THE CHURCH-ROSSER THEOREM

Thesis, Doctor of Philosophy,
Cornell University

David Edward Schroer
June 1965

The Church-Rosser Theorem

Thesis presented to the Faculty of the Graduate School of Cornell University for the Degree of Doctor of Philosophy
by
David Edward Schroer
June 1965.

A paper copy of this Thesis was kindly sent by the author to J. Roger Hindley after correspondence in February 1963: two parts, 673 numbered pages.

A microfilm version of this Thesis was made in the 1960s by University Microfilms Inc., Ann Arbor, Michigan, U.S.A., as Publication No. 66-41.

The present electronic copy (of Part 1 only, 95 numbered pages) was made in 2018 by scanning Hindley's copy of Part 1. That paper copy was over 50 years old when scanned, and some pages were marked and one (p.15) was damaged and had to be re-written by hand, so the electronic copy also shows marks.

David Schroer did the bulk of his Ph.D. work under J. Barkley Rosser in the University of Wisconsin, U.S.A., and by 1963 his thesis was complete (as far as Hindley knows). But it was in 1965 at Cornell University that he obtained the Degree of Ph.D. -- under Anil Nerode.

THE CHURCH-ROSSER THEOREM

Presented to the Faculty of the Graduate School
of Cornell University for the Degree of
Doctor of Philosophy

by
David Edward Schroer

June 1965

TABLE OF CONTENTS

PART I

SECTION 1

REDUCTION COMPLEXES

Definition 1.1. We say that C is a reduction complex (and write $\ulcorner Cx\ C \urcorner$) iff C is a quadruple,

$$C = \langle V_C, \Sigma_C, \perp_C, \top_C \rangle,$$

such that V_C is a set (called the set of C-vertices), Σ_C is a non-empty set (called the set of C-cells), and $\perp_C$, $\top_C$ are functions (called the C-initial and C-terminal boundary operators) such that

$$(\text{Arg } \perp_C) \cap (\text{Arg } \top_C) \supseteq \Sigma_C$$

and

$$(\perp_C``\Sigma_C) \cup (\top_C``\Sigma_C) \subseteq V_C \ .$$

(See Note 1.1 below.) For any $\xi \in \Sigma_C$, $(\perp_C \xi)$ is called the C-initial vertex of ξ and $(\top_C \xi)$ is called the C-terminal vertex of ξ. (See Notation 1.1(a) below.)

Note 1.1. For any function f, (Arg f) is the set of all arguments of f (i.e., the domain of definition of f). (Terminology from Rosser's Textbook; but see Notation 1.1(a) below.) For any function f and any set A included in (Arg f), f``A is the set of all values of f for arguments in A (i.e., the image of A under f). (The '``' notation is standard. See Rosser's Textbook, for example.) For later use we mention also (Val f), the set of all values of f (i.e., $f``(\mathrm{Arg}\ f)$).

Notation 1.1(a). We use simple juxtaposition to indicate application of function to argument, and hence write the expressions on the left below, instead of the more usual notations on the right:

Arg f	Arg(f)
Val f	Val(f)
$\perp_C \xi$	$\perp_C(\xi)$
$T_C \xi$	$T_C(\xi)$.

This notation is used systematically throughout this thesis.

Remark 1.1. For reasons of a technical nature, it is in some ways convenient to consider only those reduction complexes — called perhaps "normalized reduction complexes" — in which the cells are one-termed sequences. Given any reduction complex C, one can then define the normalization of C to be the normalized reduction complex $\bar{C}$ in which $\Sigma_{\bar{C}}$ is the set of all one-termed sequences of cells in Σ , i.e.,

$$\Sigma_{\bar{C}} = \{\{\langle 0,\xi\rangle\} \mid \xi \varepsilon \Sigma_C\} ,$$

and $V_{\bar{C}}$, $\perp_{\bar{C}}$, $T_{\bar{C}}$ are induced by the one-to-one correspondence between $\xi \varepsilon \Sigma_C$ and $\bar{\xi} \varepsilon \Sigma_{\bar{C}}$ such that $\bar{\xi} = \{\langle 0,\xi\rangle\}$; i.e.,

$$V_{\bar{C}} = V_C$$
$$\perp_{\bar{C}}\bar{\xi} = \perp_C \xi$$
$$T_{\bar{C}}\bar{\xi} = T_C \xi .$$

(To stratify this, define instead $V_{\bar{C}} = \mathrm{USC}\ V_C$, $\perp_{\bar{C}}\bar{\xi} = \{\perp_C \xi\}$, $T_{\bar{C}}\bar{\xi} = \{T_C \xi\}$. In this case $\perp_{\bar{C}}\bar{\xi} = \perp_C``\{\xi\}$. (See Rosser's

Textbook.)) Hence there would be no real loss of generality in altering Definition 1.1 so as to require that Σ_C be a non-empty set of one-termed sequences. For reasons which may be regarded as aesthetic, we have not chosen this alternative. This results in some "abus de langage" (see Bourbaki), but avoids some complications (as in Definition 1.11 below).

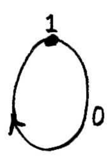
Notation 1.2. Hereafter we suppress the prefix 'C-' and subscript '$_C$' and write, for example,

$$'C = \langle V, \Sigma, \perp, \top \rangle' \;.$$

(See Definition 1.1.) Explicitly: in all formal contexts the letters 'V', 'Σ', '$\perp$', '$\top$' occurring without subscripts are to be regarded as the letters 'V_C', 'Σ_C', '$\perp_C$', '$\top_C$' respectively; likewise for the words 'vertex', 'cell', 'initial', 'terminal'; and likewise for the formulas 'Cimr', CDesc', CRedn', etc. (occurring below). *In the future such prefixes and subscripts, once introduced, will be similarly suppressed without further explicit warning.*

Theorem 1.3. There exists a reduction complex.

Proof. Take $V_C = \{1\}$, $\Sigma_C = \{0\}$, and $\perp_C = \top_C = \{\langle 0,1\rangle\}$. (See Note 1.3 below.) Then $\perp_C 0 = \top_C 0 = 1$. Verify that the resulting C is a reduction complex.



Note 1.3. A function is considered to be a class of ordered pairs (with arguments in the left position, values in the right) (as in Rosser's Textbook and other sources).

Remark 1.3. The example given in the above proof has needlessly much structure: it may be simplified by taking 1 to be the same as 0, in which case $V_C = \Sigma_C$ and $\perp_C = \top_C = \Sigma_C \times \Sigma_C$. The resulting simplified complex, while logically adequate as an example, is however insufficiently illustrative of the concept (which is barely illustrated by the example given).

Restriction 1.4. We make the following restrictions: C is a reduction complex.

Note 1.4. Our usage with respect to restricted variables is basically that of Rosser's Textbook (and others). We diverge from Rosser's usage with respect to free occurrences of restricted variables. Following the treatment of Hailperin's Restricted Quantification Paper (system QF_γ, Part II Section 5) we are careful to prove $(\exists x)P$ before using a variable restricted by the condition on x that P; it follows by Hailperin's Theorem 34 that all our theorems containing free occurrences of restricted variables are thus equivalent to theorems with additional hypotheses restricting the variables. This enables us to avoid stating the recurring hypothesis 'Cx C' without requiring us to precede each theorem concerned by universal quantification of 'C'. The resulting usage seems remarkably close to the handy (naive) intuitive usage of everyday mathematics.

<u>Theorem 1.5.</u> $(\exists\xi).\ \xi \varepsilon \Sigma :\ (\exists u).\ u \varepsilon V$.

<u>Proof.</u> By Definition 1.1, $\Sigma \neq \emptyset$. (See Note 1.5 below.) Hence $(\exists\xi).\ \xi \varepsilon \Sigma$. Fix ξ . By Definition 1.1, $\perp\xi \varepsilon V$. Hence $(\exists u).\ u \varepsilon V$.

<u>Note 1.5.</u> $\emptyset$ is the empty set.

<u>Restriction 1.6.</u> We make the following restrictions:
u, v, w, z (and variants) are C-vertices.
ξ, η, ζ, θ (and variants) are C-cells.

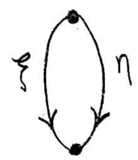
<u>Definition 1.7.</u> We say that u C-<u>immediately</u> <u>reduces</u> to v (and write $\ulcorner$u Cimr v$\urcorner$) iff $(\exists\xi).\ u = \perp\xi\ .\ v = \top\xi$.

<u>Note 1.7.</u> This definition is an example of a <u>restricted</u> <u>definition</u> (for which we know (1959) of no adequate treatment) in which the free restricted variables 'C', 'u', 'v' occur. Our usage (like the informal mathematical one) is only partially defined; the intention is that the restricted "definition" '$\ulcorner$u Cimr v$\urcorner$ for $\ulcorner(\exists\xi).\ u = \perp\xi\ .\ v = \top\xi\urcorner$' is to be essentially equivalent to the condition $\ulcorner(\forall C)::: Cx\ C$. $\supset:: (\forall u,v):.\ u,v\ \varepsilon\ V_C\ .\supset:\ u$ Cimr $v\ .\equiv.\ (\exists\xi).\ \xi \varepsilon \Sigma_C\ .\ u =$

$\perp_C \xi \;.\; v = \top_C \xi^{\urcorner}$. Since it is not the purpose of this thesis to develop a theory of restricted definitions, we will (with apologies) not specify our usage completely. (It seems likely that more than one satisfactory specification may be given.)

<u>Remark 1.8</u>. Let $\underline{R}$ be any binary relation. Then a complex C can be constructed so that $\underline{R} = \Sigma =$ Cimr . For take $V = (\mathrm{Arg}\ \underline{R}) \cup (\mathrm{Val}\ \underline{R})$, $\Sigma = \{\langle u,v\rangle \mid u\underline{R}v\} = \underline{R}$, $\perp(u,v) = u$, $\top(u,v) = v$. Obviously not every complex is isomorphic to a complex constructed in this way, since in such complexes the following condition, not satisfied in general, is always satisfied:

$$(\forall \xi,\eta):\ \perp\xi = \perp\eta \,.\, \top\xi = \top\eta \,.\!\supset\!.\ \xi = \eta \,.$$



(See diagram to right in which the condition is not satisfied.) So the general concept of a reduction complex is essentially more general than the concept of a binary relation.

<u>Remark 1.9</u>. In order to pave the way for Restriction 1.10 below, we should (in line with Note 1.4) formally assert the existence of some structure capable of being considered as the non-negative integers with an infinite element adjoined. We will assume that the reader is familiar with some such development, which need not concern us further.

! Restriction 1.10. We make the following restrictions: i, j, k are strictly positive integers.

m, n, p, q (and variants) are non-negative integers.

M, N (and variants) are non-negative integers or ∞. (See Note 1.10 below.)

Note 1.10. The precise meaning of '∞' need not concern us here. We use only the two properties that $(\forall n): n \in Nn \,.\supset.\, n < \infty$ and $(\forall n): n \in Nn \,.\supset.\, n + \infty = \infty$. Most naturally, ∞ could be taken as ω (the smallest transfinite ordinal number), except that the usual summation notation treats '∞' differently than it treats 'n'; furthermore, care would then have to be taken to indicate the correct order in additions. (We follow the usual summation notation in this thesis.)

Definition 1.11. We define the set CDesc, called the set of (connected) C-descents, to be the set of all (finite or infinite) sequences $\langle \varphi_i \rangle_{i=1}^{N}$ (see Note 1.11(a) below) such that for all $i < N$ we have that $\varphi_i \in \Sigma \,.\, \varphi_{i+1} \in \Sigma \,.\, \perp\varphi_{i+1} = \top\varphi_i$. We define the set CRedn, called the set of (connected) C-reductions, to be the set of all finite descents $\langle \rho_i \rangle_{i=1}^{n}$. In case $N = 0$, $\langle \varphi_i \rangle_{i=1}^{N}$ is the null reduction $\boxtimes$ (see Note 1.11(b) below). In case $N = 1$ it is considered part of the definition that $\varphi_1 \in \Sigma$; in this case $\langle \varphi_i \rangle_{i=1}^{1}$ is the singleton reduction $\langle \xi \rangle$ of the (unique) cell ξ such that $\varphi_1 = \xi$.

Note 1.11(a). We use here the bound-variable notation

$$\langle \varphi_i \rangle_{i=1}^{N}$$

to indicate a sequence φ with $\mathrm{Arg}\,\varphi = \{i \mid i \leq N\}$; the notation also indicates that subscript notation is to be used in indicating the application of φ to an argument. This is an explicit example of a term for a restricted variable. (See Hailperin's Restricted Quantification Paper; to suit Hailperin's notation we regard $\langle \varphi_i \rangle_{i=1}^{N}$ as an abbreviation for $\vee\varphi(\mathrm{Arg}\,\varphi = \{i \mid i \leq N\})$.) Clearly there exist such φ. Note that in case $N = \infty$, $\{i \mid i \leq N\} = \{i \mid i < N\}$ by Restriction 1.10. Note hence that while $\mathrm{Arg}\,\langle \varphi_i \rangle_{i=1}^{n} = \{i \mid i \leq n\}$, $\mathrm{Arg}\,\langle \varphi_i \rangle_{i=1}^{\infty} = \{i \mid i < \infty\}$. This agrees with the usual summation notation.

Note 1.11(b). The precise kind of sequence considered need not concern us here. The precise nature of the null reduction $\boxtimes$ depends on the precise kind of sequence considered; for some specifications we would have $\boxtimes = \emptyset$.

Remark 1.11. Many concepts concerning descents (and, specifically, reductions) can be generalized to arbitrary "chains" (and, specifically, "paths"); for expository simplicity we avoid these notions.

Theorem 1.12. $(\exists\varphi).\ \varphi \in \mathrm{Desc}\ .\ (\exists\rho).\ \rho \in \mathrm{Redn}$.

Proof. Clearly $\boxtimes \in \mathrm{Redn}$ and $\mathrm{Redn} \subseteq \mathrm{Desc}$.

<u>Restriction</u> <u>1.13</u>. We make the following restrictions:
$\varphi, \psi, \omega, \chi$ (and variants) are C-<u>descents</u>.
ρ, σ, π, τ (and variants) are C-<u>reductions</u>.

<u>Definition</u> <u>1.14</u>. We define the <u>length</u> $\#\varphi$ of a descent by the following condition:

$$\# \langle \varphi_i \rangle_{i=1}^{N} = N .$$

<u>Remark</u> <u>1.14(a)</u>. $(\forall \varphi) : \varphi \, \varepsilon \, \mathrm{Redn} \; . \equiv . \; \# \varphi < \infty$.

<u>Remark</u> <u>1.14(b)</u>. $(\forall \varphi) : \varphi = \boxtimes \; . \equiv . \; \# \varphi = 0$.

<u>Definition</u> <u>1.15</u>. Where $v \, \varepsilon \, Vx$, we define the sets $\underline{\mathrm{From}}_C v$, $\underline{\mathrm{To}}_C v$ (of descents, reductions respectively) by the following conditions:

$$\mathrm{From}_C v = \{ \langle \varphi_i \rangle_{i=1}^{N} \mid N = 0 \; . \vee : 1 \leq N \leq \infty \; . \; v = \bot \varphi_1 \} ,$$

$$\mathrm{To}_C v = \{ \langle \rho_i \rangle_{i=1}^{n} \mid n = 0 \; . \vee : 1 \leq n < \infty \; . \; v = \top \rho_n \} .$$

Informally, by "abus de langage", we write $\ulcorner \xi \, \varepsilon \, \mathrm{From} \; v \urcorner$, $\ulcorner \xi \, \varepsilon \, \mathrm{To} \; v \urcorner$ instead of $\ulcorner \langle \xi \rangle \, \varepsilon \, \mathrm{From} \; v \urcorner$, $\ulcorner \langle \xi \rangle \, \varepsilon \, \mathrm{To} \; v \urcorner$.

<u>Remark</u> <u>1.15(a)</u>. $(\forall v) . \boxtimes \, \varepsilon \, (\mathrm{From} \; v) \cap (\mathrm{To} \; v)$.

<u>Remark</u> <u>1.15(b)</u>. $(\forall \varphi) : \varphi \neq \boxtimes \; . \supset . \; (\exists_1 v) . \; \varphi \, \varepsilon \, (\mathrm{From} \; v)$.

<u>Remark</u> <u>1.15(c)</u>. $(\forall \rho) : \rho \neq \boxtimes \; . \supset . \; (\exists_1 w) \; . \; \rho \, \varepsilon \, (\mathrm{To} \; w)$.

(See Note 1.15 below.)

<u>Note 1.15</u>. Where P is any statement and x is any variable, $(\exists_1 x)P$ means that there is exactly one x such that P. (See Rosser's Textbook; Rosser writes $\ulcorner(E_1 x)P\urcorner$ where we write $\ulcorner(\exists_1 x)P\urcorner$.)

<u>Definition 1.16</u>. We define the sets <u>CCoinitial</u>, <u>CCoterminal</u> (of sets of descents, reductions respectively) by the following conditions:

$$\text{CCoinitial} = \{\Gamma \mid (\exists v).\ \Gamma \overset{\subseteq}{\cancel{\varepsilon}} \text{From } v\},$$

$$\text{CCoterminal} = \{\Gamma \mid (\exists v).\ \Gamma \overset{\subseteq}{\cancel{\varepsilon}} \text{To } v\}.$$

<u>Remark 1.16</u>. By this definition, Coinitial $\subseteq$ Desc . & . Coterminal $\subseteq$ Redn .

<u>Note 1.16</u>. By "abus de langage", we write $\ulcorner\{\xi,\eta\}\ \varepsilon$ Coinitial$\urcorner$, etc., instead of $\ulcorner\{\langle\xi\rangle,\langle\eta\rangle\}\ \varepsilon$ Coinitial$\urcorner$, etc.

<u>Theorem 1.17</u>. $(\forall\varphi)$: $\{\varphi\}\ \varepsilon$ Coinitial .

<u>Proof</u>. <u>Case 1</u>. $\varphi = \boxtimes$. By Theorem 1.5, $(\exists u).\ u\ \varepsilon\ V$. Then by Remark 1.15(a), $\varphi\ \varepsilon$ (From v), so $(\exists v)(\forall\psi)$: $\psi\ \varepsilon\ \{\varphi\}$. $\supset$. $\psi\ \varepsilon$ (From v) . <u>Case 2</u>. $\varphi \neq \boxtimes$. Then $(\exists_1 v)$. $\varphi\ \varepsilon$ (From v) , so $(\exists v)(\forall\psi)$: $\psi\ \varepsilon\ \{\varphi\}$. $\supset$. $\psi\ \varepsilon$ (From v) .

<u>Note 1.17</u>. The argument in Case 1 contains our first explicit use since Theorem 1.5 of the hypothesis that Σ is non-empty (see Definition 1.1). Clearly $\{\boxtimes\}\ \varepsilon$ Coinitial . $\equiv$

. $V \neq \emptyset$, using the rest of Definition 1.1 but not the non-emptiness of Σ. We have of course been using free variables for cells and vertices, hence have been using the non-emptiness of Σ implicitly.

Corollary 1.17. $(\exists \Phi) . \Phi \; \varepsilon$ Coinitial .

Proof. By Theorem 1.12, $(\exists \varphi) . \varphi \; \varepsilon$ Desc . Fix φ . By Theorem 1.17, $\{\varphi\} \; \varepsilon$ Coinitial .

Theorem 1.18. $(\forall \rho) : \{\rho\} \; \varepsilon$ Coterminal .

Proof. Just like the proof of Theorem 1.17.

Theorem 1.19. $(\forall \varphi) : \{\boxtimes, \varphi\} \; \varepsilon$ Coinitial .

Proof. Case 1. $\varphi = \boxtimes$. Then $\{\boxtimes, \varphi\} = \{\boxtimes\}$; so $\{\boxtimes, \varphi\} \; \varepsilon$ Coinitial . Case 2. $\varphi \neq \boxtimes$. Then $(\exists_1 u) . \varphi \; \varepsilon$ (From u) . Fix u. Then $\boxtimes \; \varepsilon$(From u) . $\therefore \{\emptyset, \varphi\} \; \varepsilon$ Coinitial .

Theorem 1.20. $(\forall \rho) : \{\boxtimes, \rho\} \; \varepsilon$ Coterminal .

Proof. Just like the proof of Theorem 1.19.

Theorem 1.21. $(\forall \varphi, \psi, \omega) : \varphi \neq \boxtimes . \{\varphi, \psi\}, \{\varphi, \omega\} \; \varepsilon$ Coinitial $. \supset . \{\psi, \omega\} \; \varepsilon$ Coinitial .

Proof. Case 1. $\psi \neq \boxtimes . \omega \neq \boxtimes$. Since none of φ, ψ, ω

is null, $(\exists_1 u). \Phi \in (\text{From } u) : (\exists_1 v). \Psi \in (\text{From } v) :$ $(\exists_1 w). \omega \in (\text{From } w)$. Fix u, v, w. Since $\{\Phi, \Psi\} \in$ Coinitial , have $u = v$, and since $\{\Phi, \omega\} \in$ Coinitial , have $u = w$. Hence $v = w$. This does it.

Case 2. If either of $\Psi, \omega = \boxtimes$, by Theorem 1.19 have $\{\Psi, \omega\} \in$ Coinitial automatically.

Theorem 1.22. $(\forall \rho, \sigma, \tau) : \rho \neq \boxtimes . \{\rho, \tau\}, \{\sigma, \tau\} \in$ Coterminal $. \supset . \{\rho, \sigma\} \in$ Coterminal .

Proof. Just like the proof of Theorem 1.21.

Definition 1.23. We say that Φ is a C-coinitial set of cells (and write $\ulcorner \Phi \in \text{CCoinit} \urcorner$) iff $\Phi \subseteq \Sigma : (\forall \xi, \eta) :$ $\xi, \eta \in \Phi . \supset . \perp\xi = \perp\eta$.

Remark 1.23. $\emptyset \in$ Coinit .

Restriction 1.23. We make the following restrictions: Φ, Ψ, Ω, X (and variants) are C-Coinitial sets of cells.

Definition 1.24. For $\sigma = \langle \sigma_i \rangle_{i=1}^{n} \in$ Redn , $\Psi =$ $\langle \Psi_i \rangle_{i=1}^{N} \in$ Desc , we define the sum $\sigma +_C \Psi$ by cases as follows:

If $\sim(\exists v) \,.\, \sigma \,\varepsilon\, \mathrm{To}\ v \,.\, \psi \,\varepsilon\, \mathrm{From}\ v$, then $\sigma +_C \psi = \boxtimes$.

If $(\exists v) \,.\, \sigma \,\varepsilon\, \mathrm{To}\ v \,.\, \psi \,\varepsilon\, \mathrm{From}\ v$, then $\sigma +_C \psi$ is the unique descent $\varphi = \langle \varphi_i \rangle_{i=1}^{n+N}$ such that $(\forall i)\colon 1 \leq i \leq n \,.\supset.\, \varphi_i = \sigma_i$ and $(\forall i)\colon n < i < N \,.\supset.\, \varphi_i = \psi_{i-n}$.

Remark 1.24. This sum is associative and $\boxtimes$ is its unique null element. There exist reduction complexes in which this sum is not commutative, since usually $\rho + \sigma \neq \boxtimes \,.\supset.\, \sigma + \rho = \boxtimes$.

$\boxtimes + \rho = \rho$
$\rho + \boxtimes = \rho$

Note 1.24. By "abus de langage", we informally write $\ulcorner \xi + \varphi \urcorner$, $\ulcorner \sigma + \xi \urcorner$ instead of $\ulcorner \langle \xi \rangle + \varphi \urcorner$, $\ulcorner \sigma + \langle \xi \rangle \urcorner$ respectively.

Remark 1.25. $(\forall \rho)(\forall \psi)\colon \rho + \psi \neq \boxtimes \,.\supset.\, \#(\rho + \psi) = (\#\rho) + (\#\psi)$.

Theorem 1.26. $(\forall \rho)(\forall \psi)\colon \{\rho, \rho + \psi\} \,\varepsilon\, \mathrm{Coinitial}$.

Proof. Case 1. $\rho = \boxtimes \,.\vee.\, \rho + \psi = \boxtimes$. Then $\{\rho, \rho + \psi\} \,\varepsilon\, \mathrm{Coinitial}$. Case 2. $\rho \neq \boxtimes \,.\, \rho + \psi \neq \boxtimes$. Then $(\exists_1 u) \,.\, \rho \,\varepsilon\, (\mathrm{From}\ u)$. Fix u. Then $\rho + \psi \,\varepsilon\, (\mathrm{From}\ u)$. So $\{\rho, \rho + \psi\} \,\varepsilon\, \mathrm{Coinitial}$.

