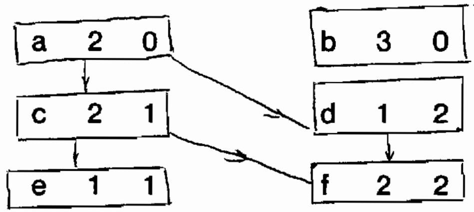

Figure 1 gives an algorithm describing our implementation. Let us remark that if it is not possible to choose a new pair in the body of the loop, that means that all the elements present in Maximal(M) and Maximal(N) are incomparable then easily $M \#_{\mathcal{M}} N$ then by Lemma 3, $M \# N$. The only problem is to prove the termination of the algorithm, but it is easy to see that although Maximal(M) and Maximal(N) can increase, they remain included in M and N which do not increase. Thus the algorithm terminates for any choice which computes a new pair (a,b) at each iteration.

6. References

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[3] Huet G., Oppen D.C.: *Equations and Rewrite Rules: a Survey*, in *Formal Languages perspectives and Open Problems*, Ed. Book R., Academic Press, (1980).

Fig. 1. An algorithm to implement Dershowitz and Manna ordering

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while "possible" do
        choose a new pair (a,b) in Maximal(M) × Maximal(N)
        if a≻b or [a = b and M(a)>N(b)] then
                for each x in succ*(b,N) do remove(x,N)

        if a≺b or [a = b and M(a)<N(b)] then
                for each x in succ*(a,M) do remove(x,M)

        if a = b and M(a) = N(b) then
                for each x in succ(a,M) do
                    nb_ant(x,N): = nb_ant(x,N) – 1
                    if nb_ant(x,N) = 0 then ad(x,Maximal(M))
                remove(a,Maximal(M))
                for each x in succ(b,N) do
                    nb_ant(x,N): = nb_ant(x,N) – 1
                    if nb_ant(x,N) = 0 then ad(x,Maximal(N))
                remove(b,Maximal(N))

if Maximal(M) = { } then if Maximal(N) = { } then return(" M = N ")
                                              else return(" M≺≺N ")
                    else if Maximal(N) = { } then return("N≺≺M")
                                              else return(" M # N ")
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