Theorem 1.27. $(\forall \sigma, \tau)\colon \{\tau, \sigma + \tau\} \,\varepsilon\, \mathrm{Coterminal}$.

Proof. Just like the proof of Theorem 1.26.

Definition 1.28. We say that u C-strictly reduces to v (and write ⌜u Cred v⌝), iff $(\exists\rho): \rho \neq \emptyset \,.\, \rho \in (\text{From } u) \cap (\text{To } v)$, and we say that u C-reduces to v (and write ⌜u $\underline{\underline{\text{Cred}}}$ v⌝) iff u = v . ∨ . u Cred v .

Remark 1.28(a). ($\underline{\underline{\text{red}}}$) ε Qord . (See Note 1.28 below.)

Note 1.28. Qord is the class of quasi-orderings (i.e., reflexive, transitive binary relations). (See Rosser's Textbook.)

Remark 1.28(b). Let S be any quasi-ordering. Then a complex C can be constructed so that S = ($\underline{\underline{\text{red}}}$) . In fact, treating S as R in Remark 1.8 does the trick. (For such a complex ($\underline{\underline{\text{red}}}$) = ($\underline{\underline{\text{imr}}}$) .) Hence the study of reduction in reduction complexes is equivalent to the study of arbitrary quasi-orderings.

Definition 1.29. We say that u is C-convertible to v (and we write ⌜u Cconv v⌝) iff either u = v or there exists a sequence $\alpha = \langle\alpha_i\rangle_{i=1}^{n}$ of vertices such that $\alpha_1 =$ u . α_n = v and $(\forall i): 1 \leq i < n \,.\supset: \alpha_i$ imr α_{i+1} . ∨. α_{i+1} imr α_i .

Definition 1.30. We define the set CNormVx of C-normal (or -end) vertices by the following condition:

CNormVx = $\{v \mid \sim(\exists u).\ v \text{ imr } u\}$.

Definition 1.31

We define the set C CoNorm Vx of C-CoNormal vertices by the following condition:

$$C\ \mathrm{Conorm}\ V_x = \{w \mid (\exists v).\ w\ \mathrm{conv}\ v\ .\ v \in \mathrm{Norm}\ V_x\}$$

We define the set C Prenorm Vx of C-prenormal vertices by the following condition:

$$C\ \mathrm{Prenorm}\ V_x = \{w \mid (\exists v).\ w\ \underline{\underline{\mathrm{red}}}\ v\ .\ v \in \mathrm{Norm}\ V_x\}$$

Definition 1.32 We say that C has the CHURCH-ROSSER PROPERTY (and write ⌜ChR C⌝) iff the following condition is satisfied:

$$(\forall u, v, w):\ u\ \underline{\underline{\mathrm{red}}}\ v\ .\ u\ \underline{\underline{\mathrm{red}}}\ w\ .\ \supset$$
$$(\exists z).\ v\ \underline{\underline{\mathrm{red}}}\ z\ .\ w\ \underline{\underline{\mathrm{red}}}\ z$$

Theorem 1.33 $\mathrm{ChR}\ C\ .\ \equiv:\ (\forall u, v):\ u\ \mathrm{conv}\ v \supset (\exists w).\left\{\begin{array}{l} u\ \underline{\underline{\mathrm{red}}}\ w\ . \\ v\ \underline{\underline{\mathrm{red}}}\ w \end{array}\right\}$

Proof See Newman's paper. The condition on the right is the classical form of the property; see the Church-Rosser paper.

Remark 1.34

$$\mathrm{ChR}\ C\ .\ \supset:\ (\forall w):\ w \in \mathrm{Conorm}\ V_x\ .\ \equiv:\ w \in \mathrm{Prenorm}\ V_x\ .$$
$$\equiv:\ (\exists_1 v).\ w\ \underline{\underline{\mathrm{red}}}\ v\ .\ v \in \mathrm{Norm}\ V_x$$
$$\equiv:\ (\exists_1 v).\ w\ \mathrm{conv}\ v\ .\ v \in \mathrm{Norm}\ V_x\ .$$

<u>Remark 1.35</u>. Define a <u>component</u> in a complex C to be an equivalence class under conversion, and let <u>S</u> be the converse of $\underline{\underline{\text{red}}}$. Then by Theorem 1.33, (ChR C) iff each connected component is directed by <u>S</u>, in the usual sense of order-theoretic topology. (See John L. Kelley, <u>General Topology</u>, New York, (Van Nostrand), 1955, p.65.) Since a connected component is itself a complex, we then have by Remark 1.28(b) that the study of reduction in connected reduction complexes with the Church-Rosser Property is equivalent to the study of directed sets.

<u>Theorem 1.36</u>. ChR C . ≡ : (∀u,v,w): u imr v . u red w . ⊃ . (∃z). v $\underline{\underline{\text{red}}}$ z . w $\underline{\underline{\text{red}}}$ z .

<u>Proof</u>. See Newman's Paper. The condition on the right is the easiest to verify. Note that the first (actual) occurrence of 'red' can be changed to an occurrence of '$\underline{\underline{\text{red}}}$' without changing the truth of the theorem.

<u>Remark 1.36</u>. As pointed out in Newman's Paper, although ChR C .⊃: (∀u,v,w): u imr v . u imr w .⊃. (∃z). v $\underline{\underline{\text{red}}}$ z . w $\underline{\underline{\text{red}}}$ z , the converse is not universally valid. So Theorem 1.36 is as far as one can go in this fashion in the direction of "localizing" the Church-Rosser Property. (See Diagram 1.40 below.)

Definition 1.37. We say that C has the *finite descent property* (and write ⌜FinDesc C⌝) iff the following condition is satisfied:

$$(\forall\varphi).\ \#\varphi < \infty .$$

Definition 1.38. We say that C is *locally-finite* (and write ⌜LocFin C⌝) iff Coinit $\subseteq$ Finite (i.e., iff $(\forall\Phi).\ \Phi \in$ Finite).

Restriction 1.39. We assemble here our restrictions on variables of the abstract theory of reduction complexes:

$u, v, w, z \in \mathrm{Vx}$,

$\xi, \eta, \zeta, \theta \in \mathrm{Cell}$,

$i, j, k \in \mathrm{Nn} - \{0\}$,

$m, n, p, q \in \mathrm{Nn}$,

$M, N \in \mathrm{Nn} \cup \{\infty\}$,

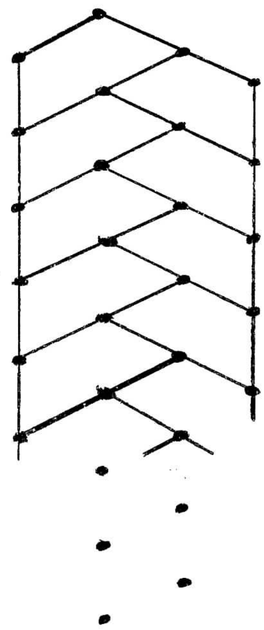
$\varphi, \psi, \omega, \chi \in \mathrm{Desc}$,

$\rho, \sigma, \pi, \tau \in \mathrm{Redn}$,

$\Phi, \Psi, \Omega, \mathrm{X} \in \mathrm{Coinit}$.

These are to be understood as affecting sub- and superscripted variants of the indicated variables.

Diagram 1.40. The diagram to the right, in which all cells are directed downward, represents a reduction complex C in which $(\forall u, v, w)$: u imr v . u imr w . $\supset$. $(\exists v)$. v $\underline{\underline{\text{red}}}$ z . w $\underline{\underline{\text{red}}}$ z , but $\sim$ChR C . This example was suggested by Professer Rosser.



SECTION 2

DERIVATION COMPLEXES

Definition 2.1. We say that the pair $\langle C,\delta\rangle$ is a derivation complex (and write $\ulcorner\mathrm{DerivCx}(C,\delta)\urcorner$) iff (Cx C) and δ is a set-valued function whose arguments include all ordered pairs of cells, such that the following five conditions are satisfied:

(δ1) $(\forall\xi,\eta)\colon\ \delta(\xi,\eta)\subseteq\Sigma$,

(δ2) $(\forall\xi,\eta,\xi')\colon\ \xi'\ \varepsilon\ \delta(\xi,\eta)\ .\supset.\ \perp\xi' = \top\eta$,

(δ3) $(\forall\xi)\colon\ \delta(\xi,\xi)=\emptyset$,

(δ4) $(\forall\xi,\eta)\colon\ \perp\xi\neq\perp\eta\ .\supset.\ \delta(\xi,\eta)=\emptyset$,

(δ5) $(\forall\xi,\eta)\colon\ \delta(\xi,\eta)\ \varepsilon$ Finite .

(See Note 2.1 below.)

Note 2.1. 'Finite' is the class of all finite sets (Rosser's 'Fin'; see Rosser's Textbook).

Remark 2.1. The concept of a derivation operation is a modification of Newman's generalization of Church and Rosser's concept of residual. (See the Church-Rosser Paper or Newman's Paper.)

Theorem 2.2. $(\exists\delta)$. DerivCx(C,δ) .

Proof. Define for all ξ, η : $\delta(\xi,\eta) = \emptyset$. For this δ, all of (δ1)-(δ5) hold, as is easily seen.

Restriction 2.3. We make the following restriction: δ is such that DerivCx(C,δ) .

Note 2.3. Thanks to Hailperin's Restricted Quantification Paper, we feel free to make this type of restriction while retaining natural modes of deduction.

Definition 2.4. Where DerivCx(C,δ) , δ can be "extended" to a unique extended derivation operation δ^*_C whose arguments are precisely all ordered pairs consisting of a coinitial set of cells and a reduction (notation: $\ulcorner(\Phi/\pi)_{C,\delta}\urcorner$ instead of $\ulcorner\delta^*_C(\Phi,\pi)\urcorner$) as defined inductively by the following two conditions:

(δ6a) $\quad (\forall\Phi): (\Phi/\emptyset) = \Phi$,

(δ6b) $\quad (\forall\Phi)(\forall\pi)(\forall\eta): \pi+\eta \neq \emptyset \,.\supset.$

$$(\Phi/\pi+\eta) = \bigsqcup\{\delta(\xi,\eta) \mid \xi \in (\Phi/\pi)\} .$$

(See Note 2.4 below.)

Note 2.4. '$\bigsqcup$' is our symbol for the set-theoretic operation of manifold union, usually denoted '$\bigcup$'. (See Rosser's Textbook.)

Remark 2.4. δ^* can be further extended to a function δ^{**} whose arguments are ordered pairs consisting of a not necessarily coinitial set of cells and a reduction, but we have no use for the further extension, and since our variables for sets of cells are restricted to be coinitial, we also have no convenient notation.

Notation 2.5. We write $\ulcorner(\xi/\eta)\urcorner$ instead of $\ulcorner(\{\xi\}/\eta)\urcorner$; and we write $\ulcorner(\Phi/\eta)\urcorner$ instead of $\ulcorner(\Phi/\langle\eta\rangle)\urcorner$. In particular, then, we write $\ulcorner(\xi/\eta)\urcorner$ instead of $\ulcorner\delta(\xi,\eta)\urcorner$. In this notation the conditions which define a derivation cx are the following five:

$(\delta 1)$ $(\forall\xi,\eta)\colon (\xi/\eta) \subseteq \Sigma$,

$(\delta 2)$ $(\forall\xi,\eta,\xi')\colon \xi' \in (\xi/\eta) \ .\supset.\ \bot\xi' = \top\eta$,

$(\delta 3)$ $(\forall\xi)\colon (\xi/\xi) = \emptyset$,

$(\delta 4)$ $(\forall\xi,\eta)\colon \bot\xi \neq \bot\eta \ .\supset.\ (\xi/\eta) = \emptyset$,

$(\delta 5)$ $(\forall\xi,\eta)\colon (\xi/\eta) \in \text{Finite}$.

Theorem 2.6. $(\forall\Phi)(\forall\pi)\colon (\Phi/\pi) \subseteq \Sigma$.

Proof. The conclusion follows from $(\delta 1)$ by $(\delta 6)$ using induction on $\#\pi$.

Note 2.6. By our conventions about restricted variables, this theorem is equivalent to

$$\text{DerivCx}(C,\delta) \ .\supset:\ (\forall\Phi)(\forall\pi)\colon (\Phi/\pi) \subseteq \Sigma$$

(since 'δ' occurs free in the unabbreviated version of (Φ/π)). A similar situation holds throughout most of the remainder of this thesis.

Lemma 2.7. $(\forall\Phi)(\forall\pi)(\forall\eta)\colon \Phi \neq \emptyset \,.\, \pi+\eta \neq \boxtimes \,.\supset.$ $\perp''(\Phi/\pi+\eta) = \{T\eta\}$. (See Note 2.7 below.)

[Note 2·7 does not exist]

Proof. By (§4) and (§6b), $(\Phi/\pi+\eta) = \bigsqcup\{(\xi/\eta) \mid \xi \,\varepsilon\, (\Phi/\pi)\} = \bigsqcup\{(\xi/\eta) \mid \xi \,\varepsilon\, (\Phi/\pi) \,.\, \perp\xi = \perp\eta\}$. By (§2), the conclusion follows.

Theorem 2.7. $(\forall\Phi)(\forall\rho)(\forall v)\colon v \,\varepsilon\, \perp''(\Phi/\rho) \,.\supset.\, \rho \,\varepsilon (\mathrm{To}\ v)$.

Proof. Case 1. $\rho = \boxtimes$. By definition, $\boxtimes \,\varepsilon\, (\mathrm{To}\ v)$. Case 2. $\rho \neq \boxtimes$. Then $(\exists\pi)(\exists\eta).\ \rho = \pi+\eta$; fix π, η and use the lemma.

Corollary 2.7. $(\forall\Phi)(\forall\rho)\colon (\Phi/\rho) \,\varepsilon\, \mathrm{Coinit}$.

Proof. Case 1. $\rho = \boxtimes$. Then $(\Phi/\rho) = \Phi$ by (§6). Case 2. $\rho \neq \boxtimes$. If $(\Phi/\rho) = \emptyset$, have $(\Phi/\rho) \,\varepsilon\, \mathrm{Coinit}$ automatically. If $(\Phi/\rho) \neq \emptyset$, $(\exists v).\ v \,\varepsilon\, \perp''(\Phi/\rho)$; fix v; then by the theorem, $\{v\} = \perp''(\Phi/\rho)$, whence $(\Phi/\rho) \,\varepsilon\, \mathrm{Coinit}$.

Theorem 2.8. $(\forall\pi)\colon (\emptyset/\pi) = \emptyset$.

Proof. The conclusion follows by (§6) using induction on $\#\pi$.

Theorem 2.9. $(\forall\xi)(\forall\pi)\colon (\xi/\xi+\pi) = \emptyset$.

Proof. The conclusion follows from (§3) and Theorem 2.8 by (§6), using induction on $\#\pi$.

Theorem 2.10. $(\forall\Phi)(\forall\pi)$: $(\Phi/\pi) = \bigsqcup\{(\xi/\pi) \mid \xi \varepsilon \Phi\}$.

Proof. The conclusion follows from (S6b) using induction on $\#\pi$.

Theorem 2.11. $(\forall\Phi)(\forall\pi)(\forall\eta)$: $\perp\eta \not\varepsilon (\perp``\Phi)$. ⊃ . $(\Phi/\eta+\pi) = \emptyset$.

Proof. The conclusion follows from (S4) and Theorem 2.8 by (S6), using induction on $\#\pi$.

Theorem 2.12. $(\forall\Phi)(\forall\pi)$: Φ ε Finite . ⊃. (Φ/π) ε Finite .

Proof. The conclusion follows from (S5) by (S6), using induction on $\#\pi$.

Theorem 2.13. $(\forall\Phi)(\forall\rho,\sigma)$: $\rho+\sigma \neq \emptyset$. ⊃ . $(\Phi/\rho+\sigma) = ((\Phi/\rho)/\sigma)$.

Proof. The conclusion follows from (S6), using induction on $\#\sigma$.

Theorem 2.14. $(\forall\Phi,\Psi)(\forall\pi)$: $\Phi\cup\Psi$ ε Coinit . ⊃ . $(\Phi\cup\Psi/\pi) = (\Phi/\pi)\cup(\Psi/\pi)$.

Proof. The conclusion follows from (S6) using induction on $\#\pi$.

Definition 2.15. We say that $\varphi(C,\delta)$-develops Φ (and write $\ulcorner\varphi(C,\delta)$develops $\Phi\urcorner$) as defined inductively on $\#\varphi$ according to the following three conditions:

(Δa) $(\forall\Phi)$: $\boxtimes$ develops Φ ,

(Δb) $(\forall\Phi)(\forall\pi)(\forall\xi)$: $(\pi+\xi)$ develops Φ . $\equiv$. πdevelops Φ . & . $\xi \in (\Phi/\pi)$.

(Δc) $(\forall\Phi)(\forall\varphi)$:. φdevelops Φ . $\equiv$: $(\forall\psi)(\forall\rho)$: $\varphi = \rho + \psi$. $\supset$. ρ develops Φ .

Note 2.15. The concept of a development is a slight modification of Newman's abstract form of Church and Rosser's "sequence of contractions on the parts of" a formula. See Newman's Paper and the Church-Rosser Paper.

Notation 2.16. In line with our previous notational conventions, the prefix '(C,δ)' and the subscript '$_{C,\delta}$', once introduced, will be suppressed.

Theorem 2.17. $(\forall\Phi)(\forall\xi)$: $\xi \in \Phi$. $\equiv$. $\langle\xi\rangle$ develops Φ .

Proof. I. Let $\xi \in \Phi$. By (Δa), $\boxtimes$ develops Φ . Since $\xi \in (\Phi/\boxtimes)$, it follows by (Δb) that $(\boxtimes+\xi)$ develops Φ . But $\boxtimes + \xi = \langle\xi\rangle$. II. Let $\langle\xi\rangle$ develop Φ . But $\boxtimes + \xi = \langle\xi\rangle$, whence $(\boxtimes + \xi)$ develops Φ . By (Δb) it follows that $\xi \in (\Phi/\boxtimes)$. But $(\Phi/\boxtimes) = \Phi$.

Notation 2.18. By "abus de langage" we sometime write $\ulcorner \xi$ develops $\Phi \urcorner$ instead of $\ulcorner \langle \xi \rangle$ develops $\Phi \urcorner$.

Theorem 2.19. $(\forall \rho) :$ ρ develops $\emptyset$. $\equiv$. $\rho = \boxtimes$.

Proof. I. Assume as hypothesis that ρ develops $\emptyset$. If $\rho \neq \boxtimes$, would have that $(\exists \pi)(\exists \xi)$. $\rho = \pi + \xi$. Fix π, ξ . Then by (Δb), $\xi \in (\emptyset/\pi)$. But $(\emptyset/\pi) = \emptyset$, whence $\xi \notin (\emptyset/\pi)$. So $\rho = \boxtimes$. II. Assume as hypothesis that $\rho = \boxtimes$. Then by (Δa), $(\forall \Phi)$. ρ develops Φ , whence ρ develops $\emptyset$.

Theorem 2.20. $(\forall \Phi, \Omega)(\forall \pi) :$ $\Omega \subseteq \Phi$. π develops Ω . $\supset$. π develops Φ .

Proof. By induction on $\# \pi$.

Lemma 2.21(a). $(\forall \Phi)(\forall \pi, \rho) :$ π develops Φ . ρ develops (Φ/π) . $\supset$. $(\pi + \rho)$ develops Φ .

Proof. By induction on $\# \rho$.

Lemma 2.21(b). $(\forall \Phi)(\forall \pi, \rho) :$ π develops Φ . $(\pi + \rho)$ develops Φ . $\supset$. ρ develops (Φ/π) .

Proof. By induction on $\# \rho$.

Theorem 2.21. $(\forall \Phi)(\forall \pi, \rho) :$ π develops Φ . $\supset$: ρ develops (Φ/π) . $\equiv$. $(\pi + \rho)$ develops Φ .

Proof. Use the lemmas.

Theorem 2.22. $(\forall\Phi)(\forall u)(\forall\rho)$: $\perp``\Phi = \{u\}$. ρ develops Φ . $\supset$. ρ ε (From u) .

Proof. By induction on $\#\rho$.

Definition 2.23. We say that u (C,δ)-strictly develops to v (and write $\ulcorner$u dev v$\urcorner$) iff

$(\exists\Phi)(\exists\pi)$: $\perp``\Phi = \{u\}$. $\pi \neq \emptyset$. π develops Φ . $\pi\ \varepsilon$(To v),

and we say that u (C,δ)-develops to v (and write $\ulcorner$u $\underline{\underline{\text{dev}}}$ v$\urcorner$) iff

u dev v . $\vee$. u = v .

Remark 2.24. $(\forall u,v)$: u imr v . $\supset$. u dev v .

Remark 2.25. $(\forall u,v)$: u dev v . $\supset$. u red v .

Theorem 2.26. $(\forall\Phi)(\forall\pi)$: π develops Φ . $\supset$. $(\exists\Omega)$. $\Omega \subseteq \Phi$. Ω ε Finite . π develops Ω .

Proof. By induction on $\#\pi$.

Theorem 2.27. $(\forall u,v)$: u dev v . $\equiv$. $(\exists\Omega)(\exists\pi)$: Ω ε Finite . $(\perp``\Omega) = \{u\}$. $\pi \neq \emptyset$. π develops Ω . π ε (To v) .

Proof. I. Assume DerivCx(C,δ) u,v ε Vx and the right hand side of the equivalence. Then u dev v follows trivially. II. Assume DerivCx(C,δ) . $u,v \in Vx$. u dev v . Fix Φ, π so that $(\perp``\Phi) = \{u\}$ . $\pi \neq \boxtimes$ . $\pi$ develops $\Phi$ . $\pi \in$ (To v) . By Theorem 2.26, $(\exists\Omega)$. $\Omega \subseteq \Phi$ . $\Omega \in$ Finite . $\pi$ develops $\Omega$. Fix $\Omega$. Since $\Omega \subseteq \Phi$, $(\perp``\Omega) \subseteq (\perp``\Phi) = \{u\}$ ; furthermore, since $\pi \neq \boxtimes$ , have that $\Omega \neq \emptyset$ , whence $(\perp``\Omega) = \{u\}$.

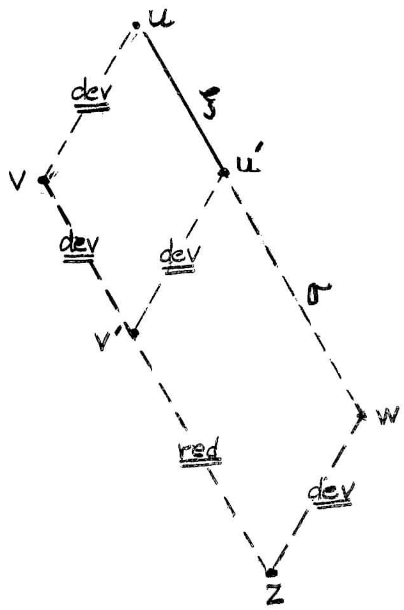
<u>Definition</u> <u>2</u>.<u>28</u>. We say that $\langle C,\delta\rangle$ is a <u>locally Church-Rosser</u> complex (and write ⌜LocChR(C,δ)⌝) iff

$$(\forall u,v,w)\colon u \underline{\underline{\text{dev}}}\ v \,.\, u \underline{\underline{\text{dev}}}\ w \,.\supset.\, (\exists z).\ v \underline{\underline{\text{dev}}}\ z \,.\, w \underline{\underline{\text{dev}}}\ z \,.$$

<u>Theorem</u> <u>2</u>.<u>29</u>. LocChR(C,δ) . $\supset$. ChR C .

<u>Proof</u>. Assume LocChR(C,δ) . Know ChR C . $\equiv$: $(\forall u,v,w)\colon u \underline{\underline{\text{imr}}}\ v \,.\, u \underline{\underline{\text{red}}}\ w \,.\supset.\, (\exists z).\ v \underline{\underline{\text{red}}}\ z \,.\, w \underline{\underline{\text{red}}}\ z$. Since $u \underline{\underline{\text{imr}}}\ v \,.\supset.\, u \underline{\underline{\text{dev}}}\ v$, it thus suffices to show that $(\forall u,v,w)\colon u \underline{\underline{\text{dev}}}\ v \,.\, u \underline{\underline{\text{red}}}\ w \,.\supset.\, (\exists z).\ v \underline{\underline{\text{red}}}\ z \,.\, w \underline{\underline{\text{red}}}\ z$. So assume $u \underline{\underline{\text{dev}}}\ v \,.\, u \underline{\underline{\text{red}}}\ w$. Use induction on $\#\rho$ where ρ is a reduction from u to w.

<u>Basis</u>. $\rho = \boxtimes$; then $u = w$, and it suffices to take $z = v$.



<u>Induction Step</u>. $\rho = \xi + \sigma$.
Let $u' = T\xi$. Then $u \ \underline{\underline{\mathrm{dev}}}\ u'$. Since also $u \ \underline{\underline{\mathrm{dev}}}\ v$, have that $(\exists v')$. $u' \ \underline{\underline{\mathrm{dev}}}\ v'$. $v \ \underline{\underline{\mathrm{dev}}}\ v'$. Fix v'. Then $u' \ \underline{\underline{\mathrm{dev}}}\ v'$ and σ descends from u' to w; by induction hypothesis, $(\exists z)$. $v' \ \underline{\underline{\mathrm{red}}}\ z$. $w \ \underline{\underline{\mathrm{red}}}\ z$. Fix z. Then $v \ \underline{\underline{\mathrm{red}}}\ v'$. $v' \ \underline{\underline{\mathrm{red}}}\ z$ and $w \ \underline{\underline{\mathrm{red}}}\ z$, q.e.d.

<u>Definition 2.30</u>. We say that φ (C,δ)-<u>completely develops</u> Φ (and write ⌜φ (C,δ)comdevelops Φ⌝) iff φ develops Φ and

$$(\forall\xi): \xi \ \varepsilon\ \Phi \ . \supset . \ (\exists n). \ (\xi/\langle\varphi_i\rangle_{i=1}^{n}) = \emptyset \ .$$

(Terminology from Newman's Paper.)

<u>Remark 2.30</u>. $(\exists n). \ (\xi/\langle\varphi_i\rangle_{i=1}^{n}) = \emptyset : \equiv : (\exists\pi)(\exists\psi). \ \varphi = \pi + \psi \ . \ (\xi/\pi) = \emptyset$.

<u>Remark 2.31</u>. $(\forall\Phi)(\forall\rho):$ ρ comdevelops Φ $. \equiv .$ ρ develops Φ . $(\Phi/\rho) = \emptyset$.

<u>Lemma 2.32(a)</u>. $(\forall\Phi,\Psi)(\forall\rho,\sigma'): (\Phi\cup\Psi) \ \varepsilon$ Coinit$_\cap$Finite . ρ comdevelops Φ . σ' comdevelops (Ψ/ρ) $. \supset .$ $(\rho+\sigma')$ comdevelops $(\Phi\cup\Psi)$.

Proof. Assume $(\Phi \cup \Psi) \in \mathrm{Coinit}_\cap\mathrm{Finite}$. ρ develops Φ . σ' develops (Ψ/ρ) . $(\Phi/\rho) = ((\Psi/\rho)/\sigma') = \emptyset$. Then by Theorem 2.20, ρ develops $(\Phi \cup \Psi)$ and σ' develops $(\Phi/\rho) \cup (\Psi/\rho)$; from the latter it follows by Theorem 2.14 that σ' develops $(\Phi \cup \Psi/\rho)$. Hence by Lemma 2.21(a) we have that $(\rho + \sigma')$ develops $(\Phi \cup \Psi)$. Furthermore, $(\Phi \cup \Psi/\rho + \sigma') =$ $(\Phi/\rho + \sigma') \cup (\Psi/\rho + \sigma') = ((\Phi/\rho)/\sigma') \cup ((\Psi/\rho)/\sigma') = (\ /\sigma') \cup \emptyset =$ $\emptyset \cup \emptyset = \emptyset$.

Lemma 2.32(b). $(\forall \Phi, \Psi)(\forall \rho, \sigma')$: $(\Phi \cup \Psi) \in \mathrm{Coinit}_\cap\mathrm{Finite}$. ρ comdevelops Φ . $(\rho + \sigma')$ comdevelops $(\Phi \cup \Psi)$. $\supset$. σ' comdevelops (Ψ/ρ) .

Proof. Assume $(\Phi \cup \Psi) \in \mathrm{Coinit}_\cap\mathrm{Finite}$. & . ρ develops Φ . $(\rho + \sigma')$ develops $(\Phi \cup \Psi)$. & . $(\Phi/\rho) = (\Phi \cup \Psi/\rho + \sigma') =$ $\emptyset$. Then by Lemma 2.21(b), σ' develops $(\Phi \cup \Psi/\rho)$. But $(\Phi \cup \Psi/\rho) = (\Phi/\rho) \cup (\Psi/\rho) = \emptyset \cup (\Psi/\rho) = (\Psi/\rho)$. So σ' develops (Ψ/ρ) . Furthermore, $((\Psi/\rho)/\sigma') = (\Psi/\rho + \sigma')$; but $\emptyset =$ $(\Phi \cup \Psi/\rho + \sigma') = (\Phi/\rho + \sigma') \cup (\Psi/\rho + \sigma') = ((\Phi/\rho)/\sigma') \cup$ $(\Psi/\rho + \sigma') = (\emptyset/\sigma') \cup (\Psi/\rho + \sigma') = \emptyset \cup (\Psi/\rho + \sigma') = (\Psi/\rho + \sigma')$. So σ' comdevelops (Ψ/ρ) .

Theorem 2.32. $(\forall \Phi, \Psi)(\forall \rho, \sigma')$: $(\Phi \cup \Psi) \in \mathrm{Coinit}_\cap\mathrm{Finite}$. ρ comdevelops Φ . $\supset$: σ' comdevelops (Ψ/ρ) . $\equiv$. $(\rho + \sigma')$ comdevelops $(\Phi \cup \Psi)$.

Proof. Use the lemmas.

Definition 2.33. Consider the following (C,δ)Condition (M) :

(M) $(\forall\Phi): \Phi \in$ Finite $.\supset.$ $(\exists\rho).\ \rho$ comdevelops Φ .

Theorem 2.34. (M) $.\equiv.$ $(\forall\Phi)(\forall\rho): \Phi \in$ Finite $.\ \rho$ develops Φ $.\supset.$ $(\exists\sigma').\ (\rho+\sigma')$ comdevelops Φ .

Proof. I. Assume (M) and $\Phi \in$ Finite $.\ \rho$ develops Φ . Then $(\Phi/\rho) \in$ Coinit$\cap$Finite , so by (M), $(\exists\sigma'): \sigma'$ develops (Φ/ρ) $.\ ((\Phi/\rho)/\sigma') = \emptyset$. Fix σ' . By Lemma 2.21(a), $(\rho+\sigma')$ develops Φ . Furthermore, $(\Phi/\rho+\sigma') = ((\Phi/\rho)/\sigma') = \emptyset$. II. Assume the right hand side of the equivalence, and let $\Phi \in$ Finite . As always, $\boxtimes$ develops Φ . Hence $(\exists\sigma').\ \sigma'$ develops $(\Phi/\boxtimes)$ $.\ (\Phi/\boxtimes+\sigma') = \emptyset$. Since $(\Phi/\boxtimes) = \Phi$ $.\ (\boxtimes+\sigma') = \sigma'$; this does it.

Definition 2.35. Consider the following (C,δ)Condition (P) :

(P) $(\forall\Phi)(\forall\rho,\sigma): \rho,\sigma$ comdevelops Φ $.\supset.$ $\{\rho,\sigma\} \in$ Coterminal

Theorem 2.36. (M) . (P) $.\supset.$ LocChR(C,δ) .

Proof. Assume the antecedent and suppose $u \underline{\underline{\text{dev}}}\ v$ $.\ u \underline{\underline{\text{dev}}}\ w$. To prove $(\exists z).\ v \underline{\underline{\text{dev}}}\ z\ .\ w \underline{\underline{\text{dev}}}\ z$. In case $u = v$ or $u = w$, take $z = w$ or $z = v$ respectively. It remains to consider when u dev v . u dev w . Then by Theorem 2.27, $(\exists\Phi,\Psi)(\exists\rho,\sigma): \Phi,\Psi \in$ Finite $.\ (\bot``\Phi) = (\bot``\Psi) = \{u\}\ .\ \rho \neq$

$\boxtimes \neq \sigma$. ρ develops Φ . σ develops Ψ . $\rho \varepsilon (\mathrm{To}\ v)$. $\sigma \varepsilon (\mathrm{To}\ w)$. Fix Φ, Ψ, ρ, σ. Then $(\Phi \cup \Psi) \varepsilon$ Coinit$_\cap$Finite . Hence $(\Phi/\sigma), (\Psi/\rho) \varepsilon$ Coinit$_\cap$Finite . By (M), $(\exists \rho', \sigma')$: ρ' develops $(\Phi \cup \Psi/\sigma)$. σ' develops $(\Phi \cup \Psi/\rho)$. $(\Phi \cup \Psi/\sigma + \rho') = (\Phi \cup \Psi/\rho + \sigma') = \emptyset$. (Recall $(\forall \Phi)(\forall \rho, \sigma)$: $(\Phi/\rho + \sigma) = ((\Phi/\rho)/\sigma)$.) Fix ρ', σ' . Then by (P), $\{\rho + \sigma', \sigma + \rho'\} \varepsilon$ Coterminal . In case $\sigma' = \boxtimes$ or $\rho' = \boxtimes$ take $z = v$ or $z = w$ respectively; otherwise there is a unique z such that $\rho', \sigma' \varepsilon (\mathrm{To}\ z)$. Obviously v $\underline{\underline{\mathrm{dev}}}$ z . w $\underline{\underline{\mathrm{dev}}}$ z . This does it.

<u>Corollary 2.36</u>. (M) . (P) . $\supset$. ChR C .

<u>Proof</u>. By Theorem 2.29, LocChR(C,δ) . $\supset$. ChR C .

SECTION 3

NORMAL DERIVATION COMPLEXES

Definition 3.1. We say that π, σ are (C,δ)-equivalent (and write $\ulcorner \pi(C,\delta) \doteq \sigma \urcorner$) iff $\{\pi, \sigma\} \varepsilon$ Coinitial$\cap$Coterminal . $(\forall \xi).\ (\xi/\pi) = (\xi/\sigma)$.

Note 3.1(a). Since $(\forall \xi)(\forall \pi)$: $\{\xi, \pi\} \not\varepsilon$ Coinitial $. \supset .$ $(\xi/\pi) = \emptyset$, the clause '$(\forall \xi)$: $(\xi/\pi) = (\xi/\sigma)$' in this definition could be replaced by '$(\forall \xi)$: $\{\xi, \pi, \sigma\}$ ε Coinitial $. \supset .$ $(\xi/\pi) = (\xi/\sigma)$' without changing the meaning.

Note 3.1(b). This concept of equivalence of reductions is more or less inherent in the discussion of the Church-Rosser Paper, and is implicit in Newman's Paper. Its present sharp form is essentially as given in Curry's Church-Rosser Paper.

Note 3.1(c). Equivalence of reductions can be relativized as follows: where $\{\pi, \rho\} \cup \Phi$ ε Coinital , write $\ulcorner \pi \doteq \rho \bmod \Phi \urcorner$ whenever π, ρ ε Coinitial$\cap$Coterminal : $(\forall \xi)$: $\xi \ \varepsilon\ \Phi\ . \supset .\ (\xi/\pi) = (\xi/\rho)$.

Remark 3.1(a). $(\forall \pi)$: $\pi \doteq \pi$.

Remark 3.1(b). $(\forall\pi,\sigma)\colon \pi \doteq \sigma \,.\supset.\, \sigma \doteq \pi$.

Remark 3.1(c). $(\forall\pi,\rho,\sigma)\colon \pi \doteq \rho \,.\, \rho \doteq \sigma \,.\supset.\, \pi \doteq \sigma$.

Remark 3.1(d). $(\forall\pi,\rho,\sigma,\tau)\colon \pi \doteq \rho \,.\, \sigma \doteq \tau \,.\supset.\, \pi+\sigma \doteq \rho+\tau$.

Theorem 3.2. $(\forall\rho)\colon \rho \doteq \boxtimes \,.\equiv.\, \rho = \boxtimes$.

Proof. I. Let $\rho = \boxtimes$. Then clearly $\rho \doteq \boxtimes$. II. Conversely, let $\rho \neq \boxtimes$. Then $(\exists\xi)(\exists\sigma).\ \rho = \xi + \sigma$. Fix ξ, σ . Then for $\Phi = \{\xi\}$, we see that $(\Phi/\rho) = \emptyset \neq \Phi$. But $(\Phi/\boxtimes) = \Phi$, whence $\sim(\rho \doteq \boxtimes)$.

Lemma 3.3. $(\forall v)(\forall\Phi)(\forall\tau)\colon \Phi = \{\xi \mid \perp\xi = v\} \,.\, \tau \,\varepsilon\, (\mathrm{From}\ v) \,.\, \tau \neq \boxtimes \,.\supset\colon (\forall\Psi)\colon (\Psi/\tau) \subseteq (\Phi/\tau)$.

Proof. Assume $\xi \,\varepsilon\, \Psi$. If $\perp\xi \neq v$, then $(\xi/\tau) = \emptyset \subseteq (\Phi/\tau)$. If $\perp\xi = v$, then $(\xi/\tau) = (\{\xi\}/\tau) \subseteq (\Phi/\tau)$ since $\{\xi\} \subseteq \Phi$. In either case $(\xi/\tau) \subseteq (\Phi/\tau)$. This holds for all $\xi \,\varepsilon\, \Psi$; hence by Theorem 2.10, $(\Psi/\tau) \subseteq (\Phi/\tau)$.

Theorem 3.3. (M). LocFin C $.\supset\colon (\forall\rho,\sigma)\colon \rho \neq \boxtimes \neq \sigma \,.\, \{\rho,\sigma\} \,\varepsilon\, \mathrm{Coinitial}_\cap\mathrm{Coterminal} \,.\supset.\, (\exists\tau).\ \rho + \tau \doteq \sigma + \tau$.

Proof. Recall that LocFin C $.\equiv.$ Coinit $\subseteq$ Finite (by Definition 1.38). By Remark 1.15, $(\exists_1 u).\ \rho,\sigma \,\varepsilon\, (\mathrm{From}\ u)\colon (\exists_1 v).\ \rho,\sigma \,\varepsilon\, (\mathrm{To}\ v)$. Fix u, v. Let $\Phi = \{\xi \mid \perp\xi = v\}$. Then $\Phi \,\varepsilon$ Coinit , whence by LocFin C, $\Phi \,\varepsilon$ Finite . By (M),

$(\exists\tau)$, τ comdevelops Φ. Fix τ.

Case 1. $\tau = \boxtimes$. Then $\rho + \tau = \rho$. $\sigma + \tau = \sigma$, so $\{\rho + \tau, \sigma + \tau\} \in$ Coinitial$\cap$Coterminal by hypothesis. (Proof completed below.)

Case 2. $\tau \neq \boxtimes$. Since $\rho, \sigma \in$ (To v) . $\tau \in$ (From v), $\rho + \tau \neq \boxtimes \neq \sigma + \tau$. Now we have by Theorem 1.26 that $\{\rho + \tau, \rho\}, \{\sigma + \tau, \sigma\} \in$ Coinitial . Hence by Theorem 1.21, $\{\rho + \tau, \sigma + \tau\} \in$ Coinitial . Also, we have by Theorem 1.27 that $\{\rho + \tau, \sigma + \tau\} \in$ Coterminal . So $\{\rho + \tau, \sigma + \tau\} \in$ Coterminal .

Now in either Case 1 or Case 2, we have, using the lemma, that $(\forall\xi)$: $(\xi/\rho + \tau) = ((\xi/\rho)/\tau) \subseteq (\Phi/\tau) = \emptyset$. $(\xi/\sigma + \tau) = ((\xi/\sigma)/\tau) \subseteq (\Phi/\tau) = \emptyset$. So $(\forall\xi)$. $(\xi/\rho + \tau) = (\xi/\sigma + \tau)$. This completes the proof.

Definition 3.4. Consider the following (C, δ)Conditions $(\underline{N})$, $(\underline{N}')$, $(\underline{N}'')$:

(N) $(\forall\Phi)(\forall\rho, \sigma)$: $\Phi \in$ Finite . ρ, σ comdevelops Φ. $\supset$. $\rho \doteq \sigma$.

(N') $(\forall\Phi, \Psi)(\forall\rho, \sigma, \rho', \sigma')$: $(\Phi \cup \Psi) \in$ Coinit$\cap$Finite . ρ comdevelops Φ . σ comdevelops Ψ . ρ' comdevelops (Φ/σ) . σ' comdevelops (Ψ/ρ) . $\supset$. $\rho + \sigma' \doteq \sigma + \rho'$.

(N'') $(\forall\Phi, \Psi)(\forall\rho, \sigma)$: $(\Phi \cup \Psi) \in$ Coinit$\cap$Finite . ρ comdevelops Φ . σ comdevelops Ψ . $\supset$. $(\exists\rho', \sigma')$ ρ' comdevelops (Φ/σ) . σ' comdevelops (Ψ/ρ) . $\rho + \sigma' \doteq \sigma + \rho'$.

<u>Lemma 3.5(a)</u>. (N) . ≡ . (N') .

<u>Proof</u>. I. Assume (N) and $(\Phi \cup \Psi)$ ε Coinit$_\cap$Finite . ρ comdevelops Φ . σ comdevelops Ψ . ρ' comdevelops (Φ/σ) . σ' comdevelops (Ψ/ρ) . Then by Lemma 2.32(a), $(\rho + \sigma')$ comdevelops $(\Phi \cup \Psi)$. $(\sigma + \rho')$ comdevelops $(\Phi \cup \Psi)$, whence by (N), $\rho + \sigma' \doteq \sigma + \rho'$. II. Assume (N') and Φ ε Finite . ρ, σ comdevelop Φ . Then $\boxtimes$ comdevelops (Φ/σ) . $\boxtimes$ comdevelops (Φ/ρ) , whence taking $\Phi = \Psi$ in (N'), $\rho + \boxtimes \doteq \sigma + \boxtimes$, whence $\rho \doteq \sigma$.

<u>Lemma 3.5(b)</u>. (N'') . ⊃ . (N) .

<u>Proof</u>. Assume (N'') and let Φ ε Coinit$_\cap$Finite . ρ, σ comdevelop Φ. Then by (N''), $(\exists \rho', \sigma')$. ρ' comdevelops (Φ/σ) . σ' comdevelops (Ψ/ρ) . $\rho + \sigma' \doteq \sigma + \rho'$. Fix ρ', σ' . Then ρ', σ' develop $\emptyset$, whence $\rho' = \sigma' = \boxtimes$, whence $\rho = \rho + \sigma' \doteq \sigma + \rho' = \sigma$.

<u>Theorem 3.5</u>. (M) . ⊃: (N) . ≡ . (N') . ≡ . (N'') .

<u>Proof</u>. I. By Lemma 3.5(a), (M) . (N) . ⊃. (N') .

II. Assume (M) . (N') , and let $(\Phi \cup \Psi)$ ε Coinit$\cap$ Finite . ρ comdevelops Φ . σ comdevelops Ψ . Then by (M), $(\exists \rho', \sigma')$: ρ' comdevelops (Φ/σ) . σ' comdevelops (Ψ/ρ) . Fix ρ', σ' . By (N'), $\rho + \sigma' \doteq \sigma + \rho'$. So (M) . (N') . ⊃. (N'') .

III. By Lemma 3.5(b), (M) . (N'') . ⊃. (N) .

Thus we have (M) . ⊃ : (N) . ⊃. (N') . ⊃. (N'') . ⊃ . (N) , from which the conclusion follows.

<u>Definition 3.6.</u> We say that $\langle C,\delta\rangle$ is a <u>normal complex</u> (and write $\ulcorner$NormCx$(C,\delta)^\urcorner$) iff conditions (M) and (N) are both satisfied:

(M) $(\forall\Phi):\Phi \varepsilon \text{Finite} .\supset. (\exists\rho). \rho$ comdevelops Φ .

(N) $(\forall\Phi):\Phi \varepsilon \text{Finite} .\supset: (\forall\rho,\sigma): \rho,\sigma$ comdevelop Φ . $\supset. \rho \doteq \sigma$.

<u>Theorem 3.7.</u> NormCx(C,δ) $.\supset.$ LocChR(C,δ) .

<u>Proof.</u> Obviously (N) $\supset$ (P) . Hence NormCx(C,δ) $.\supset.$ (M) . (P) . Hence LocChR(C,δ) by Theorem 2.36.

<u>Corollary 3.7.</u> NormCx(C,δ) $.\supset.$ ChR C .

<u>Proof.</u> By Theorem 2.29, Loc ChR(C,δ) $.\supset.$ ChR C .

<u>Definition 3.8.</u> We define the set (C,δ)<u>Dev</u>, called the set of (C,δ)-<u>developments</u>, as follows:

Dev $= \{\varphi \mid (\exists\Phi). \varphi$ develops $\Phi\}$.

<u>Remark 3.8.</u> By Theorem 2.26, we have $(\forall\rho): \rho \varepsilon$ Dev . $\equiv . (\exists\Phi).\Phi \varepsilon$ Finite . ρ develops Φ .

<u>Definition 3.9.</u> We define the set (C,δ)<u>ComDev</u>, called the set of (C,δ)-<u>complete</u> <u>developments</u>, as follows:

ComDev $= \{\varphi \mid (\exists\Phi). \varphi$ comdevelops $\Phi\}$.

(See Definition 2.30 and Remarks 2.30, 2.31.)

Definition 3.10. We define the set (C,δ)ComDevFin, called the set of (C,δ)-complete developments of finite sets, as follows:

ComDevFin $= \{\varphi \mid (\exists \Phi) . \Phi \subseteq$ Finite . φ comdevelops $\Phi\}$.

Remark 3.10. LocFin C $. \supset .$ ComDev = ComDevFin . (See Definition 1.38.)

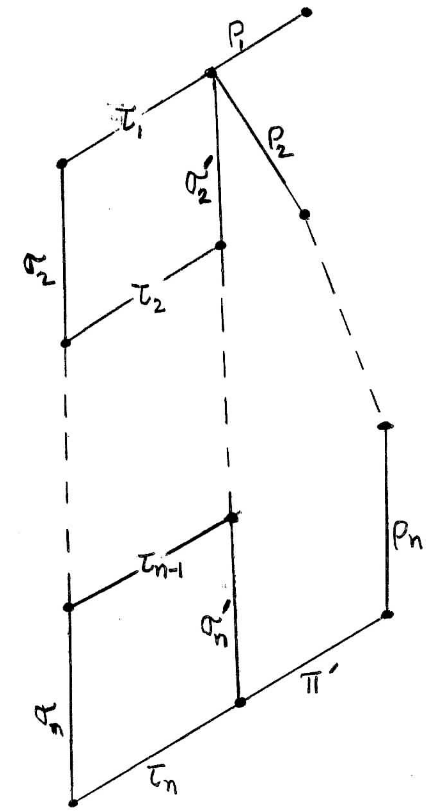
Lemma 3.11. NormCx(C,δ) $. \supset :$ $(\forall \rho_1,\ldots,\rho_n) : \rho = \rho_1 + \cdots + \rho_n \, . \, \rho_1,\ldots,\rho_n \in$ Dev $. \supset . (\exists \pi,\sigma_1,\ldots,\sigma_n) . \; \rho + \pi \doteq \sigma_1 + \cdots + \sigma_n \, . \, \pi,\sigma_1,\ldots,\sigma_n \in$ ComDevFin .

Proof. By induction on n.

Basis. $n = 0$. Then $\rho = \rho_1 + \cdots + \rho_0 = \boxtimes$; take $\pi = \boxtimes$ and note $\sigma_1 + \cdots + \sigma_0 = \boxtimes$. Since $\boxtimes \in$ ComDevFin, everything is as it should be.

Induction Step. $n \geq 1$. By induction hypothesis, $(\exists \pi', \sigma_2', \ldots, \sigma_n') . \; \rho_2 + \cdots + \rho_n + \pi' \doteq \sigma_2' + \cdots + \sigma_n' \, . \, \pi', \sigma_2', \ldots, \sigma_n' \in$ ComDevFin . Fix $\pi', \sigma_2', \ldots, \sigma_n'$. (May have $\rho_1 = \boxtimes$; then $\rho = \rho_2 + \cdots + \rho_n$, whence $\rho + \pi' \doteq \sigma_1' + \cdots + \sigma_n'$ and we are done. However, the general argument covers this case.) Since $\rho_1 \in$ Dev , $(\exists \Phi) . \; \rho_1$ develops Φ . Fix Φ . Then by Theorem 2.26, $(\exists \Phi_0) . \; \Phi_0 \subseteq \Phi \, . \, \Phi_0 \in$ Finite . ρ_1 develops Φ_0 . Fix Φ_0; note $\Phi_0 \in$ Coinit$\cap$Finite . For $1 \leq i \leq n$, define $\Phi_i = (\Phi_0 / \rho_1 + \sigma_2' + \cdots + \sigma_i')$; then each $\Phi_i \in$ Coinit$\cap$Finite , so $(\exists \tau_i) . \; \tau_i$ comdevelops Φ_i . Fix $\tau_1, \ldots, \tau_n$. Now since each $\sigma_i' \in$ ComDevFin , $(\exists \Psi_i') .$ $\Psi_i' \in$ Finite . σ_i' comdevelops Ψ_i' . Note each $\Psi_i' \in$

Coinit$\cap$Finite . Fix $\Psi_2', \ldots, \Psi_n'$. Define $\Psi_i = (\Psi_i'/\tau_{i-1})$. Then each $\Psi_i \in$ Coinit$\cap$Finite , so by (M),



$(\exists\sigma_i)$. σ_i comdevelops Ψ_i . Fix $\sigma_2, \ldots, \sigma_n$. Have then for all i, $1 \leq i \leq n$: $\tau_{i-1} + \sigma_i$, $\sigma_i' + \tau_i$ comdevelop $\Phi_{i-1} \cup \Psi_i'$. So $\tau_{i-1} + \sigma_i \doteq \sigma_i' + \tau_i$. It follows by induction that $\tau_1 + \sigma_2 + \cdots + \sigma_n \doteq \sigma_2' + \cdots + \sigma_n' + \tau_n$.

Now let $\sigma_1 = \rho_1 + \tau_1$ and let $\pi = \pi' + \tau_n$. By construction, $\sigma_1, \sigma_2, \ldots, \sigma_n \in$ ComDevFin ; furthermore, $\rho + \pi = \rho_1 + \rho_2 + \cdots + \rho_n + (\pi' + \tau_n) \doteq \rho_1 + \sigma_2' + \cdots + \sigma_n' + \tau_n \doteq \rho_1 + \tau_1 + \sigma_2 + \cdots + \sigma_n = \sigma_1 + \sigma_2 + \cdots + \sigma_n$. Lastly, since $\pi' \in$ ComDevFin , $(\exists\Omega)$. $\Omega \in$ Finite . π' comdevelops Ω. Fix Ω. Furthermore, $\Phi_n = (\Phi_0/\rho_1 + \sigma_2' + \cdots + \sigma_n') = (\Phi_0/\rho + \pi') = ((\Phi_0/\rho)/\pi')$. τ_n comdevelops Φ_n ;

hence by Theorem 2.32, $\pi = \pi' + \tau_n$ comdevelops $\Omega \cup (\Phi_0/\rho)$, where $(\Omega \cup (\Phi_0/\rho)/\pi) = (\Omega/\pi' + \tau_n) \cup ((\Phi_0/\rho)/\pi' + \tau_n) = (\emptyset/\tau_n) \cup (\Phi_n/\tau_n) = \emptyset$. This completes the proof.

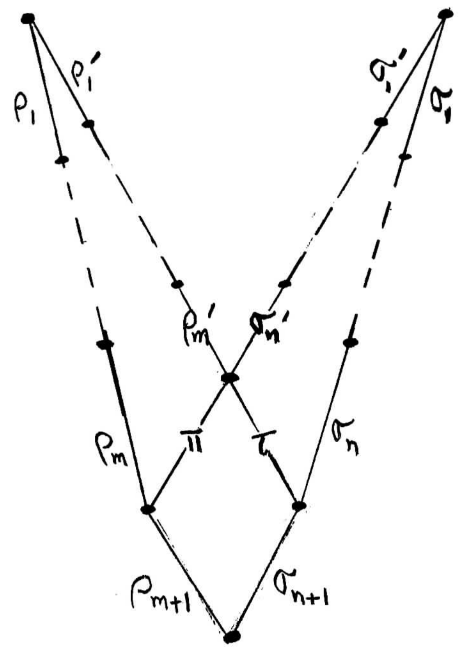
<u>Theorem 3.11.</u> NormCx(C,δ) . $v \in$ NormVx . $\rho \in$ (To v) . $\rho = \rho_1 + \cdots + \rho_n$. & . $\rho_1,\ldots,\rho_n \in$ Dev . $\supset$. $(\exists\sigma_1,\ldots,\sigma_n)$. $\rho \doteq \sigma_1 + \cdots + \sigma_n$. & . $\sigma_1,\ldots,\sigma_n \in$ ComDevFin .

<u>Proof.</u> Apply the lemma. Since $v \in$ NormVx, $\pi = \boxtimes$. This does it.

<u>Theorem 3.12.</u> NormCx(C,δ) . $u,v \in$ Vx . u conv v . $\supset$. $(\exists\rho,\rho_1,\ldots,\rho_{m+1},\sigma,\sigma_1,\ldots,\sigma_{n+1})$. $\rho = \rho_1 + \cdots + \rho_{m+1}$. $\sigma = \sigma_1 + \cdots + \sigma_{n+1}$. & . $\rho \in$ (From u) . $\sigma \in$ (From v) . $\{\rho,\sigma\} \in$ Coterminal . & . $\rho_1,\ldots,\rho_{m+1},\sigma_1,\ldots,\sigma_{n+1} \in$ ComDevFin .

<u>Proof.</u> Assume the antecedent. Since NormCx(C,δ), have by Corollary 3.7 that ChR C. Hence by Theorem 1.33, since u conv v, have $(\exists\rho',\sigma')$. $\rho' \in$ (From u) . $\sigma' \in$ (From v) . $\{\rho',\sigma'\} \in$ Coterminal . Fix ρ', σ'. Then $(\exists\rho_1',\ldots,\rho_m',\sigma_1',\ldots,\sigma_n')$. $\rho' = \rho_1' + \cdots + \rho_m'$. $\sigma' = \sigma_1' + \cdots + \sigma_n'$. $\rho_1',\ldots,\rho_m',\sigma_1',\ldots,\sigma_n' \in$ Dev . Now by Lemma 3.11, $(\exists\pi,\rho_1,\ldots,\rho_m,\tau,\sigma_1,\ldots,\sigma_n)$. $\rho' + \pi \doteq \rho_1 + \cdots + \rho_m$. & . $\sigma' + \tau \doteq \sigma_1 + \cdots + \sigma_n$. $\pi,\rho_1,\ldots,\rho_m,\tau,\sigma_1,\ldots,\sigma_n \in$ ComDevFin . Fix π, $\rho_1,\ldots,\rho_m$, τ, $\sigma_1,\ldots,\sigma_n$. Fix Φ, Ψ such that $\Phi,\Psi \in$ Finite . π comdevelops Φ . τ comdevelops Ψ . Let $\Phi' = (\Phi/\tau)$. $\Psi' = (\Psi/\pi)$. Then since NormCx(C,δ), have (N''), whence $(\exists\rho_{m+1},\sigma_{n+1})$. ρ_{m+1} comdevelops (Ψ/π) .

σ_{n+1} comdevelops (Φ/τ) . $\pi + \rho_{m+1} \doteq \tau + \sigma_{n+1}$. Fix ρ_{m+1}, σ_{n+1} ; note ρ_{m+1}, σ_{n+1} ε ComDevFin . $\{\rho_{m+1}, \sigma_{n+1}\}$ ε Coterminal . Now let $\rho = \rho_1 + \cdots + \rho_m + \rho_{m+1}$, $\sigma = \sigma_1 + \cdots + \sigma_n + \sigma_{n+1}$. Verify that the desired conditions are all satisfied.



<u>Definition 3.13</u>. We define the set <u>CMaxCoinit</u>, called the set of C-<u>maximal coinitial sets</u> (of cells), as follows:

$$\text{MaxCoinit} = \{\Phi \mid \Phi \in \text{Coinit} : (\forall \xi): \Phi \cup \{\xi\} \in \text{Coinit} . \supset . \xi \in \Phi\} .$$

<u>Definition 3.14</u>. We define the set (C,δ)<u>TotDev</u>, called the set of (C,δ)-<u>total developments</u>, as follows:

$$\text{TotDev} = \varphi \mid (\exists \Phi): \Phi \in \text{MaxCoinit} . \varphi \text{ comdevelops } \Phi\} .$$

<u>Remark 3.14(a)</u>. $(\exists v) . v \in \text{NormVx}: \supset . \emptyset \in \text{TotDev}$.

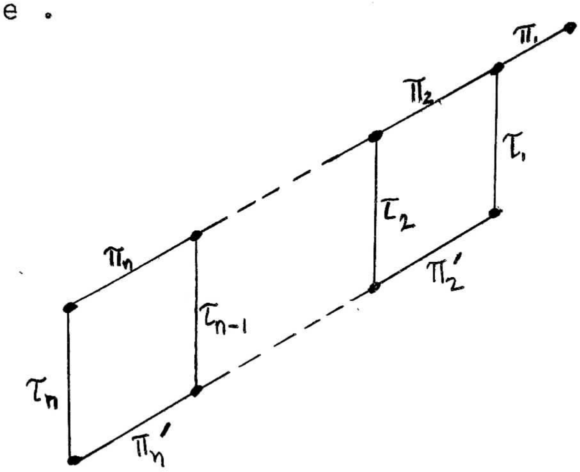
Remark 3.14(b). (N) $.\supset:$ $(\forall\rho,\sigma)$: $\rho,\sigma \in$ TotDev . $\rho \neq \emptyset \neq \sigma$. $\{\rho,\sigma\} \in$ Coinitial $.\supset.$ $\rho \doteq \sigma$.

Theorem 3.15. NormCx(C, δ) . LocFin C $.\supset:$ $(\forall\rho,\rho_1,\ldots,\rho_n)(\forall v)$: $\rho = \rho_1 + \cdots + \rho_n$. $\rho_1,\ldots,\rho_n \in$ Dev . $\rho \in$ To v . v $\in$ NormVx $.\supset.$ $(\exists\sigma_1,\ldots,\sigma_n)$. $\rho \doteq \sigma_1 + \cdots + \sigma_n$. $\sigma_1,\ldots,\sigma_n \in$ TotDev .

Proof. Assume NormCx(C,δ) . LocFin C. I.e., (M) . (N) . $(\forall\Phi)$. $\Phi \in$ Finite . In case $\rho = \boxtimes$, take $\sigma_1 = \cdots = \sigma_n = \boxtimes$; by Remark 3.14(a), the theorem follows. So assume $\neq \boxtimes$. In this case use induction on n.

Basis. n = 0 . Assume the antecedent. Then $\rho = \rho_1 + \cdots + \rho_n = \boxtimes$ and we are done .

Induction Step. n = m+1 . Assume the antecedent. Then by Theorem 3.11, $(\exists\pi_1,\ldots\pi_n)$. $\rho \doteq \pi_1 + \cdots + \pi_n$. $\pi_1,\ldots,\pi_n \in$ ComDevFin . Fix $\pi_1,\ldots,\pi_n$. Then $(\exists\Phi_1,\ldots,\Phi_n)$ $(\forall i)$: $1 \leq i \leq n$ $.\supset.$ $\Phi_i \in$ Finite . π_i comdevelops Φ_i . Fix $\Phi_1,\ldots,\Phi_n$. Let $\Omega = \{\xi \mid \{\xi,\rho\} \in$ Coinitial$\}$. Then since $\rho \neq \boxtimes$, $\Omega \in$ Coinit , whence since LocFin C have $\Omega \in$ Finite . Since $\rho \neq \boxtimes$, see that $\Omega \neq \emptyset$. Clearly $\Omega \in$ MaxCoinit . $\Phi_1 \subseteq \Omega$. Let $\Psi_0 = \Omega$ and for $1 \leq i \leq n$ let $\Psi_i = (\Psi_{i-1}/\pi_i)$. By (M), $(\exists\pi_2',\ldots,\pi_n')(\forall i)$: $1 < i \leq n$ $.\supset.$ π_i' comdevelops (Φ_i/τ_{i-1}) . Fix $\pi_2',\ldots,\pi_n'$. By (N), $(\forall i)$: $1 < i \leq n$ $.\supset.$ $\pi_i + \tau_i = \tau_{i-1} + \pi_i'$. So $\pi_2 + \cdots + \pi_n + \tau_n \doteq \tau_1 + \pi_2' + \cdots + \pi_n'$; since v $\in$ NormVx , it follows that $\tau_n = \boxtimes$, whence $\pi_2 + \cdots + \pi_n \doteq \tau_1 + \pi_2' + \cdots + \pi_n'$. Let $\sigma_1 = \pi_1 + \tau_1$. Since π_1 comdevelops Φ_1 . τ_1 comdevelops (Ω/π_1) , it follows by Lemma 2.32(a) that σ_1 comdevelops Ω. Then $\sigma_1 \varepsilon$ TotDev. Furthermore, the induction hypothesis applies to $\pi_2' + \dots + \pi_n'$, so $(\exists\sigma_2,\dots,\sigma_n)$. $\pi_2' + \dots + \pi_n' \doteq \sigma_2 + \dots + \sigma_n$. $\sigma_2,\dots,\sigma_n \varepsilon$ TotDev . Have now $\rho \doteq \pi_1 + \pi_2 + \dots + \pi_n \doteq \pi_1 + \tau_1 + \pi_2' + \dots + \pi_n' \doteq \sigma_1 + \sigma_2 + \dots + \sigma_n$, and we are done .



Remark 3.15. It is not excluded that some of the σ_i's be null; should this occur, it is clear that there will be a j such that $(\forall i)$: $i \le j$.$\supset$. $\sigma_i \ne \boxtimes$: $j < i \le n$.$\supset$. $\sigma_i = \boxtimes$.

Definition 3.16. We say that $\langle C,\delta\rangle$ has the *local finite descent* property (and write ⌜LocFinDesc(C,δ)⌝) iff the following condition is satisfied:

$$(\forall\Phi)(\forall\varphi): \Phi \ \varepsilon \ \text{Finite} \ . \ \varphi \ \text{develops} \ \Phi \ .\supset. \ \#\varphi < \infty .$$

Definition 3.17. We define the set (C,δ)*SysDesc*, called the set of (C,δ)-*systematic descents*, as follows:

$$\text{SysDesc} = \Big\{\textstyle\sum_{i=1}^{N} \varphi_i \,\Big|\, N \leq \infty \,.\, (\forall n): 0 \leq n < N \,.\, \supset . \; \varphi_{n+1} \in \text{Dev} : (\forall i): 1 \leq i < N \,.\, \supset . \; \varphi_i \in \text{TotDev} \cap \text{Redn}\Big\} .$$

Theorem 3.18. NormCx(C,δ) . LocFinDesc(C,δ) . LocFin C . $\supset$: $(\forall u)(\forall\varphi)$: $u \in$ PrenormVx . $\varphi \in$ From u . $\varphi \in$ SysDesc . $\supset$. $\#\varphi < \infty$.

Proof. Assume NormCx(C,δ) . LocFinDesc(C,δ) . LocFin C , and let $u \in$ PrenormVx . $\varphi \in$ From u . $\varphi \in$ SysDesc . If $u \in$ NormVx , $\#\varphi = 0$ and we are done . Otherwise by Definition 1.31 and 1.28, we have that $(\exists v)(\exists\rho)$. $v \in$ NormVx . $\rho \neq \boxtimes$. $\rho \in$ (From u)$\cap$(To v) . Fix v,ρ . By Theorem 3.15, $(\exists\sigma_1,\ldots,\sigma_m)$. $\rho \doteq \sigma_1 + \cdots + \sigma_m$. $\sigma_1,\ldots,\sigma_m \in$ TotDev . Fix $\sigma_1,\ldots,\sigma_m$. Let $\varphi = \sum_{i=1}^{N} \varphi_i$ as in Definition 3.17.

Case 1. $N < \infty$. Then use induction on N.

Basis. $N = 0$. Trivial.

Induction Step. $N = n+1$. Then $\varphi_1,\ldots,\varphi_n \in$ Redn ; since LocFinDesc(C,δ) . LocFin C , $\varphi_N = \varphi_{n+1} \in$ Redn .

Case 2. $N = \infty$. Then $(\forall i)$: $1 \leq i < \infty$. $\supset$. $\varphi_i \in$ TotDev$\cap$Redn , whence by induction on m (where $\rho \doteq \sigma_1 + \cdots + \sigma_m$), we see that $(\forall i)$: $1 \leq i \leq m$. $\supset$. $\varphi_i = \sigma_i$, whence $\varphi_m \in$ To v . Hence by induction on i we see that $(\forall i)$: $m < i < \infty$. $\supset$. $\varphi_i = \boxtimes$, whence $\#\varphi < \infty$, q.e.d.

Corollary 3.18. NormCx(C,δ) . LocFinDesc(C,δ) . LocFin C . $\supset$: $(\forall u)(\forall\varphi)$: $u \in$ PrenormVx . $\varphi \in$ From u . $\varphi \in$ TotDev . $\supset$. $\#\varphi < \infty$.

SECTION 4

REGULAR DERIVATION COMPLEXES

Definition 4.1. We say that $\langle C, \delta \rangle$ is a regular complex (and write $\ulcorner \mathrm{RegCx}(C,\delta) \urcorner$) iff the following (C,δ)Condition (Reg) is satisfied:

(Reg) $(\forall \Psi)(\forall \pi)(\forall \xi, \eta, \zeta) : \pi \text{ develops } \Psi \,.\, \{\xi, \eta\} \subseteq (\zeta/\pi) \,.\, \xi \neq \eta \,.\supset.\, \mathrm{Nc}(\xi/\eta) = 1$

(See Notation 4.1 below.)

Notation 4.1. As in Rosser's Textbook, we write $\ulcorner \mathrm{Nc}\, \alpha \urcorner$ to denote the cardinal number of α.

Remark 4.1. (Reg) $. \equiv :$ $(\forall \Psi)(\forall \pi)(\forall \xi, \eta, \zeta) : \pi \text{ develops } \Psi \,.$ $\{\xi, \eta\} \subseteq (\zeta/\pi) \,.\, \xi \neq \eta \,.\supset.\, \mathrm{Nc}(\xi/\eta) = \mathrm{Nc}(\eta/\xi) = 1$.

Definition 4.2. We say that J is a relation of relative separation on $\langle C, \delta \rangle$ (and write $\ulcorner (C,\delta)\mathrm{RelSep}\ J \urcorner$) iff J is a relation holding between sets Φ of cells and ordered pairs of cells (notation: $\ulcorner J_\Phi(\xi, \eta) \urcorner$; read $\ulcorner \Phi$ J-separates ξ from $\eta \urcorner$) satisfying the following (C,δ)Conditions (J1) - (J6):

(J1)✓ $(\forall \Phi)(\forall \xi) : \Phi \cup \{\xi\} \in \mathrm{Coinit} \,.\supset.\, J_\Phi(\xi, \xi)$,

(J2) $(\forall \Phi)(\forall \xi, \eta) : J_\Phi(\xi, \eta) \,.\supset.\, J_\Phi(\eta, \xi)$,

(J3) $(\forall \Phi)(\forall \xi, \eta) : J_\Phi(\xi, \eta) \,.\supset.\, \Phi \cup \{\xi, \eta\} \in \mathrm{Coinit}$,

(J4)✓ $(\forall \Phi)(\forall \xi, \eta) : J_\Phi(\xi, \eta) \,.\, \xi \neq \eta \,.\supset.\, \mathrm{Nc}(\xi/\eta) = 1$,

(J5)✓ $(\forall \Phi)(\forall \xi, \eta, \Theta)(\forall \xi', \eta') : J_\Phi(\xi, \eta) \,.\, \Theta \in \Phi \,.\, \xi' \in (\xi/\Theta) \,.\, \eta' \in (\eta/\Theta) \,.\supset.\, J_{(\Phi/\Theta)}(\xi', \eta')$,

(J6) $(\forall \Phi, \Omega)(\forall \xi, \eta) : J_\Phi(\xi, \eta) \,.\, \Omega \subseteq \Phi \,.\supset.\, J_\Omega(\xi, \eta)$.

The conditions (J1), (J4), (J5) are of special importance and have been marked with a check for emphasis. (See Corollary 4.4 below.)

Note 4.2. In Newman's Paper, Newman introduced a relation J of absolute separation (or "non-interference") between coinitial cells, and proved (ChR C) and other properties for complexes C satisfying certain conditions involving J. In Rosser's Review of Curry's Paper a counterexample is given which shows that such a relation does not exist in systems of λ-conversion. We show in Part II that a relation of relative separation does exist in all the interesting systems of λ-conversion.

Remark 4.2. The additional condition (J7), $(\forall\Phi,\Psi)(\forall\xi,\eta)$: $(\Phi\cup\Psi)\in\text{Coinit} \,.\, J_\Phi(\xi,\eta) \,.\, J_\Psi(\xi,\eta) \,.\supset.\, J_{(\Phi\cup\Psi)}(\xi,\eta)$, is satisfied in our applications.

Theorem 4.3. $(\exists J).\ (J1)\,.\,(J4)\,.\,(J5)\ ;\supset:\ \text{RegCx}(C,\delta)$.

Proof. Assume the antecedent, and let π develop Ψ . & . $\{\xi,\eta\}\subseteq(\zeta/\pi)$. $\xi\neq\eta$. Since $(\zeta/\pi)\neq\emptyset$, it follows that $\Psi\cup\{\zeta\}\in\text{Coinit}$. Hence $J_\Psi(\zeta,\zeta)$ by (J1) . Hence by (J5) have by induction on $\#\pi$ that $J_{(\Psi/\pi)}(\xi,\eta)$. Therefore $\text{Nc}(\xi/\eta)=1$ by (J4) . This completes the proof.

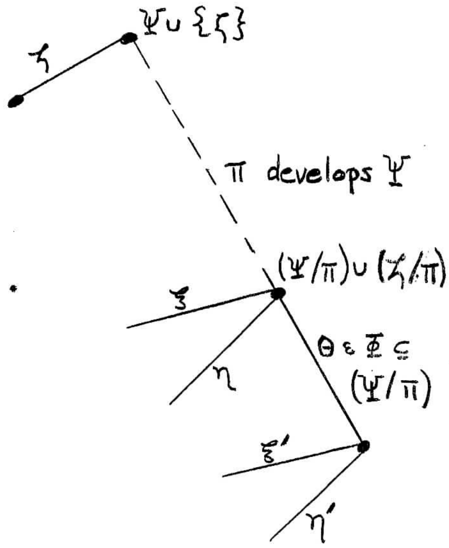
Theorem 4.4. $\text{RegCx}(C,\delta)\ .\supset.\ (\exists J).\ \text{RelSep}\ J$.

Proof. Assume $\text{RegCx}(C,\delta)$; define J by the following equation:

$$J = \{\langle \Phi, \xi, \eta \rangle \mid (\exists \Psi)(\exists \pi)(\exists \zeta) : \Psi \cup \{\zeta\} \varepsilon \text{ Coinit} . \pi \text{ develops } \Psi . \Phi \subseteq (\Psi/\pi) . \{\xi, \eta\} \subseteq (\zeta/\pi)\} \cup \{\langle \Phi, \xi, \xi \rangle \mid \Phi \cup \{\xi\} \varepsilon \text{ Coinit}\} .$$

(J1), (J3), (J6) follow automatically for this J. (J2) for this J follows from the fact that $(\forall \xi, \eta) : \{\xi, \eta\} = \{\eta, \xi\}$. (J4) for this J follows from Condition (Reg). Finally we have for this J that (J5) is implied by the following condition:

$$(\forall \Phi, \Psi)(\forall \xi, \eta, \zeta, \Theta)(\forall \pi) :. \Psi \cup \{\zeta\} \varepsilon \text{ Coinit} . \pi \text{ develops } \Psi . \& . \Phi \subseteq (\Psi/\pi) . \{\xi, \eta\} \subseteq (\zeta/\pi) . \Theta \varepsilon \Phi . \xi' \varepsilon (\xi/\Theta) . \eta' \varepsilon (\eta/\Theta) : \supset . \Psi \cup \{\zeta\} \varepsilon \text{ Coinit} . (\pi + \Theta) \text{ develops } \Psi . (\Phi/\Theta) \subseteq (\Psi/\pi + \Theta) . \{\xi', \eta'\} \subseteq (\zeta/\pi + \Theta) .$$



$\{\xi, \eta\} \subseteq (\zeta/\pi) . \xi' \varepsilon (\xi/\Theta) . \eta' \varepsilon (\eta/\Theta) .$

This in turn follows from the following two conditions, the latter of which is to be applied as indicated alphabetically and also with '$\{\zeta\}$' for 'Ψ':

$$(\forall \Psi)(\forall \pi)(\forall \Theta) : \pi \text{ develops } \Psi . \Theta \varepsilon (\Psi/\pi) . \supset . (\pi + \Theta) \text{ develops } \Psi ,$$

$$(\forall \Psi)(\forall \pi)(\forall \Theta) : \pi + \Theta \neq \boxtimes . \supset . (\Psi/\pi + \Theta) = ((\Psi/\pi)/\Theta) .$$

The first of these is immediate from Definition 2.15; the second is Theorem 2.13. This completes the proof.

Corollary 4.4. $(\exists J).(J1).(J4).(J5) :\supset: (\exists J).$ RelSep J .

Proof. Immediate using Theorems 4.3 and 4.4.

Definition 4.5. Where RelSep J, we define for each $\Phi \varepsilon$ Coinit the set $(C,\delta,J)\Phi$Sep, called the set of sets J-separated by Φ, as follows:

$$(C,\delta,J)\Phi\mathrm{Sep} = \{\Psi \mid (\forall \xi,\eta): \xi,\eta \varepsilon \Psi . \supset . J_{\Phi}(\xi,\eta)\} .$$

Note 4.5. This concept is central in our further development, which uses Newman's method of proof. (See Newman's Paper.) A related concept is an extended relation J* defined as follows: $J^* = \{\langle\Phi,\Psi,\Omega\rangle \mid (\forall\xi,\eta): \xi \varepsilon \Psi . \eta \varepsilon \Omega . \supset . J_{\Phi}(\xi,\eta)\}.$ This would be central in an abstract development following Church-Rosser's method of proof. (See Church's Monograph, p. 23.) These two approaches are so-to-speak "perpendicular" to each other.

Notation 4.6. Hereafter 'J', as well as 'C' and 'δ', will be suppressed in the usual manner.

Remark 4.7. RelSep J . $\supset$. $(\forall\Phi)$. $\emptyset \varepsilon \Phi$Sep .

Theorem 4.8. RelSep J $.\supset: (\forall\Phi)(\forall\xi,\eta): J_{\Phi}(\xi,\eta) .. \equiv .$ $\{\xi,\eta\} \varepsilon \Phi$Sep .

Proof. Assume that RelSep J and that $\Phi \varepsilon$ Coinit and $\xi, \eta \ \varepsilon$ Cell . I. Assume $J_{\Phi}(\xi, \eta)$. Then by (J3), $\Phi \cup \{\xi, \eta\} \ \varepsilon$ Coinit ; hence by (J1), $J_{\Phi}(\xi, \xi)$. $J_{\Phi}(\eta, \eta)$. Also, by (J2), $J_{\Phi}(\eta, \xi)$. With $J_{\Phi}(\xi, \eta)$ these show that $\{\xi, \eta\} \ \varepsilon \ \Phi$Sep . II. Assume $\{\xi, \eta\} \ \varepsilon \ \Phi$Sep ; then directly from the definition, $J_{\Phi}(\xi, \eta)$. This completes the proof.

Remark 4.8. This theorem enables us to avoid the $\ulcorner J_{\Phi} \urcorner$ notation in practice. We illustrate by giving conditions equivalent to those which define a relation of relative separation, as follows:

(J1) $(\forall \Phi)(\forall \xi) : \Phi \cup \{\xi\} \ \varepsilon$ Coinit $. \supset . \{\xi\} \ \varepsilon \ \Phi$Sep ,

(J3) $(\forall \Phi)(\forall \xi, \eta) : \{\xi, \eta\} \ \varepsilon \ \Phi$Sep $. \supset . \Phi \cup \{\xi, \eta\} \ \varepsilon$ Coinit ,

(J4) $(\forall \Phi)(\forall \xi, \eta) : \{\xi, \eta\} \ \varepsilon \ \Phi$Sep $. \ \xi \neq \eta \ . \supset .$ Nc$(\xi / \eta) = 1$,

(J5) $(\forall \Phi)(\forall \xi, \eta, \zeta)(\forall \xi', \eta') : \{\xi, \eta\} \ \varepsilon \ \Phi$Sep $. \ \zeta \ \varepsilon \ \Phi . \ \& \ .$ $\xi' \ \varepsilon \ (\xi / \zeta) \ . \ \eta' \ \varepsilon \ (\eta / \zeta) \ . \supset . \ \{\xi', \eta'\} \ \varepsilon \ (\Phi / \zeta)$Sep ,

(J6) $(\forall \Phi, \Omega)(\forall \xi, \eta) : \{\xi, \eta\} \ \varepsilon \ \Phi$Sep $. \ \Omega \subseteq \Phi . \supset . \ \{\xi, \eta\} \ \varepsilon$ ΩSep .

The condition (J2) is trivial in this formulation; it is of course essential in the proof of the equivalence of the two formulations.

Remark 4.9. RelSep J $. \supset : (\forall \Phi, \Psi) : \Psi \varepsilon \ \Phi$Sep $. \equiv .$ $(\forall \xi, \eta) : \xi, \eta \ \varepsilon \ \Psi . \supset . \ \{\xi, \eta\} \ \varepsilon \ \Phi$Sep .

Theorem 4.10. RelSep J $. \supset : (\forall \Phi)(\forall \xi, \zeta) : \Phi \cup \{\xi\} \ \varepsilon$ Coinit $. \ \zeta \ \varepsilon \ \Phi . \supset . \ (\xi / \zeta) \ \varepsilon \ (\Phi / \zeta)$Sep .

<u>Proof</u>. Assume RelSep J . $\Phi \cup \{\xi\} \varepsilon$ Coinit . $\zeta \varepsilon \Phi$. By (J1), $\{\xi\} \varepsilon \Phi$Sep . By (J5), $(\forall \xi_1', \xi_2')$: $\xi_1', \xi_2' \varepsilon (\xi/\zeta)$. $\supset$. $\{\xi_1', \xi_2'\} \varepsilon (\Phi/\zeta)$Sep . Hence $(\xi/\zeta) \varepsilon (\Phi/\zeta)$Sep , q.e.d.

<u>Theorem 4.11</u>. RelSep J . $\supset$: $(\forall \Phi, \Psi)$: $\Psi \varepsilon \Phi$Sep . $\supset$. $(\Phi \cup \Psi) \varepsilon$ Coinit .

<u>Proof</u>. By (J3). In case $\Psi = \emptyset$ (whence $\Phi \cup \Psi = \Phi$), the proof depends essentially on the fact that Φ is restricted so that $\Phi \varepsilon$ Coinit.

<u>Remark 4.12</u>. RelSep J . $\supset$: $(\forall \Phi, \Psi, \Omega)$: $\Psi \varepsilon \Phi$Sep . $\Omega \subseteq \Psi$. $\supset$. $\Omega \varepsilon \Phi$Sep .

<u>Remark 4.13</u>. RelSep J . $\supset$: $(\forall \Phi, \Psi, \Omega)$: $\Psi \varepsilon \Phi$Sep . $\Omega \subseteq \Phi$. $\supset$. $\Psi \varepsilon \Omega$Sep .

<u>Corollary 4.13</u>. RelSep J . $\supset$: $(\forall \Phi, \Omega)$: $\Omega \subseteq \Phi$. $\supset$. ΦSep $\subseteq \Omega$Sep .

<u>Lemma 4.14</u>. RelSep J . $\supset$: $(\forall \Phi, \Psi)(\forall \xi)$: $\Psi \varepsilon \Phi$Sep . & . $\xi \varepsilon \Phi$. $\supset$. $(\Psi/\xi) \varepsilon (\Phi/\xi)$Sep .

<u>Proof</u>. By (J5).

<u>Theorem 4.14</u>. RelSep J . $\supset$: $(\forall \Phi, \Psi)(\forall \pi)$: $\Psi \varepsilon \Phi$Sep . & . π develops Φ . $\supset$. $(\Psi/\pi) \varepsilon (\Phi/\pi)$Sep .

<u>Proof</u>. From the lemma, using induction on $\#\pi$.

Lemma 4.15. RelSep J . $\supset$: $(\forall\Phi,\Psi)(\forall\xi)$: $\Psi \cup \{\xi\} \varepsilon \Phi$Sep . $\Psi\varepsilon$Finite . $\xi\varepsilon\Phi$. $\supset$. Nc$(\Psi/\xi) \leq$ Nc Ψ .

Proof. The proof is by two cases as follows:

Case 1. $\xi \not\varepsilon \Psi$. Then for any $\eta\varepsilon\Psi$ we have by (J4) that Nc(η/ξ) = 1 ; hence there is a many-to-one correspondence from Ψ into (Ψ/ξ) . Hence Nc$(\Psi/\xi) \leq$ Nc Ψ .

Case 2. $\xi\varepsilon\Psi$. Let $\Psi' = \Psi - \{\xi\}$. Then by Case 1, Nc$(\Psi'/\xi) \leq$ Nc Ψ' = -1 + Nc Ψ . Since $(\Psi/\xi) = (\Psi'/\xi)$, it follows that Nc$(\Psi/\xi) \leq -1$ + Nc Ψ , whence Nc$(\Psi/\xi) \leq$ Nc Ψ .

Remark 4.15. The presence of the clause '$\Psi\varepsilon$Finite' in our statement of Lemma 4.15 is not essential to the truth of the lemma, but enables us to avoid the axiom of choice in the proof, thus preserving the effective character of the theory.

Note 4.16. The assumption that $(\forall\xi,\eta,\Theta)$: $\xi\neq\eta$. $\supset$. $(\xi/\zeta)\cap(\eta/\zeta) = \emptyset$ would enable us to strengthen the inequality in some of the next several theorems to an equality. This assumption is actually satisfied by derivation in the systems of lambda-conversion which we will consider.

Theorem 4.17. RelSep J . $\supset$: $(\forall\Phi,\Psi,\Omega)(\forall\pi)$: $\Psi\cup\Omega\ \varepsilon\ \Phi$Sep. $\Psi\ \varepsilon$ Finite . $\Omega\subseteq\Phi$. π develops Ω . $\supset$. Nc$(\Psi/\pi) \leq$ Nc Ψ.

Proof. From the lemma, using induction on $\#\pi$.

Lemma 4.18(a). RelSep J . $\supset$: $(\forall\Phi)(\forall\xi)$: $\Phi \in \text{Finite} \cap \Phi\text{Sep}$ $\xi \in \Phi$. $\supset$. $\text{Nc}\,\Phi \geq 1 + \text{Nc}(\Phi/\xi)$.

Proof. Let $\Psi = \Phi - \{\xi\}$. Then $\text{Nc}\,\Phi = 1 + \text{Nc}\,\Psi \geq 1 + \text{Nc}(\Psi/\xi)$ by Lemma 4.17. But $\text{Nc}(\Psi/\xi) = \text{Nc}(\Phi/\xi)$ since $\text{Nc}(\xi/\xi) = 0$. This does it.

Lemma 4.18(b). RelSep J . $\supset$: $(\forall\Phi)(\forall\rho)$: $\Phi \in \text{Finite} \cap \Phi\text{Sep}$ ρ develops Φ . $\supset$. $\text{Nc}\,\Phi \geq (\#\rho) + \text{Nc}(\Phi/\rho)$.

Proof. From Lemma 4.18(a), using induction on $\#\rho$.

Theorem 4.18. RelSep J . $\supset$: $(\forall\Phi)(\forall\varphi)$: $\Phi \in \text{Finite} \cap \Phi\text{Sep}$. φ develops Φ . $\supset$. $\#\varphi \leq \text{Nc}\,\Phi$.

Proof. By Lemma 4.18(b), $(\forall\rho)$: ρ develops Φ . $\#\rho = \text{Nc}\,\Phi$. $\supset$. $(\Phi/\rho) = \emptyset$. Hence $\sim(\exists\rho)$. ρ develops Φ . $\#\rho > \text{Nc}\,\Phi$. This does it.

Remark 4.18. This proof shows that for $\Phi \in \Phi\text{Sep}$, $(\text{Nc}\,\Phi)$ is a uniform bound for the number of steps in a development of Φ .

Corollary 4.18. RelSep J . $\supset$: $(\forall\Phi)(\forall\varphi)$: $\Phi \in \text{Finite} \cap \Phi\text{Sep}$. φ develops Φ . $\supset$. $\#\varphi < \infty$.

Definition 4.19. Where RelSep(C,δ,J) . $\Phi \in \text{Coinit} \cap \text{Finite}$, define deg Φ, the degree of Φ as follows:

$$\deg\Phi = \min\{m \mid (\exists\Phi_1,\Phi_2,\ldots,\Phi_m).\ \Phi = \Phi_1 \cup \Phi_2 \cup \cdots \cup \Phi_m\ .\ \Phi_1,\Phi_2,\ldots,\Phi_m \in \Phi\text{Sep}\}\ .$$

Remark 4.19. Assume $\Phi \varepsilon \text{Coinit}_\cap\text{Finite}$ .. Let $\Phi = \{\xi_1,\ldots,\xi_n\}$. Then $(\forall i): 1 \leqslant i \leqslant n \,.\supset.\, \{\xi_i\} \varepsilon \Phi\text{Sep}$, by (J1). Hence $(\exists\Phi_1,\ldots,\Phi_n).\ \Phi = \Phi_1 \cup \cdots \cup \Phi_n \,.\, \Phi_1,\ldots,\Phi_n \varepsilon \Phi\text{Sep}$. So deg Φ exists $\leqslant n$.

Lemma 4.20. RelSep J $.\supset: (\forall\Phi): \Phi \varepsilon \text{Finite}_\cap\Phi\text{Sep} \,.\supset.\, (\exists\rho).\ \rho$ comdevelops Φ .

Proof. By induction on Nc Φ, using Lemma 4.18(b).

Theorem 4.20. RelSep J $.\supset.$ Condition (M) .

Proof. Assume the antecedent. Let $\Phi \varepsilon$ Finite; to prove $(\exists\rho).\ \rho$ comdevelops Φ. . Use induction on deg Φ .

Basis. Deg $\Phi = 0$. Then $\Phi = \emptyset$, whence $\boxtimes$ comdevelops Φ; so take $\rho = \boxtimes$.

Induction Step. Deg $\Phi = n+1$. Fix $\Phi_1,\ldots,\Phi_{n+1} \varepsilon \Phi\text{Sep}$ such that $\Phi = \Phi_1 \cup \cdots \cup \Phi_{n+1}$. Since $\Phi \varepsilon$ Finite so is every $\Phi_i \varepsilon$ Finite . By the lemma, $(\exists\rho_1).\ \rho_1$ comdevelops Φ_1 .. By Theorem 4.14, $(\Phi_i/\rho_1) \varepsilon (\Phi/\rho_1)\text{Sep}$, so where $\Phi_i' = (\Phi_i/\rho_1)$ and where $\Phi' = \Phi_1' \cup \cdots \cup \Phi_{n+1}'$, we have that $\Phi' = \emptyset \cup \Phi_2' \cup \cdots \cup \Phi_{n+1}' = \Phi_2' \cup \cdots \cup \Phi_{n+1}'$, giving that deg $\Phi' \leqslant n$. So the induction hypothesis applies, whence $(\exists\rho').\ \rho'$ comdevelops Φ' . Fix ρ'. Clearly $(\rho_1 + \rho')$ comdevelops Φ .

Corollary 4.20. RegCx$(C,\mathcal{S})$ $.\supset.$ Condition (M) .

Proof. Assume the antecedent. Then $(\exists J)$. RelSep J . Use the theorem.

SECTION 5

NORMALITY AND RELATED PROPERTIES IN REGULAR COMPLEXES

Part A. Normal Relations of Relative Separation

Definition 5.1. We say that J is a normal relation of relative separation on $\langle C, \delta \rangle$ (and write $\ulcorner (C,\delta)\mathrm{NormRelSep}\ J \urcorner$) iff RelSep J and the following (C,δ,J)Condition (JN) holds:

(JN) $(\forall \Phi, \Psi)(\forall \rho, \sigma, \rho', \sigma') : \Phi \cup \Psi \in \mathrm{Coinit} \cap \mathrm{Finite}\ .$ $\Phi, \Psi \in (\Phi \cup \Psi)\mathrm{Sep}\ .\ \rho$ comdevelops $\Phi\ .\ \sigma$ comdevelops $\Psi .\ \rho'$ comdevelops $(\Phi/\sigma)\ .\ \sigma'$ comdevelops $(\Psi/\rho)\ .$ $\supset .\ \rho + \sigma' \doteq \sigma + \rho'\ .$

Theorem 5.2. $\mathrm{NormCx}(C,\delta)\ .\ \mathrm{RelSep}\ J\ . \supset .\ \mathrm{NormRelSep}\ J\ .$

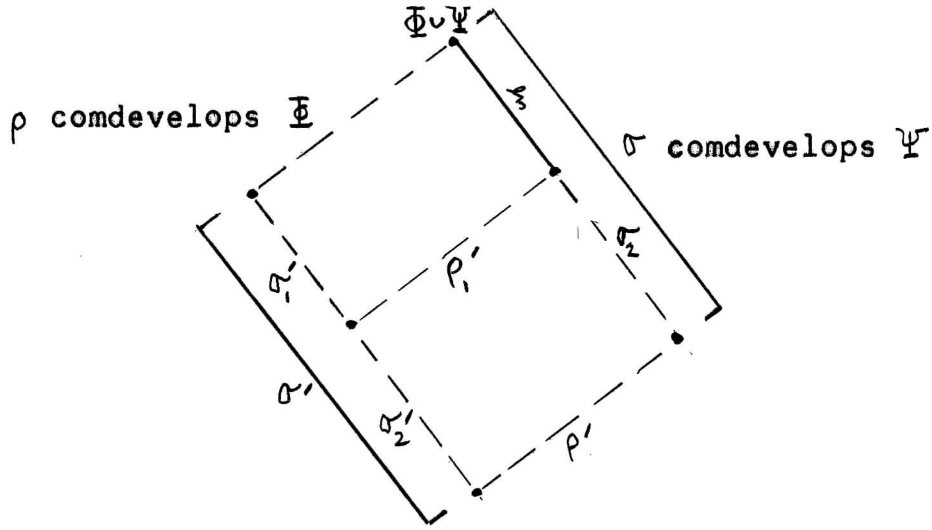
Proof. Consequent follows directly from (N').

Lemma 5.3(a). $\mathrm{NormRelSep}\ J\ . \supset : (\forall \Phi, \Psi) : \Phi \cup \Psi \in \mathrm{Coinit} \cap \mathrm{Finite}\ .\ \Phi \in (\Phi \cup \Psi)\mathrm{Sep}\ .\ \rho$ comdevelops $\Phi\ .\ \sigma$ comdevelops $\Psi . \supset . (\exists \rho', \sigma').\ \rho'$ comdevelops $(\Phi/\sigma)\ .\ \sigma'$ comdevelops $(\Psi/\rho)\ .$ $\rho + \sigma' \doteq \sigma + \rho'\ .$

Proof. Assume $\mathrm{NormRelSep}\ J\ .\ \Phi \cup \Psi \in \mathrm{Coinit} \cap \mathrm{Finite}\ .\ \Phi \in (\Phi \cup \Psi)\mathrm{Sep}\ .\ \rho$ comdevelops $\Phi\ .\ \sigma$ comdevelops $\Psi .$ Use induction on $\# \sigma$, as follows:

Basis. $\sigma = \boxtimes$. Then $(\Phi/\sigma) = \Phi$; also $\Psi = \emptyset$, whence $(\Psi/\rho) = \emptyset$. Take $\rho' = \rho\ .\ \sigma' = \boxtimes$. This does it.

Induction Step. $\sigma = \xi + \sigma_2$. Then $\xi \,\varepsilon\, \Psi$. Since Φ ε $(\Phi \cup \Psi)$Sep , have by Remark 4.13 that $\Phi \,\varepsilon\, (\Phi \cup \{\xi\})$Sep ; and by (J1), $\{\xi\} \,\varepsilon\, (\Phi \cup \{\xi\})$Sep . Clearly also $(\Phi \cup \{\xi\}) \,\varepsilon$ Coinit$\cap$Finite . Now since RelSep J we have Condition (M). So let ρ_1' comdevelop (Φ/ξ) . σ_1' comdevelop (ξ/ρ) . It follows by (JN) that $\rho + \sigma_1' \doteq \xi + \rho_1'$. Furthermore $(\Phi/\xi) \cup (\Psi/\xi)$ ε Coinit$_\cap$Finite , and, since $\xi \,\varepsilon\, (\Phi \cup \Psi)$, we have that $(\Phi/\xi) \,\varepsilon$ $(\Phi \cup \Psi/\xi)$Sep = $((\Phi/\xi) \cup (\Psi/\xi)/\xi)$Sep . Also, ρ_1' comdevelops (Φ/ξ) . σ_2 comdevelops (Ψ/ξ) . Hence the induction hypothesis



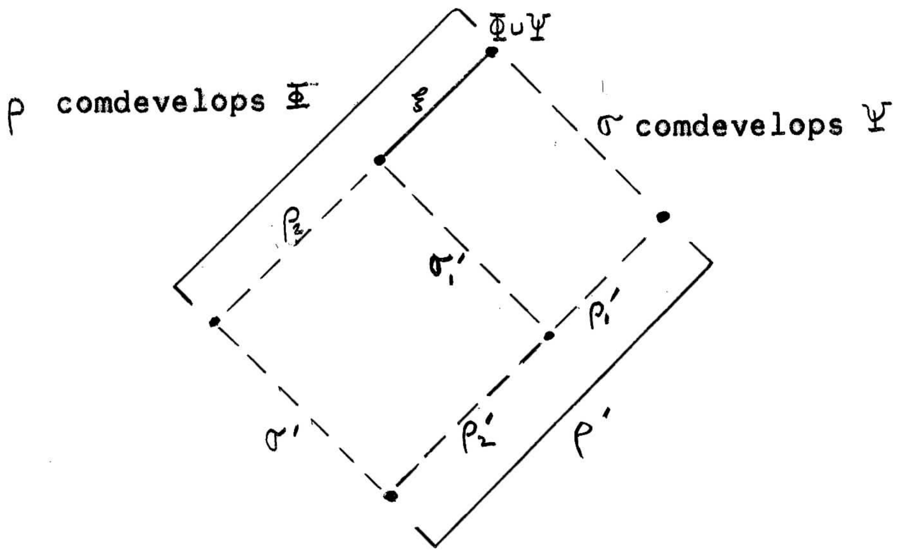
applies, whence $(\exists \rho', \sigma_2')$. ρ' comdevelops $((\Phi/\xi)/\sigma_2)$. σ_2' comdevelops $((\Psi/\xi)/\rho_1')$. $\rho_1' + \sigma_2' \doteq \sigma_2 + \rho'$. Fix ρ', σ_2', and let $\sigma' = \sigma_1' + \sigma_2'$. Then $\rho + \sigma' = \rho + \sigma_1' + \sigma_2' \doteq \xi + \rho_1' + \sigma_2' \doteq \xi + \sigma_2 + \rho' = \sigma + \rho'$, q.e.d.

<u>Lemma</u> <u>5.3</u>(<u>b</u>). NormRelSep J $.\supset.$ Condition (N'') .

<u>Proof.</u> Assume NormRelSep J . $\Phi \cup \Psi \in \mathrm{Coinit} \cap \mathrm{Finite}$. ρ comdevelops Φ . σ comdevelops Ψ . Use induction on $\#\rho$ to prove $(\exists \rho', \sigma')$. ρ' comdevelops (Φ/σ) . σ' comdevelops (Ψ/ρ) . $\rho + \sigma' \doteq \sigma + \rho'$.

<u>Basis.</u> $\rho = \boxtimes$. Take $\rho' = \boxtimes$. $\sigma' = \sigma$.

<u>Induction</u> <u>Step.</u> $\rho = \xi + \rho_2$. Since $\{\xi\} \in ((\{\xi\} \cup \Psi)\mathrm{Sep}$, the preceding lemma applies with '$\{\xi\}$' for 'Φ' . So fix ρ_1', σ_1' such that ρ_1' comdevelops (ξ/σ) . σ_1' comdevelops (Ψ/ξ) . $\rho + \sigma_1' \doteq \sigma + \rho_1'$. Since $(\Phi/\xi) \cup (\Psi/\xi) = (\Phi \cup \Psi/\xi) \in$ Coinit . ρ_2 comdevelops (Φ/ρ_1) . σ_1' comdevelops (Ψ/ρ_1) , the



induction hypothesis applies, whence $(\exists \rho_2', \sigma')$. ρ_2' comdevelops $((\Phi/\xi)/\sigma_1')$. σ' comdevelops $((\Psi/\xi)/\rho_2)$. $\sigma_1' + \rho_2' \doteq \rho_2 + \sigma'$. Fix ρ_2', σ', and let $\rho' = \rho_1' + \rho_2'$. Then $\rho + \sigma' = \xi + \rho_2 + \sigma' \doteq \xi + \sigma_1' + \rho_2' \doteq \sigma + \rho_1' + \rho_2' = \sigma + \rho'$, q.e.d.

Theorem 5.3. $\mathrm{NormRelSep}\ J\ .\supset.\ \mathrm{NormCx}(C,\delta)$.

Proof. Assume the antecedent. Then RelSep J, whence RegCx(C,δ), whence Condition (M). Also, by Lemma 5.3(b), we have Condition (N''). So NormCx(C,δ) by Definition 3.6.

Corollary 5.3(a). $(\exists J)[\mathrm{NormRelSep}\ J]\ .\supset.\ \mathrm{NormCx}(C,\delta)$.

Corollary 5.3(b).

$\mathrm{RegCx}(C,\delta)\ .\ \mathrm{NormCx}(C,\delta)$:

$\equiv : (\exists J).\ \mathrm{NormRelSep}\ J$:

$\equiv : (\exists J) : \mathrm{RelSep}\ J : (\forall J) : \mathrm{RelSep}\ J\ .\supset.\ \mathrm{NormRelSep}\ J$.

Corollary 5.3(c). $\mathrm{RegCx}(C,\delta)\ .\supset ::$

$\mathrm{NormCx}(C,\delta)$:

$\equiv : (\exists J).\ \mathrm{NormRelSep}\ J$:

$\equiv : (\forall J) : \mathrm{RelSep}\ J\ .\supset.\ \mathrm{NormRelSep}\ J$.

Part B. Normality for Self-Separated Sets

Definition 5.4. Consider the following (C,δ,J) Conditions (R), (S), (S'), (S''):

(R) $(\forall\Phi): \Phi \in \mathrm{Finite}_\cap \Phi\mathrm{Sep} .\supset. (\exists\rho). \rho$ comdevelops Φ .

(S) $(\forall\Omega)(\forall\pi,\tau): \Omega \in \mathrm{Finite}_\cap \Omega\mathrm{Sep} . \pi,\tau$ comdevelops $\Omega . \supset. \pi \doteq \tau$.

(S') $(\forall\Phi,\Psi)(\forall\rho,\sigma,\rho',\sigma'): (\Phi\cup\Psi) \in \mathrm{Coinit}_\cap \mathrm{Finite}_\cap (\Phi\cup\Psi)\mathrm{Sep} . \rho$ comdevelops $\Phi . \sigma$ comdevelops Ψ . ρ' comdevelops $(\Phi/\sigma) . \sigma'$ comdevelops $(\Psi/\rho) .\supset. \rho+\sigma' \doteq \sigma+\rho'$.

(S'') $(\forall\Phi,\Psi)(\forall\rho,\sigma): \Phi\cup\Psi \in \mathrm{Coinit}_\cap \mathrm{Finite}_\cap (\Phi\cup\Psi)\mathrm{Sep} . \rho$ comdevelops $\Phi . \sigma$ comdevelops $\Psi .\supset:$ $(\exists\rho',\sigma'): \rho'$ comdevelops $(\Phi/\sigma) . \sigma'$ comdevelops $(\Psi/\rho) . \rho+\sigma' \doteq \sigma+\rho'$.

Remark 5.4. By Theorem 4.11, $(\Phi\cup\Psi) \in (\Phi\cup\Psi)\mathrm{Sep} .\supset. (\Phi\cup\Psi) \in \mathrm{Coinit}$ (on the premise RelSep J), so that the occurrences of 'Coinit' in (S'), (S'') can be omitted.

Note 5.4. Condition (S') is quite similar to Condition (JN); however, we there assume only that $\Phi,\Psi \in (\Phi\cup\Psi)\mathrm{Sep}$, whereas the corresponding (stronger) assumption in (S') is that $(\Phi\cup\Psi) \in (\Phi\cup\Psi)\mathrm{Sep}$. The difference is crucial.

Lemma 5.5. RelSep J $.\supset.$ Condition (R) .

Proof. Clearly Condition $(M) \supset (R)$, but by Theorem

4.20, RelSep J $.\supset.$ (M) .

Theorem 5.5. RelSep J $.\supset:$ (S) $.\equiv.$ (S') $.\equiv.$ (S'') .

Proof. I. Assume RelSep J . (S) , and let $\Phi \cup \Psi \in$ $\mathrm{Coinit}_\cap \mathrm{Finite}_\cap (\Phi \cup \Psi)\mathrm{Sep}$. ρ comdevelops Φ . σ comdevelops Ψ . ρ' comdevelops (Φ/σ) . σ' comdevelops (Ψ/ρ) . To prove $\rho + \sigma' \doteq \sigma + \rho'$. Take $\Omega = \Phi \cup \Psi$, $\pi = \rho + \sigma'$, $\tau = \sigma + \rho'$ and apply (S). So (S) $.\supset.$ (S') . II. Assume RelSep J . (S') , and let $(\Phi \cup \Psi) \in \mathrm{Coinit}_\cap \mathrm{Finite}_\cap (\Phi \cup \Psi)\mathrm{Sep}$. ρ comdevelops Φ . σ comdevelops Ψ. Then by Remark 4.12, $\Phi, \Psi \in (\Phi \cup \Psi)\mathrm{Sep}$, whence we have by Theorem 4.14 that $(\Phi/\sigma) \in (\Phi/\sigma)\mathrm{Sep}$. (Ψ/ρ) $\in (\Psi/\rho)\mathrm{Sep}$. So by the lemma, $(\exists \rho', \sigma'):$ ρ' comdevelops (Φ/σ) . σ' comdevelops (Ψ/ρ) . Fix ρ', σ'. Now apply (S'), giving $\rho + \sigma' \doteq \sigma + \rho'$. So (S') $.\supset.$ (S'') . III. Assume RelSep J . (S'') , and let $\Omega \in \mathrm{Coinit}_\cap \mathrm{Finite}_\cap \Omega\mathrm{Sep}$. π, τ comdevelop Ω. To prove $\pi \doteq \tau$. Take $\Phi = \Psi = \Omega$, $\rho = \pi$, $\sigma = \tau$ and apply (S'') . Hence $(\exists \rho', \sigma'):$ ρ' comdevelops (Ω/τ) . σ' comdevelops (Ω/π) . $\pi + \sigma' \doteq \tau + \rho'$. Fix ρ', σ'. Then since $(\Omega/\tau) = (\Omega/\pi) = \emptyset$, we have that $\rho' = \sigma' = \boxtimes$. Hence $\pi = \pi + \sigma' \doteq \tau + \rho' = \tau$. So (S'') $.\supset.$ (S) . This completes the proof of the theorem.

Definition 5.6. We say that J is a relation of relative separation on $\langle C, \delta \rangle$ normal for self-separated sets (and write $\ulcorner(C, \delta)\mathrm{NormRelSelfSep}\ J\urcorner$) iff RelSep J . Condition (S) .

<u>Theorem 5.7.</u> NormRelSelfSep J $. \supset :$ $(\forall \Phi_L, \Phi_R):$ $(\Phi_L \cup \Phi_R) \varepsilon$ Coinit$\cap$Finite$\cap(\Phi_L \cup \Phi_R)$Sep . ρ comdevelops $(\Phi_L \cup \Phi_R)$ $. \supset .$ $(\exists \rho_L, \rho_R) .$ $\rho \doteq \rho_L + \rho_R$. ρ_L comdevelops Φ_L . ρ_R comdevelops (Φ_R/ρ_L) .

<u>Proof.</u> Assume the antecedent. Since RegCx(C, δ) , have Condition (M). Hence $(\exists \rho_L) .$ ρ_L comdevelops Φ_L . Fix ρ_L. By (M) again, $(\exists \rho_R) .$ ρ_R comdevelops (Φ_R/ρ_L) . Fix ρ_R. Now ρ , $\rho_L + \rho_R$ comdevelop $(\Phi_L \cup \Phi_R)$. Since $(\Phi_L \cup \Phi_R) \varepsilon$ $(\Phi_L \cup \Phi_R)$Sep , it follows by (S) that $\rho \doteq \rho_L + \rho_R$.

Part C. Normality for Self-Separated Pairs

Definition 5.8. Consider the following (C,δ,J) Conditions (C), (D), $(D^!_J)$, $(D^{!\prime}_J)$:

(C) $(\forall\xi,\eta)(\exists\rho')$. ρ' comdevelops (ξ/η) .

(D) $(\forall\xi,\eta)(\forall\rho',\sigma')$: ρ',σ' comdevelop (ξ/η) .⊃. $\rho' = \sigma'$.

$(D^!_J)$ $(\forall\xi,\eta)(\forall\rho',\sigma')$: $\{\xi,\eta\} \varepsilon \{\xi,\eta\}\mathrm{Sep}$. ρ' comdevelops (ξ/η) . σ' comdevelops (η/ξ) .⊃. $\eta + \rho' \doteq \xi + \sigma'$.

$(D^{!\prime}_J)$ $(\forall\xi,\eta)$: $\{\xi,\eta\} \varepsilon \{\xi,\eta\}\mathrm{Sep}$.⊃. $(\exists\rho',\sigma')$. ρ' comdevelops (ξ/η) . σ' comdevelops (η/ξ) . $\eta + \rho' \doteq \xi + \sigma'$.

Remark 5.8.

(a) Condition (M) .⊃. Condition (C) .

∴ (b) $\mathrm{NormCx}(C,\delta)$.⊃. Condition (C) ;

and (c) $\mathrm{RegCx}(C,\delta)$.⊃. Condition (C) ,

whence (d) RelSep J .⊃. Condition (C) .

Also, (e) $\mathrm{NormCx}(C,\delta)$.⊃. Condition (D) ,

and (f) $\mathrm{NormCx}(C,\delta)$. RelSep J .⊃. $(D^!_J)$,

and (g) $\mathrm{NormCx}(C,\delta)$. RelSep J .⊃. $(D^{!\prime}_J)$.

Theorem 5.9(a). RelSep J .⊃: $(D^!_J)$.⊃. $(D^{!\prime}_J)$.

Proof. Assume RelSep J and also $(D^!_J)$. Then by Remark 5.8(d), (C) . $(D^!_J)$; i.e., $(\forall\xi,\eta)(\exists\rho')$. ρ' comdevelops

(ξ/η) : $(\forall\xi,\eta)(\forall\rho',\sigma')$: $\{\xi,\eta\}$ ε $\{\xi,\eta\}$Sep . ρ' comdevelops (ξ/η) . σ' comdevelops (η/ξ) .$\supset$. $\eta + \rho' \doteq \xi + \sigma'$. Let $\{\xi,\eta\}$ ε $\{\xi,\eta\}$Sep . To prove $(\exists\rho',\sigma')$. ρ' comdevelops (ξ/η) . σ' comdevelops (η/ξ) . $\eta + \rho' \doteq \xi + \sigma'$. By (C), $(\exists\rho',\sigma')$. ρ' comdevelops (ξ/η) . σ' comdevelops (η/ξ) . Fix ρ', σ'. By $(D_J^!)$, $\eta + \rho' \doteq \xi + \sigma'$.

<u>Theorem</u> <u>5.9(b)</u>. RelSep J .$\supset$: $(D_J^{!'})$.$\supset$. $(D_J^!)$.

<u>Proof</u>. Assume RelSep J and also $(D_J^{!'})$; i.e., $(\forall\xi,\eta)$: $\{\xi,\eta\}$ ε $\{\xi,\eta\}$Sep .$\supset$. $(\exists\rho',\sigma')$. ρ' comdevelops (ξ/η) . σ' comdevelops (η/ξ) . $\eta + \rho' \doteq \xi + \sigma'$. Let $\{\xi,\eta\}$ ε $\{\xi,\eta\}$Sep . ρ_1' comdevelop (ξ/η) . σ_1' comdevelop (η/ξ) . To prove $\eta + \rho_1' \doteq \xi + \sigma_1'$. Since $\{\xi,\eta\}$ ε $\{\xi,\eta\}$Sep , it follows by (J4) that $(\exists\eta',\xi')$. $(\eta/\xi) = \{\eta'\}$. $(\xi/\eta) = \{\xi'\}$. Fix η', ξ' . Hence $(\forall\rho')$: ρ' develops (ξ/η) .$\supset$. $\rho' = \langle\xi'\rangle$: $(\forall\sigma')$: σ' develops (η/ξ) .$\supset$. $\sigma' = \langle\eta'\rangle$. But by our hypothesis $(\exists\rho',\sigma')$. ρ' comdevelops (ξ/η) . σ' comdevelops (η/ξ) . $\eta + \rho' \doteq \xi + \sigma'$. Fix ρ', σ'. Then $\rho' = \langle\xi'\rangle$. $\sigma' = \langle\eta'\rangle$; but for the same reason $\rho_1' = \langle\xi'\rangle$. $\sigma_1' = \langle\eta'\rangle$. so $\rho_1' = \rho'$. $\sigma_1' = \sigma'$, whence $\eta + \rho_1' \doteq \xi + \sigma_1'$, q.e.d.

<u>Corollary</u> <u>5.9</u>. RelSep J .$\supset$: $(D_J^!) \equiv (D_J^{!'})$.

<u>Definition</u> <u>5.10</u>. We say that J is a <u>pairwise</u> <u>normal</u> <u>relation</u> <u>of</u> <u>relative</u> <u>separation</u> <u>on</u> $\langle C,\delta\rangle$ (and write $\ulcorner(C,\delta)$2NormRelSep J$\urcorner$) iff RelSep J . Condition $(D_J^!)$.

Remark 5.10. By Corollary 5.9,

2NormRelSep J . ≡ . RelSep J . Condition $(D_J^{\prime\prime})$.

Theorem 5.11. 2NormRelSep J . ⊃. NormRelSelfSep J .

Proof. Assume 2NormRelSep J . I.e., RelSep J . $(D_J^{\prime})$, where $(D_J^{\prime})$ is as follows:

$(\forall \xi,\eta)(\forall \rho',\sigma')$: $\{\xi,\eta\}$ ε $\{\xi,\eta\}$Sep . ρ' comdevelops (ξ/η) . σ' comdevelops (η/ξ) .⊃. $\eta + \rho' \doteq \xi + \sigma'$.

To prove NormRelSelfSep J . I.e., RelSep J . (S) , where (S) is as follows:

$(\forall\Omega)(\forall\pi,\tau)$: Ω ε $\text{Finite}_\cap\Omega$Sep . π,τ comdevelop Ω .⊃. $\pi \doteq \tau$.

Use induction on Nc Ω . Assume Ω ε $\text{Finite}_\cap\Omega$Sep . π,τ comdevelop Ω. To prove $\pi \doteq \tau$.

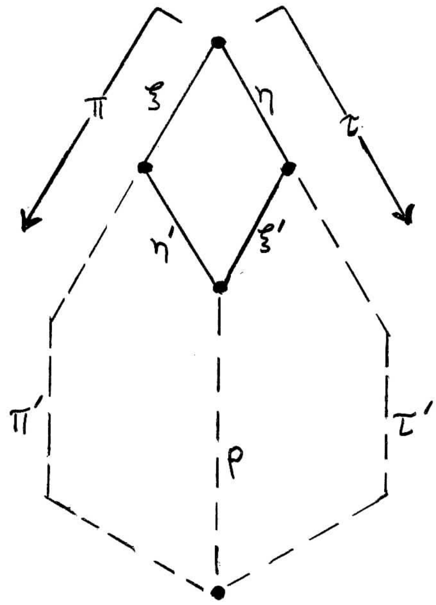
Basis. Nc Ω = 0 . Then Ω = ∅ , whence π = τ = ⊠ .

Induction Step. Nc Ω = n+1 . Then Ω ≠ ∅ , whence π ≠ ⊠ ≠ τ . Let π = ξ + π' , τ = η + τ' . Then ξ,η ε Ω , and since Ω ε $\text{Finite}_\cap\Omega$Sep it follows by Lemma 4.18(a) that Nc(Ω/ξ) ≤ n . Nc(Ω/η) ≤ n ; also, (Ω/ξ) ε $\text{Finite}_\cap(\Omega/\xi)$Sep . (Ω/η) ε $\text{Finite}_\cap(\Omega/\eta)$Sep by Lemma 4.14; also, π' comdevelops (Ω/ξ) . τ' comdevelops (Ω/η) .

In case ξ = η , it follows by the induction hypothesis that $\pi' \doteq \tau'$, whence $\pi = \xi + \pi' \doteq \eta + \tau' = \tau$ and we are done.

So assume ξ ≠ η . Then since Ω ε ΩSep it follows by (J4) that $(\exists\eta',\xi')$. $(\eta/\xi) = \{\eta'\}$. $(\xi/\eta) = \{\xi'\}$. Fix η', ξ'. Also since Ω ε ΩSep , it follows by Remarks 4.12, 4.13

that $\{\xi,\eta\} \varepsilon \{\xi,\eta\}\mathrm{Sep}$. So by $(D^!_J)$ we have that $\xi + \eta' = \eta + \xi'$. Hence in particular $(\Omega/\xi + \eta') = (\Omega/\eta + \xi')$. Now by Condition (M) (which holds by Theorem 4.20), it follows that $(\exists\rho).\ \rho$ comdevelops $(\Omega/\xi + \eta')$. Fix ρ . Then $\eta' + \rho$



comdevelops (Ω/ξ) . $\xi' + \rho$ comdevelops (Ω/η) . So $\pi' \doteq \eta' + \rho$, $\tau' \doteq \xi' + \rho$ by the induction hypothesis. So $\pi = \xi + \pi' \doteq \xi + \eta' + \rho \doteq \eta + \xi' + \rho \doteq \eta + \tau' = \tau$, q.e.d.

<u>Corollary</u> <u>5</u>.<u>11</u>(<u>a</u>). 2NormRelSep J .⊃. Condition (D) .

<u>Proof</u>. Theorem 5.11 asserts that 2NormRelSep J .⊃. RelSep J . (S) . But clearly RelSep J . (S) .⊃. (D) , since $(\forall\xi,\eta).\ (\xi/\eta)\ \varepsilon\ \mathrm{Finite}_\cap(\xi/\eta)\mathrm{Sep}$ by (J1) and (J5).

<u>Corollary</u> <u>5</u>.<u>11</u>(<u>b</u>). 2NormRelSep J . ≡ . RelSep J . (C) . (D) . $(D^!_J)$.

Part D. Pairwise Normality

Definition 5.12. Consider the following (C,δ,J) Conditions (C), (D), (D'), (D''):

(C) $(\forall\xi,\eta)(\exists\rho')$. ρ' comdevelops (ξ/η) .

(D) $(\forall\xi,\eta)(\forall\rho',\sigma')$: ρ',σ' comdevelop (ξ/η) .⊃. $\rho' \doteq \sigma'$.

(D') $(\forall\xi,\eta)(\forall\rho',\sigma')$: ρ' comdevelops (ξ/η) . σ' comdevelops (η/ξ) .⊃. $\eta+\rho' \doteq \xi+\sigma'$.

(D'') $(\forall\xi,\eta)(\exists\rho',\sigma')$: ρ' comdevelops (ξ/η) . σ' comdevelops (η/ξ) . $\eta+\rho' \doteq \xi+\sigma'$.

(These conditions are meaningful in an arbitrary derivation complex; nothing in Part D depends on regularity or relative separation.)

Remark 5.12.

(a) NormCx(C,δ) .⊃. Condition (D') .

(b) NormCx(C,δ) .⊃. Condition (D'') .

(c) Condition (D') .⊃. Condition (D'_J) .

(d) Condition (D'') .⊃. Condition (D''_J) .

See also Remark 5.8.

Theorem 5.13(a). (C) . (D') .⊃. (D'') .

Proof. Assume (C) . (D') and consider ξ,η . Then, by (C), $(\exists\rho',\sigma')$. ρ' comdevelops (ξ/η) . σ' comdevelops (η/ξ) Fix ρ', σ'. By (D'), $\eta+\rho' \doteq \xi+\sigma'$.

<u>Theorem</u> <u>5.13(b)</u>. (D) . (D'') .⊃. (D') .

<u>Proof</u>. Assume (D) . (D'') and let ρ_1' comdevelop (ξ/η) . σ_1' comdevelop (η/ξ) . To prove $\eta + \rho_1' \doteq \xi + \sigma_1'$. By (D''), $(\exists \rho', \sigma')$. ρ' comdevelops (ξ/η) . σ' comdevelops (η/ξ) . $\eta + \rho' \doteq \xi + \sigma'$. Fix ρ', σ'. By (D), $\rho' \doteq \rho_1'$. $\sigma' \doteq \sigma_1'$. Hence $\eta + \rho_1' \doteq \eta + \rho' \doteq \xi + \sigma' \doteq \xi + \sigma_1'$, q.e.d.

<u>Corollary</u> <u>5.13</u>. (C) . (D) .⊃: (D') . ≡ . (D'') .

<u>Definition</u> <u>5.14</u>. We say that ⟨C,$\mathfrak{S}$⟩ is a <u>pairwise normal</u> <u>complex</u> (and write ⌜2NormCx(C,$\mathfrak{S}$)⌝) iff all of Conditions (C), (D), (D') hold. (See Definition 5.12.)

<u>Remark</u> <u>5.14</u>.

(a) NormCx(C,$\mathfrak{S}$) .⊃. 2NormCx(C,$\mathfrak{S}$) .

(b) 2NormCx(C,$\mathfrak{S}$) . ≡ . (C) . (D) . (D'') .

Part E. Regular Pairwise Normal Complexes

Definition 5.15. We say that $\langle C,\mathcal{S}\rangle$ is a regular pairwise normal complex (and write ⌜Reg2NormCx$(C,\mathcal{S})$⌝) iff RegCx$(C,\mathcal{S})$. (D'') . (See Definition 5.12.)

Theorem 5.16. Reg2NormCx$(C,\mathcal{S})$. RelSep J .⊃. 2NormRelSep J .

Proof. Assume Reg2NormCx$(C,\mathcal{S})$. RelSep J . Then since (D''), we have (D_J'') by Remark 5.12(d), whence by Remark 5.10, we have 2NormRelSep J .

Corollary 5.16(a). Reg2NormCx$(C,\mathcal{S})$. RelSep J .⊃. NormRelSelfSep J .

Proof. Immediate by Theorem 5.11.

Corollary 5.16(b). Reg2NormCx$(C,\mathcal{S})$. RelSep J .⊃. $(\forall \Phi_L,\Phi_R)$: $(\Phi_L \cup \Phi_R)$ ε Coinit$_\cap$Finite$_\cap(\Phi_L \cup \Phi_R)$Sep . ρ comdevelops $(\Phi_L \cup \Phi_R)$.⊃. $(\exists \rho_L,\rho_R)$. $\rho \doteq \rho_L + \rho_R$. ρ_L comdevelops Φ_L . ρ_R comdevelops (Φ_R/ρ_L) .

Proof. Immediate by Theorem 5.7.

Theorem 5.17. Reg2NormCx(C, δ) . ≡ . RegCx(C, δ) . & . 2NormCx(C, δ) .

Proof. I. Assume Reg2NormCx(C, δ) . 2NormCx(C, δ) . (See Definition 5.14.) Then by Remark 5.14(b), have (D'') . So Reg2NormCx(C, δ) . II. Assume Reg2NormCx(C, δ). Then RegCx(C, δ) immediately. By Remark 5.8(c), we have Condition (C). Since RegCx(C, δ), ($\exists$J). RelSep J . Fix J. Hence by Theorem 5.16, we have 2NormRelSep J . Hence Condition (D), by Corollary 5.11(a). Thus by Remark 5.14(b), we have 2NormCx(C, δ). This completes the proof.

SECTION 6

(ORDINARY) CHURCH COMPLEXES

<u>Definition</u> <u>6.1</u>. Consider the following (C,δ)<u>Conditions</u> $(\underline{\lambda 1})$, $(\underline{\lambda 2})$:

$(\lambda 1)$ $\quad (\forall \xi,\eta,\zeta): \xi \neq \zeta . \supset . (\xi/\eta)\cap(\zeta/\eta) = \emptyset$.

$(\lambda 2)$ $\quad (\forall \xi,\eta): \perp\xi = \perp\eta . \xi \neq \eta . \supset . (\xi/\eta) \neq \emptyset$.

<u>Definition</u> <u>6.2</u>. We say that $\langle C,\delta\rangle$ is an <u>ordinary</u> <u>restricted</u> <u>Church</u> <u>complex</u> (and write $\ulcorner$ORCh$(C,\delta)^\urcorner$) iff

Reg2NormCx(C,δ) . Condition $(\lambda 1)$. Condition $(\lambda 2)$.

<u>Theorem</u> <u>6.3</u>. Condition $(\lambda 2)$ $. \supset : (\forall \xi,\eta): \xi \doteq \eta . \equiv . \xi = \eta$.

<u>Proof</u>. Assume $(\lambda 2)$. I. Let $\xi \neq \eta$. If $\perp\xi \neq \perp\eta$, then automatically $\sim[\xi \doteq \eta]$. So assume $\perp\xi = \perp\eta$. Then by $(\lambda 2)$, it follows that $(\xi/\eta) \neq \emptyset$. Since $(\xi/\xi) = \emptyset$, we see that $(\exists\zeta). (\zeta/\xi) \neq (\zeta/\eta)$, whence $\sim[\xi \doteq \eta]$. II. Assume $\xi = \eta$. Then $\xi \doteq \eta$ automatically.

<u>Remark</u> <u>6.3</u>. No special properties of $\langle C,\delta\rangle$ are assumed in Theorem 6.3 other than that DerivCx(C,δ) .

<u>Definition</u> <u>6.4</u>. Where Λ, K are any sets such that $\Sigma = \Lambda \cup K$, consider the following (C,δ,Λ,K)<u>Conditions</u>

(C1) - (C5):

(C1) $(\forall\xi,\zeta,\eta): \xi,\zeta,\eta \in \Lambda .\ \xi \neq \zeta\ .\supset.\ (\xi/\eta)\cap(\zeta/\eta) = \emptyset$

(C2) $(\forall\xi,\eta): \xi,\eta \in \Lambda .\ \bot\xi = \bot\eta\ .\ \xi \neq \eta\ .\supset.\ (\xi/\eta) \neq \emptyset$

(C3) $(\forall\xi,\eta): \xi,\eta \in \Lambda\ .\supset.\ (\xi/\eta) \subseteq \Lambda$.

(C4) $(\forall\xi,\eta): \eta \in K\ .\supset.\ \mathrm{Nc}(\xi/\eta) \leq 1$.

(C5) $(\forall\xi,\eta): \xi \in K\ .\supset.\ (\xi/\eta) \subseteq K$.

Definition 6.5. We define the set of ordinary covering pairs for $\langle C,\delta\rangle$ (for which we write $\ulcorner(C,\delta)\mathrm{OrdCovPr}\urcorner$) as follows:

$$(C,\delta)\mathrm{OrdCovPr} = \{\langle\Lambda,K\rangle \mid \Sigma = \Lambda\cup K\ .\ (C1)\ .\ (C2)\ .\ (C3)\ .\ (C4)\ .\ (C5)\} .$$

Definition 6.6. We say that $\langle C,\delta\rangle$ is an ordinary unrestricted Church complex (and write $\ulcorner\mathrm{OUCh}(C,\delta)\urcorner$) iff

$$\mathrm{Reg2NormCx}(C,\delta)\ .\ (\exists\Lambda,K).\ \langle\Lambda,K\rangle \in \mathrm{OrdCovPr} .$$

Remark 6.6. $\mathrm{ORCh}(C,\delta)\ .\supset.\ \langle\Sigma,\emptyset\rangle \in \mathrm{OrdCovPr}$.

Corollary 6.6. $\mathrm{ORCh}(C,\delta)\ .\supset.\ \mathrm{OUCh}(C,\delta)$.

Theorem 6.7. $\langle\Lambda,K\rangle \in \mathrm{OrdCovPr}\ .\supset:\ (\forall\Phi,\Psi): \Phi\cup\Psi \in \mathrm{Coinit}\ .\ \Phi\cup\Psi \subseteq \Lambda .\ \sigma$ develops $\Psi\ .\supset.\ (\Phi/\sigma) \subseteq \Lambda$.

Proof. By (C3), using induction on $\#\sigma$.

Theorem 6.8. $\langle \Lambda, K \rangle \varepsilon \text{ OrdCovPr} . \supset : (\forall \Phi, \Psi, \Omega)(\forall \pi):$ $\Phi \cup \Psi \cup \Omega \; \varepsilon \text{ Coinit} . \Phi \cup \Psi \cup \Omega \subseteq \Lambda . \pi \text{ develops } \Omega . \Phi \cap \Psi = \emptyset . \supset .$ $(\Phi/\pi) \cap (\Psi/\pi) = \emptyset$.

Proof. By (C1) and (C3), using induction on $\# \pi$.

Corollary 6.8. $\text{ORCh}(C, \xi) . \supset : (\forall \Phi, \Psi, \Omega)(\forall \pi) : \Phi \cup \Psi \cup \Omega \; \varepsilon$ $\text{Coinit} . \pi \text{ develops } \Omega . \Phi \cap \Psi = \emptyset . \supset . (\Phi/\pi) \cap (\Psi/\pi) = \emptyset$.

Proof. Use Remark 6.6.

Lemma 6.9. $\langle \Lambda, K \rangle \varepsilon \text{ OrdCovPr} . \supset : (\forall \Phi)(\forall \eta) : \Phi \cup \{\eta\} \; \varepsilon$ $\text{Coinit} \cap \text{Finite} . \Phi \cup \{\eta\} \subseteq \Lambda . \eta \notin \Phi . \supset . \text{Nc}(\Phi/\eta) \geq \text{Nc } \Phi$.

Proof. Assume the antecedent. Then since $\eta \notin \Phi$, we have by (C2) for $\xi \varepsilon \Phi$ that $(\xi/\eta) \neq \emptyset$; and in addition we have by (C1) for $\xi_1, \xi_2 \; \varepsilon \; \Phi . \xi_1 \neq \xi_2$ that $(\xi_1/\eta) \cap (\xi_2/\eta)$ $= \emptyset$. It follows that $\text{Nc}(\Phi/\eta) \geq \text{Nc } \Phi$, completing the proof.

Note 6.9. The proof of Lemma 6.9 shows that there is a function from (Φ/η) onto Φ ; the assumption $\Phi \cup \{\eta\} \; \varepsilon$ Finite enables us to avoid the axiom of choice in concluding from this that $\text{Nc}(\Phi/\eta) \geq \text{Nc } \Phi$, thus preserving the effective character of the theory.

Theorem 6.9. $\langle \Lambda, K \rangle \varepsilon \text{ OrdCovPr} . \supset : (\forall \Phi, \Psi)(\forall \sigma) : \Phi \cup \Psi \; \varepsilon$ $\text{Coinit} \cap \text{Finite} . \Phi \cup \Psi \subseteq \Lambda . \Phi \cap \Psi = \emptyset . \sigma \text{ develops } \Psi . \supset . \text{Nc}(\Phi/\sigma)$ $\geq \text{Nc } \Phi$.

Proof. By Lemma 6.9 and Theorem 6.7 using induction on $\#\sigma$.

Corollary 6.9(a). $\langle\Lambda, K\rangle \varepsilon$ OrdCovPr $.\supset:(\forall\Psi)(\forall\xi)(\forall\pi):$ $\{\xi\}\cup\Psi \,\varepsilon$ Coinit $.\,\{\xi\}\cup\Psi \subseteq \Lambda.\ \pi$ develops $\Psi.\ (\xi/\eta) = \emptyset\ .\supset.$ $\xi\,\varepsilon\,\Psi$.

Proof. Assume the antecedent. Then by Theorem 2.26, $(\exists\Omega).\,\Omega \subseteq \Psi\,.\,\Omega\ \varepsilon$ Finite $.\ \pi$ develops Ω. Fix Ω, and let $\Phi = \{\xi\}$. Then $\Phi\cup\Omega\ \varepsilon$ Coinit$_\cap$Finite $.\ \Phi\cup\Omega \subseteq \Lambda.\ \pi$ develops Ω. Hence by Theorem 6.7, $\Phi_\cap\Omega = \emptyset\ .\supset.$ Nc$(\Phi/\pi) \geqslant$ Nc Φ ; equivalently, since $\Phi = \{\xi\}$, we have $\xi \not\varepsilon\ \Omega\ .\supset.$ Nc$(\xi/\eta) \geq$ Nc $\{\xi\}$ $= 1$, whence $\xi \not\varepsilon\ \Omega\ .\supset.\ (\xi/\eta) \neq \emptyset$, whence $(\xi/\pi) = \emptyset\ .\supset.$ $\xi\ \varepsilon\ \Omega$. So $\xi\,\varepsilon\,\Omega$; but $\Omega \subseteq \Psi$. So $\xi\,\varepsilon\,\Psi$, q.e.d.

Corollary 6.9(b). ORCh$(C,\S)\ .\supset:(\forall\Phi,\Psi)(\forall\sigma):\Phi\cup\Psi\ \varepsilon$ Coinit$_\cap$Finite $.\ \Phi_\cap\Psi = \emptyset\ .\ \sigma$ develops $\Psi\ .\supset.$ Nc$(\Phi/\sigma) \geqslant$ Nc Φ .

Proof. Use Remark 6.6.

Theorem 6.10. $\langle\Lambda, K\rangle\ \varepsilon$ OrdCovPr $.\supset:(\forall\Phi,\Psi)(\forall\sigma):\Phi\cup\Psi$ ε Coinit $.\ \Phi \subseteq K\ .\ \sigma$ develops $\Psi\ .\supset.\ (\Phi/\sigma) \subseteq K$.

Proof. By (C5), using induction on $\#\sigma$.

Theorem 6.11. $\langle\Lambda, K\rangle\ \varepsilon$ OrdCovPr $.\supset:(\forall\Phi,\Psi)(\forall\sigma):\Phi\cup\Psi$ ε Coinit $.\ \Psi \subseteq K\ .\ \sigma$ develops $\Psi\ .\supset.$ Nc$(\Phi/\sigma) \leq$ Nc Φ .

Proof. By (C4) and (C5), using induction on $\#\sigma$.

Note 6.11. In the proof of Theorem 6.11 it is shown that there is a one-to-one correspondence from (Φ/σ) into Φ; unlike the situation with respect to Lemma 6.9, no cardinality assumptions and no axiom of choice are used to conclude from this that $\mathrm{Nc}(\Phi/\sigma) \leq \mathrm{Nc}\,\Phi$.

Theorem 6.12. $\langle\Lambda,K\rangle \in \mathrm{OrdCovPr}$ $.\supset:$ $(\forall\Phi)(\forall\rho):$ $\Phi \subseteq K$. ρ develops Φ $.\supset.$ $(\exists\Omega)$. $\Omega \subseteq \Phi$. $\Omega \in \mathrm{Finite}$. ρ comdevelops Ω .

Proof. Assume $\langle\Lambda,K\rangle \in \mathrm{OrdCovPr}$. Use induction on $\#\rho$.

Basis. If $\#\rho = 1$, $(\exists\xi)$. $\rho = \langle\xi\rangle$. Fix ξ . Since ρ develops Φ , it follows that $\xi \in \Phi$, whence $\xi \in K$; note that $(\xi/\rho) = (\xi/\xi) = \emptyset$. Take $\Omega = \{\xi\}$. Then $\Omega \subseteq \Phi$. $\Omega \in$ Finite . ρ comdevelops Ω.

Induction Step. If $\#\rho = k+1$, $(\exists\sigma)(\exists\xi')$. $\rho = \sigma + \xi'$. Fix σ, ξ' . Then σ develops Φ . $\#\sigma = k$. So by induction hypothesis, $(\exists\Omega')$. $\Omega' \subseteq \Phi$. $\Omega' \in \mathrm{Finite}$. ρ comdevelops Ω' . Fix Ω' . Since ρ develops Φ , it follows that $(\exists\xi)$. $\xi \in \Phi$. $\xi' \in (\xi/\rho)$. Fix ξ. By Theorem 6.11, $\mathrm{Nc}(\xi/\sigma) \leq \mathrm{Nc}\{\xi\} = 1$. So $(\xi/\eta) = \{\xi'\}$, whence $(\xi/\rho) = (\xi'/\xi') = \emptyset$, whence $(\Omega' \cup \{\xi\}/\rho) = (\Omega'/\rho) \cup (\xi/\rho) = \emptyset \cup \emptyset = \emptyset$. Take $\Omega = \Omega' \cup \{\xi\}$. Then $\Omega \subseteq \Phi$. $\Omega \in$ Finite . ρ comdevelops Ω. This completes the proof.

Definition 6.13. We write $H(C,\mathfrak{S},J,\Lambda,K)$ for $\mathrm{Reg2NormCx}(C,\mathfrak{S})$. RelSep J . $\langle\Lambda,K\rangle \in \mathrm{OrdCovPr}$. ,

Theorem 6.13. $H(C,\delta,J,\Lambda,K) \;.\supset:\; (M) \;.\; (S) \;.\; (D')$.
(See Definitions 2.33, 5.4, 5.12.)

Proof. Assume $H(C,\delta,J,\Lambda,K)$. Then Reg2NormCx(C,δ), whence by Definition 5.15 we have RegCx(C,δ) . (D'') . From RegCx(C,δ) we get (M) by Corollary 4.20. By Corollary 5.16(a) we have NormRelSelfSep J, which by Definition 5.6 gives (S). To get (D'), use Theorem 5.17 to deduce that 2NormCx(C,δ), and apply Remark 5.14(b).

Corollary 6.13. $H(C,\delta,J,\Lambda,K) \;.\supset.\; (C) \;.\; (D)$.
(See Definition 5.8 or 5.12.)

Proof. $(M) \supset (C)$, while RelSep J . $(S) \;.\supset.\; (D)$.

Definition 6.14. We wrote $\ulcorner K_{C,\delta,J}(\Phi,\Psi,\rho,\sigma,\rho',\sigma')\urcorner$ for $\Phi\cup\Psi \;\varepsilon$ Coinit$\cap$Finite . $\Phi,\Psi \;\varepsilon\; (\Phi\cup\Psi)$Sep . ρ comdevelops Φ . σ comdevelops Ψ . ρ' comdevelops (Φ/σ) . σ' comdevelops (Ψ/ρ) .

Remark 6.14. (JN) . $\equiv$.
$(\forall\Phi,\Psi)(\forall\rho,\sigma,\rho',\sigma'):\; K(\Phi,\Psi,\rho,\sigma,\rho',\sigma') \;.\supset.\; \rho+\sigma' \doteq \sigma+\rho'$.
(See Definition 5.1.)

We now initiate a sequence of theorems which leads to Theorem 6.18, stating that $H(C,\delta,J,\Lambda,K)\ .\supset.\ (JN)$. This is the fundamental result of this Part; from it follow our significant results about the logistic systems of lambda-conversion.

Theorem 6.15. $H(C,\delta,J,\Lambda,K)\ .\supset:\ (\forall\Phi,\Psi)(\forall\rho,\sigma,\rho',\sigma'):$ $K(\Phi,\Psi,\rho,\sigma,\rho',\sigma')\ .\ \Phi\cup\Psi\subseteq\Lambda\ .\supset.\ \rho+\sigma'\doteq\sigma+\rho'$.

Proof. Assume $H(C,\delta,J,\Lambda,K)\ .\ K(\Phi,\Psi,\rho,\sigma,\rho',\sigma')\ .\ \Phi\cup\Psi \subseteq\Lambda$. By Theorem 6.13 we have Conditions (M), (S), (D').

Case 1. $\Phi=\emptyset\ .\vee.\ \Psi=\emptyset$. Without loss of generality, assume $\Phi=\emptyset$. Then $\rho=\boxtimes$ and, since $(\Phi/\sigma)=\emptyset$, so does $\rho'=\boxtimes$. So $\rho+\sigma'=\sigma'$. $\sigma+\rho'=\sigma$. By hypothesis, σ comdevelops Ψ. σ' comdevelops (Ψ/ρ) . Since $(\Psi/\rho)=(\Psi/\boxtimes)$ $=\Psi$, it follows that σ,σ' comdevelop Ψ, whence by Definition 5.4, $\sigma\doteq\sigma'$. So $\rho+\sigma'=\sigma'\doteq\sigma=\sigma+\rho'$.

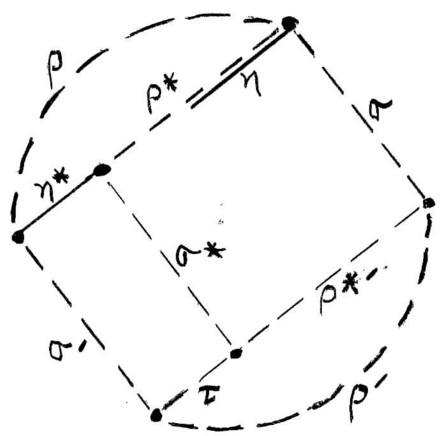
Case 2. $\Phi\neq\emptyset\ .\ \Psi\neq\emptyset\ .\ \Phi\cap\Psi=\emptyset$. Use strong induction on $m=Nc(\Psi/\rho)+Nc(\Phi/\sigma)$.

Basis. $m=0,1$. By Theorem 6.9, $Nc\,\Phi+Nc\,\Psi\leq$ $Nc(\Phi/\sigma)+Nc(\Psi/\rho)=m\leq 1$, whence $\Phi=\emptyset\ .\vee.\ \Psi=\emptyset$, contradicting the hypothesis. So the basis is vacuously satisfied.

Induction Step. $m\geq 2$.

Subcase A. $Nc\,\Phi=Nc\,\Psi=1$. Then $\#\rho=\#\sigma=1$, and the conclusion follows by Condition (D'). (Subcase A, using the hypothesis that $\Phi\neq\emptyset\neq\Psi$ and the argument used in the Basis, covers the case where $m=2$.)

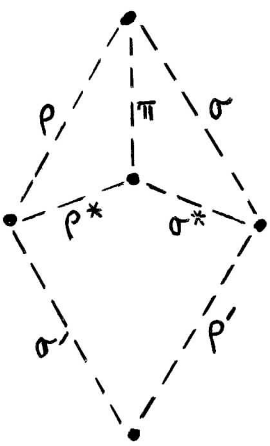
Subcase B. $\mathrm{Nc}\,\Phi > 1 . \vee . \mathrm{Nc}\,\Psi > 1$. Without loss of generality take $\mathrm{Nc}\,\Phi > 1$. Then $(\exists\eta)(\exists\Phi^*).\ \Phi = \Phi^* \cup \{\eta\}.\ \Phi^* \neq \emptyset . \eta \notin \Phi^*$. Fix η, Φ^*; note $\Phi^* \subseteq \Lambda$. By Condition (M), we get that $(\exists\rho^*).\ \rho^*$ comdevelops Φ^* . Fix ρ^*; note $(\Psi/\rho^*) \subseteq \Lambda$. Since ρ^* develops $\Phi . \Phi \in \Phi\mathrm{Sep}$, $(\exists\eta^*).\ (\eta/\rho^*) = \{\eta^*\}$. Fix η^*; clearly $\eta^* \in \Lambda$, and by Condition (S), $\rho \doteq \rho^* + \eta^*$. By (M), $(\exists\sigma^*, \rho^{*\prime}).\ \sigma^*$ comdevelops $(\Psi/\rho^*) . \rho^{*\prime}$ comdevelops (Φ^*/σ) . Fix $\sigma^*, \rho^{*\prime}$. Since $\mathrm{Nc}(\Psi/\rho^*) \leq \mathrm{Nc}(\Psi/\rho^* + \eta^*) = \mathrm{Nc}(\Psi/\rho)$ and $\mathrm{Nc}(\Phi^*/\sigma) < \mathrm{Nc}(\Phi^*/\sigma) + \mathrm{Nc}(\eta/\sigma) = \mathrm{Nc}(\Phi/\sigma)$ (using Theorem 6.8), the induction hypothesis applies to $\Phi^*, \Psi, \rho^*, \sigma, \rho^{*\prime}, \sigma^*$, whence $\rho^* + \sigma^* \doteq \sigma + \rho^{*\prime}$. Again by (M), $(\exists\tau).\ \tau$ comdevelops (η^*/σ^*) . Fix τ ; note that



by (S), $\rho' \doteq \rho^{*\prime} + \tau$. Since $\rho \doteq \rho^* + \eta^*$, have that σ' comdevelops $(\Psi/\rho^* + \eta^*)$. Since $\Phi^* \neq \emptyset . \mathrm{Nc}(\Phi^*/\sigma) \geqslant \mathrm{Nc}\,\Phi^* .\ \rho^{*\prime}$ comdevelops (Φ^*/σ) , it follows that $\#(\rho^{*\prime}) > 0$. Since $\Phi \in (\Phi \cup \Psi)\mathrm{Sep}$, have $(\Phi/\sigma) \in (\Phi/\sigma)\mathrm{Sep}$. By Lemma 4.18(b), $\mathrm{Nc}(\Phi/\sigma) \geqslant \#(\rho^{*\prime}) + \mathrm{Nc}(\Phi/\sigma + \rho^{*\prime})$, whence $\mathrm{Nc}(\Phi/\sigma + \rho^{*\prime}) < \mathrm{Nc}(\Phi/\sigma)$. On the other hand, $\mathrm{Nc}(\Psi/\rho^* + \eta^*) = \mathrm{Nc}(\Psi/\rho)$ since

$\rho \doteq \rho^* + \eta^*$. So the induction hypothesis applies to $\{\eta^*\}$, (Ψ/ρ^*), η^*, σ^*, τ, σ', whence $\eta^* + \sigma' \doteq \sigma^* + \tau$. Thus $\rho + \sigma' \doteq \rho^* + \eta^* + \sigma' \doteq \rho^* + \sigma^* + \tau \doteq \sigma + \rho^{*\prime} + \tau \doteq \sigma + \rho'$, q.e.d Case 2.

<u>Case 3</u>. $\Phi \cap \Psi \neq \emptyset$. Let $\Omega = \Phi \cap \Psi$. By (M), $(\exists \pi)$. π comdevelops Ω . Let $\Phi^* = (\Phi/\pi) = (\Phi - \Omega/\pi)$. $\Psi^* = (\Psi/\pi) = (\Psi - \Omega/\pi)$. Since $(\Phi - \Omega) \cap (\Psi - \Omega) = \emptyset$, it follows by Theorem 6.8 that $\Phi^* \cap \Psi^* = \emptyset$. Furthermore, $\Phi^* \cup \Psi^*$ ε Coinit$\cap$Finite . $\Phi^* \cup \Psi^* \subseteq \Lambda$. Φ^*, Ψ^* ε $(\Phi^* \cup \Psi^*)$Sep . By (M), $(\exists \rho^*, \sigma^*)$. ρ^* comdevelops Φ^* . σ^* comdevelops Ψ^* . Fix ρ^*, σ^*. Since by (S), $\rho \doteq \pi + \rho^*$. $\sigma = \tau + \sigma^*$, we see that



ρ' comdevelops (Φ^*/σ^*) . σ' comdevelops (Ψ^*/ρ^*) . By Case 2, $\rho^* + \sigma' \doteq \sigma^* + \rho'$. Hence $\rho + \sigma' \doteq \pi + \rho^* + \sigma' \doteq \pi + \sigma^* + \rho'$ $\doteq \sigma + \rho'$.

The reader should verify that Cases 1 - 3 are exhaustive; this completes the proof.

<u>Note 6.15</u>. The proof of this theorem is essentially the

same as the proof of Newman's Lemma 5.1 in Newman's Paper; the intricate proof of Case 2 is attributed by Newman to J. H. C. Whitehead.

Corollary 6.15(a). ORCh(C, δ) . RelSep J . $\supset$. NormRelSep J .

Corollary 6.15(b). ORCh(C, δ) . $\supset$. NormCx(C, δ) .

Proof. Assume ORCh(C, δ). Then Reg2NormCx(C, δ). Hence RegCx(C, δ), whence ($\exists$J). RelSep J . Fix J. Then by Corollary 6.15(a), NormRelSep J. Hence NormCx(C, δ) by Theorem 5.3.

Corollary 6.15(c). ORCh(C, δ) . $\supset$. ChR C .

Proof. Use Corollary 3.7.

Lemma 6.16(a). $H(C,\delta,J,\Lambda,K)$. $\supset$: $(\forall\Phi,\Psi)(\forall\rho,\sigma,\rho',\sigma')$: $K(\Phi,\Psi,\rho,\sigma,\rho',\sigma')$. Nc $\Psi = 0$. $\supset$. $\rho + \sigma' \doteq \sigma + \rho'$.

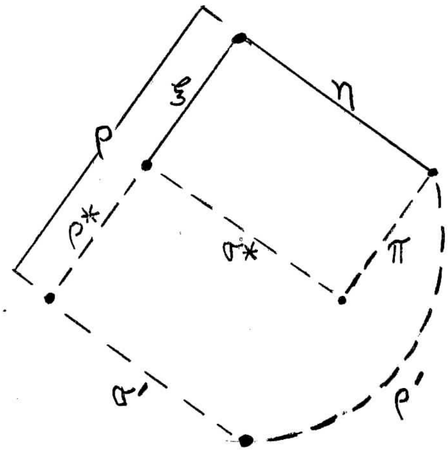
Proof. Assume $H(C,\delta,J,\Lambda,K)$. Nc $\Psi = 0$. Then $\Psi = \emptyset$, whence $\sigma = \boxtimes$. $(\Psi/\rho) = \emptyset$, whence $\sigma = \boxtimes$. $\sigma' = \boxtimes$. $(\Phi/\sigma) = \Phi$. Since $\Phi = \Phi \cup \Psi$, have that $\Phi \in \mathrm{Finite}_\cap\Phi\mathrm{Sep}$; and clearly ρ, ρ' comdevelop Φ . So by (S) we have $\rho + \sigma' = \rho \doteq \rho' = \sigma + \rho'$, q.e.d.

Lemma 6.16(b). $H(C,\delta,J,\Lambda,K)$. $\supset$: $(\forall\Phi,\Psi)(\forall\rho,\sigma,\rho',\sigma')$: $K(\Phi,\Psi,\rho,\sigma,\rho',\sigma')$: $\Phi \subseteq K$. Nc $\Psi = 1$. $\supset$. $\rho + \sigma' \doteq \sigma + \rho'$.

<u>Proof.</u> By induction on $\#\rho$.

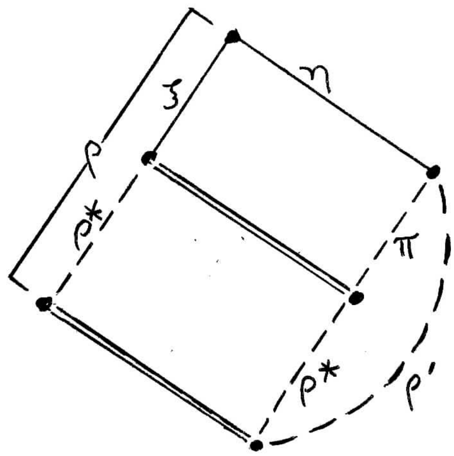
<u>Basis.</u> $\#\rho = 0$. Then $\rho = \boxtimes$. Hence $\Phi = \emptyset$, whence $\mathrm{Nc}\ \Phi = 0$ and Lemma 6.16(a) applies.

<u>Induction Step.</u> $\#\rho > 0$.

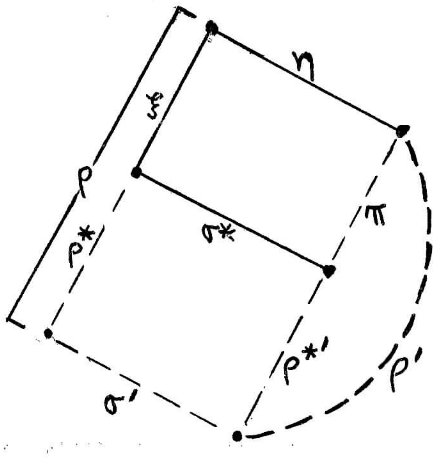


Then $(\exists \xi)(\exists \rho^*)$. $\rho = \xi + \rho^*$. Fix ξ, ρ^*; note $\xi \varepsilon K$. Since $\mathrm{Nc}\ \Psi = 1$, it follows that $(\exists \eta)$. $\Psi = \{\eta\}$. Fix η ; then $\sigma = \langle \eta \rangle$. By (M) , $(\exists \pi, \sigma^*)$. π comdevelops (ξ/η) . σ^* comdevelops (η/ξ) . Fix π, σ^*. By (D'), $\xi + \sigma^* \doteq \eta + \pi$. By (C4), $\mathrm{Nc}(\Psi/\xi) \leqslant 1$. Now we consider two cases.

<u>Case A.</u> $\mathrm{Nc}(\Psi/\xi) = 0$. Then $\sigma^* = \boxtimes$. $\sigma' = \boxtimes$. So $\xi = \xi + \sigma^* \doteq \eta + \pi$. Since ρ^* comdevelops (Φ/ξ) and $(\Phi/\xi) = (\Phi/\eta + \pi)$, it follows that ρ^* comdevelops $(\Phi/\eta + \pi)$. So $\pi + \rho^*$ comdevelops (Φ/η) . Also, $(\Phi/\eta)\ \varepsilon\ (\Phi/\eta)\mathrm{Sep}$. Hence by (S), $\pi + \rho^* \doteq \rho'$. We have $\rho + \sigma' = \rho + \boxtimes = \rho = \xi + \rho^* \doteq \eta + \pi + \rho^* \doteq \eta + \rho' = \sigma + \rho'$, q.e.d. Case A.



Case B. $Nc(\Psi/\zeta) = 1$. By (M). $(\exists\rho^{*\prime})$. $\rho^{*\prime}$ comdevelops $(\Phi/\eta + \pi)$. Fix $\rho^{*\prime}$. Since $(\Phi/\eta) \varepsilon (\Phi/\eta)\mathrm{Sep}$, it follows by (S) that $\pi + \rho^{*\prime} \doteq \rho'$. Since $\zeta + \sigma^* \doteq \eta + \pi$, it follows that $(\Phi/\zeta + \sigma^*) = (\Phi/\eta + \pi)$, whence $\rho^{*\prime}$ comdevelops $(\Phi/\zeta + \sigma^*)$. So by the induction hypothesis, $\rho^* + \sigma' \doteq \sigma^* + \rho^{*\prime}$. Now we have that $\rho + \sigma' = \zeta + \rho^* + \sigma'$ $\doteq \zeta + \sigma^* + \rho^{*\prime} \doteq \eta + \pi + \rho^{*\prime} \doteq \eta + \rho'$, q.e.d. Case B.



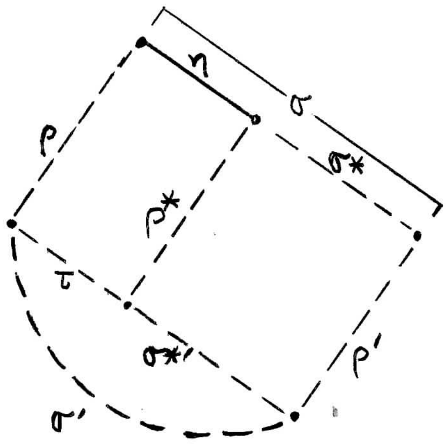
Since $\zeta \varepsilon K$, it follows that $Nc(\Psi/\zeta) \leq Nc\,\Psi = 1$, whence Cases A, B are exhaustive. This completes the proof.

Theorem 6.16. $H(C,\delta,J,\Lambda,K) \,.\supset::\, (\forall\Phi,\Psi)(\forall\rho,\sigma,\rho',\sigma')::$ $K(\Phi,\Psi,\rho,\sigma,\rho',\sigma') : \Phi \in K \,.\vee.\, \Psi \in K \,.:\supset.\, \rho + \sigma' \doteq \sigma + \rho'$.

Proof. Without loss of generality, assume that $\Phi \in K$. Use induction on $\#\sigma$.

Basis. $\#\sigma = 0$. Then $Nc\,\Psi = 0$ and Lemma 6.16(a) applies.

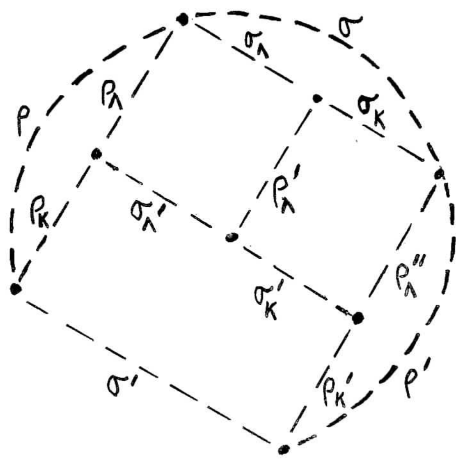
Induction Step. $\#\sigma > 0$. Then $(\exists\eta)(\exists\sigma^*)$. $\sigma = \eta + \sigma^*$. Fix η, σ^*. By (M), $(\exists\rho^*,\tau)$. ρ^* comdevelops (Φ/η). τ comdevelops (η/ρ). Fix ρ^*, τ. By Lemma 6.16(b), $\rho + \tau \doteq \eta + \rho^*$. By (M), $(\exists\sigma^{*\prime})$. $\sigma^{*\prime}$ comdevelops $(\Psi/\rho + \tau)$. Fix $\sigma^{*\prime}$. Now σ', $\tau + \sigma^{*\prime}$ comdevelop (Ψ/ρ); since $(\Psi/\rho) \varepsilon (\Psi/\rho)\mathrm{Sep}$, it follows that $\sigma' \doteq \tau + \sigma^{*\prime}$. Now $(\Phi/\eta) \in K$ by (C5). Hence the induction



hypothesis applies, whence $\rho^* + \sigma^{*\prime} \doteq \sigma^* + \rho'$. Now we have $\rho + \sigma' \doteq \rho + \tau + \sigma^{*\prime} \doteq \eta + \rho^* + \sigma^{*\prime} \doteq \eta + \sigma^* + \rho' = \sigma + \rho'$.

<u>Theorem 6.17</u>. $H(C,\delta,J,\Lambda,K) \;.\supset:\; (\forall\Phi,\Psi)(\forall\rho,\sigma,\rho',\sigma'):$ $K(\Phi,\Psi,\rho,\sigma,\rho',\sigma') \,.\, (\Phi\cup\Psi)\cap K \neq \emptyset \,.\, \Phi\cap\Lambda \neq \emptyset \neq \Psi\cap\Lambda \,.\supset.\, \rho + \sigma' \doteq \sigma + \rho'$.

<u>Proof</u>. Assume $H(C,\delta,J,\Lambda,K)$ and the antecedent. Let $\Phi_\Lambda = \Phi\cap\Lambda \,.\, \Psi_\Lambda = \Psi\cap\Lambda \,.\, \Phi_K = \Phi - \Lambda \,.\, \Psi_K = \Psi - \Lambda$. Then $\Phi_K \cup \Psi_K \subseteq K$, and $\Phi_\Lambda \cap \Phi_K = \Psi_\Lambda \cap \Psi_K = \emptyset \,.\, \Phi_\Lambda \cup \Phi_K = \Phi \,.\, \Psi_\Lambda \cup \Psi_K = \Psi$. Since $\Phi_\Lambda \cup \Phi_K \in \mathrm{Coinit}_\cap \mathrm{Finite}_\cap (\Phi_\Lambda \cup \Phi_K)\mathrm{Sep} \,.\, \rho$ comdevelops $(\Phi_\Lambda \cup \Phi_K)$, it follows by Corollary 5.16(b) that $(\exists \rho_\Lambda, \rho_K) .$ $\rho \doteq \rho_\Lambda + \rho_K \,.\, \rho_\Lambda$ comdevelops $\Phi_\Lambda \,.\, \rho_K$ comdevelops (Φ_K/ρ_Λ) . Fix ρ_Λ , ρ_K . Similarly, fix $\sigma_\Lambda , \sigma_K$ such that $\sigma \doteq \sigma_\Lambda + \sigma_K$. σ_Λ comdevelops Ψ_Λ . σ_K comdevelops (Ψ_K/σ_Λ) . By (M), $(\exists\, \rho_\Lambda', \sigma_\Lambda')$. ρ_Λ' comdevelops $(\Phi_\Lambda/\sigma_\Lambda)$. σ_Λ' comdevelops $(\Psi_\Lambda/\rho_\Lambda)$. Fix $\rho_\Lambda', \sigma_\Lambda'$. By Theorem 6.15, $\rho_\Lambda + \sigma_\Lambda' \doteq \sigma_\Lambda + \rho_\Lambda'$. By (M), $(\exists\, \rho_\Lambda'', \sigma_K')$. ρ_Λ'' comdevelops $(\Phi_\Lambda/\sigma_\Lambda + \sigma_K)$. σ_K' comdevelops $(\Psi_K/\sigma_\Lambda + \rho_\Lambda')$. Since $\Psi_K \subseteq K$, have $(\Psi_K/\sigma_\Lambda) \subseteq K$, whence by Theorem 6.16,



$\rho_\Lambda' + \sigma_K' \doteq \sigma_K + \rho_\Lambda''$. By (M), $(\exists \rho_K')$. ρ_K' comdevelops $(\Phi_K/\rho_\Lambda + \sigma_\Lambda' + \sigma_K')$. Fix ρ_K'. Since $\Phi_K \subseteq K$, have $(\Phi_K/\rho_\Lambda) \subseteq K$, whence by Theorem 6.16, $\rho_K + \sigma' \doteq \sigma_\Lambda' + \sigma_K' + \rho_K'$.

Lastly, since $\Phi \in (\Phi \cup \Psi)\mathrm{Sep}$. σ develops $(\Phi \cup \Psi)$, it follows by Theorem 4.14 that $(\Phi/\sigma) \in (\Phi/\sigma)\mathrm{Sep}$; but certainly also $(\Phi/\sigma) \in \mathrm{Coinit}_\cap \mathrm{Finite}$, and we have that ρ', $\rho_\Lambda'' + \rho_K'$ comdevelop (Φ/σ) . So by (S), $\rho' = \rho_\Lambda'' + \rho_K'$. Now we have that $\rho + \sigma' \doteq \rho_\Lambda + \rho_K + \sigma' \doteq \rho_\Lambda + \sigma_\Lambda' + \sigma_K' + \rho_K' \doteq \sigma_\Lambda + \rho_\Lambda' + \sigma_K' + \rho_K' \doteq \sigma_\Lambda + \sigma_K + \rho_\Lambda'' + \rho_K' \doteq \sigma + \rho'$, q.e.d.

<u>Theorem 6.18</u>. $H(C,\delta,J,\Lambda,K)$.⊃. (JN) .
(See Definition 5.1.)

<u>Proof</u>. (See Remark 6.14.) Assume $H(C,\delta,J,\Lambda,K)$. & . $K(\Phi,\Psi,\rho,\sigma,\rho',\sigma')$. To prove $\rho + \sigma' \doteq \sigma + \rho'$.

<u>Case 1</u>. $\Phi \cup \Psi \subseteq \Lambda$. Conclusion follows by Theorem 6.15.

<u>Case 2</u>. $\Phi \subseteq K$. v . $\Psi \subseteq K$. Conclusion follows by Theorem 6.16.

<u>Case 3</u>. $(\Phi \cup \Psi)_\cap K \neq \emptyset$. $\Phi_\cap \Lambda \neq \emptyset \neq \Psi_\cap \Lambda$. Conclusion follows by Theorem 6.17.

Since Cases 1,2,3 exhaust all possibilities, this completes the proof.

<u>Corollary 6.18(a)</u>. $\mathrm{OUCh}(C,\delta)$. RelSep J .⊃. NormRelSep J .

<u>Proof</u>. Assume the antecedent. Then by Definition 6.6, $\mathrm{Reg2NormCx}(C,\delta)$. RelSep J . $(\exists \Lambda, K)$. $\langle \Lambda, K\rangle \in \mathrm{OrdCovPr}$. Fix Λ, K. Then by Definition 6.13, $H(C,\delta,J,\Lambda,K)$. Hence by Theorem 6.18, (JN) holds, whence NormRelSep J by Definition 5.1.

<u>Corollary</u> <u>6.18(b)</u>. $\mathrm{OUCh}(C,\delta)\ .\supset.\ \mathrm{NormCx}(C,\delta)$.

<u>Proof</u>. Assume $\mathrm{OUCh}(C,\delta)$. Then by Definition 6.6, $\mathrm{Reg2NormCx}(C,\delta)$, whence by Definition 5.15, $\mathrm{RegCx}(C,\delta)$; but by Corollary 6.18(a), $(\forall J)\colon \mathrm{RelSep}\ J\ .\supset.\ \mathrm{NormRelSep}\ J$. So by Corollary 5.3(c), it follows that $\mathrm{NormCx}(C,\delta)$, q.e.d.

<u>Corollary</u> <u>6.18(c)</u>. $\mathrm{OUCh}(C,\delta)\ .\supset.\ \mathrm{ChR}\ C$.

<u>Proof</u>. Use Corollary 3.7.

<u>Definition</u> <u>6.19</u>. We say that C is a <u>choice</u> <u>complex</u> (and write ⌜Choice C⌝) iff the following condition holds:

$(\exists f)\colon.\ f\ \varepsilon\ \mathrm{Funct}\ \colon \mathrm{Arg}\ f = \{\Phi \mid \Phi\ \varepsilon\ \mathrm{Finite}\ .\ \Phi \neq \emptyset\}\ \colon$
$(\forall\Phi)\colon \Phi\ \varepsilon\ \mathrm{Arg}\ f\ .\supset.\ (f\Phi)\ \varepsilon\ \Phi$.

<u>Note</u> <u>6.19</u>. We have ignored stratification in formulating this definition. For use in a typewise restricted set theory, replace '$(f\Phi)\ \varepsilon\ \Phi$' by '$\mathrm{Val}\ f \subseteq \mathrm{USC}\ V\ .\ (f\Phi) \subseteq \Phi$' . See Rosser's Textbook.

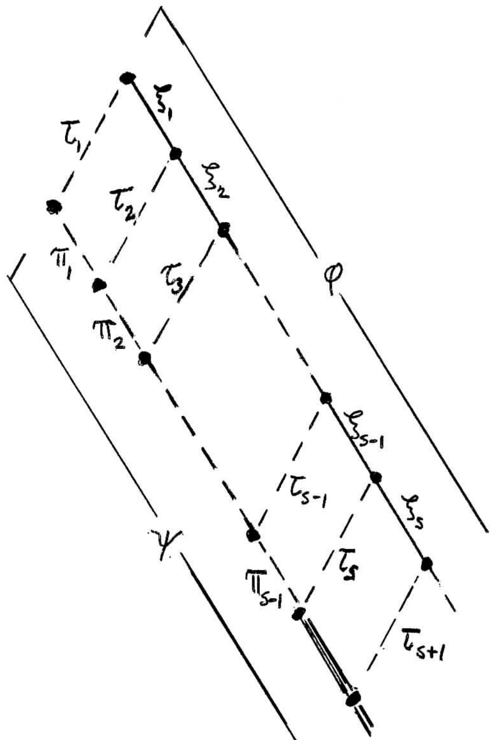
<u>Theorem</u> <u>6.20</u>. $\mathrm{OUCh}(C,\delta)\ .\ \mathrm{Choice}\ C\ .\supset.\ \mathrm{LocFinDesc}(C,\delta)$. (See Definition 3.16.) (P. 42)

<u>Proof</u>. Assume $\mathrm{OUCh}(C,\delta)$. Choice C . Then $\mathrm{Reg2NormCx}(C,\delta)\ .\ (\exists\Lambda,K).\ \langle\Lambda,K\rangle\ \varepsilon\ \mathrm{OrdCovPr}$. Fix Λ, K. Since $\mathrm{RegCx}(C,\delta)$, we have that $(\exists J)$. RelSep J . Fix J. Since Choice C, we have that $(\exists f)\colon.\ f\ \varepsilon\ \mathrm{Funct}\ \colon \mathrm{Arg}\ f = \{\Phi \mid \Phi\ \varepsilon\ \mathrm{Finite}\ .\ \Phi \neq \emptyset\}\ \colon (\forall\Phi)\colon \Phi\ \varepsilon\ \mathrm{Arg}\ f\ .\supset.\ f\Phi\ \varepsilon\ \Phi$.

Fix f. Let $\Phi \in \text{Coinit}_\cap\text{Finite}$. φ develop Φ . To prove $\#\varphi <$ ∞. Use induction on $\deg \Phi$ (see Definition 4.19).

<u>Basis</u>. $\deg \Phi = 1$. Then $\Phi \in \text{Finite}_\cap \Phi\text{Sep}$, and Corollary 4.18 applies. (This covers the case when $\Phi = \emptyset$.)

<u>Induction Step</u>. For convenience of construction, assume $\#\varphi = \infty$. $\deg \Phi = k_o + 1$, where $k_o \geqslant 1$. Then $(\exists \Omega, \Psi_1, \ldots, \Psi_{k_o})$. $\Phi = \Omega \cup \Psi_1 \cup \ldots \cup \Psi_{k_o}$. $\Omega, \Psi_1, \ldots, \Psi_{k_o} \in \Phi\text{Sep}$. Fix $\Omega, \Psi_1, \ldots, \Psi_{k_o}$. Let $\Psi = \Psi_1 \cup \ldots \cup \Psi_{k_o}$. Let $\varphi =$ $(\xi_1 + \xi_2 + \ldots)$. Define by induction $\Omega_1 = \Omega$, $\Omega_{m+1} =$ $(\Omega_m/\xi_m) = (\Omega/\xi_1 + \ldots + \xi_m)$. Let $\tau_m = [f\Omega_m + f(\Omega_m/f\Omega_m) +$ $f(\Omega_m/f\Omega_m + f(\Omega_m/f\Omega_m)) + \ldots]$. Then τ_m develops Ω_m ; since $\Omega_m \in \Omega_m\text{Sep}$ it follows that $\#\tau_m < \infty$ by Corollary 4.18. So $(\Omega_m/\tau_m) = \emptyset$, whence τ_m comdevelops Ω_m . Similarly, define



π_m so that π_m comdevelops (ξ_m/τ_m) . Let $\psi = (\pi_1 + \pi_2 + \ldots)$.

Since OUCh(C,δ), we have Condition (N), whence $(\forall i): 1 \leq i . \supset . \tau_i + \pi_i \doteq \zeta_i + \tau_{i+1}$; from this it follows by induction that Ψ develops (Ψ/τ_1) . Since also $\deg(\Psi/\tau_1) = \deg \Psi \leq k_o$, the induction hypothesis applies. Hence $\# \Psi < \infty$. Let $s = \min \{n \mid (\forall m): m \geq n . \supset . \pi_m = \emptyset\}$. Then $(\forall m): s \leq m . \supset . (\zeta_m/\tau_m) = \emptyset$.

<u>Case 1</u>. $(\Phi/\zeta_1 + \ldots + \zeta_{s-1}) \subseteq \Lambda$. By Corollary 6.9(a), we have that $(\forall \Omega)(\forall \zeta)(\forall \pi): \{\zeta\} \cup \Omega \in$ Coinit $. \{\zeta\} \cup \Omega \subseteq \Lambda . \pi$ develops $\Omega . (\zeta/\pi) = \emptyset . \supset . \zeta \in \Omega$. Hence $(\forall m): s \leq m . \supset . \zeta_m \in \Omega_m$. Thus $(\zeta_s + \zeta_{s+1} + \ldots)$ develops Ω_s ; since $\Omega_s \in \Omega_s$Sep , $\#(\zeta_s + \zeta_{s+1} + \ldots) < \infty$, whence $\# \Phi < \infty$. [As a corollary to Case 1, it follows that $\Phi \subseteq \Lambda . \deg \Phi = k_o+1 . \supset . \# \Phi < \infty$. We use this result in Case 3 below.]

<u>Case 2</u>. $(\Phi/\zeta_1 + \ldots + \zeta_{s-1}) \subseteq K$. Then $Nc(\Phi/\zeta_1 + \ldots + \zeta_{s-1}) > Nc(\Phi/\zeta_1 + \ldots + \zeta_{s-1} + \zeta_s) > Nc(\Phi/\zeta_1 + \ldots + \zeta_s + \zeta_{s+1}) > \ldots$; so eventually $(\Phi/\zeta_1 + \ldots + \zeta_{s+p}) = \emptyset$, whence $\# \Phi < \infty$.

<u>Case 3</u>. $(\Phi/\zeta_1 + \ldots + \zeta_{s-1}) \cap \Lambda \neq \emptyset \neq (\Phi/\zeta_1 + \ldots + \zeta_{s-1}) \cap K$. Use proof by contradiction from the assumption that $\#\Phi = \infty$. First, we effectively define by induction an infinite sequence $\omega_o, \omega_1, \omega_2, \ldots$ of descents satisfying the following four conditions (in which we use the notation that $\omega_n = [\Theta_1^n + \Theta_2^n + \ldots]$, where each $\Theta_i^n \in \Sigma$):

(1) $\#\omega_n = \infty$.

(2) ω_n develops $(\Phi/\zeta_1 + \ldots + \zeta_{s-1})$.

(3) $\Theta_1^n + \ldots + \Theta_{n-1}^n = \Theta_1^{n-1} + \ldots + \Theta_{n-1}^{n-1}$.

(4) $\Theta_1^n + \ldots + \Theta_n^n$ develops $(\Phi/\zeta_1 + \ldots + \zeta_{s-1}) \cap \Lambda$.

Basis (for the definition). $n = 0$. Take $\omega_0 = (\xi_s + \xi_{s+1} + \dots)$. (1), (2) are obvious, while (3) holds since $\boxtimes = \boxtimes$, and (4) holds since $(\forall\Phi).\ \boxtimes$ develops Φ.

Induction Step (for the definition). $n = m+1$. By the corollary to Case 1 (in square brackets), we have (using (C5)) that $(\exists i).\ \Theta_i^m \not\in \Lambda$. Let $i = \min\{i \mid \Theta_i^m \not\in \Lambda\}$. By induction hypotheses (3), (4), we have that $m < i$. By Case 2 and the assumption that $\#\varphi = \infty$, we see that $(\exists j).\ j > i.\ \Theta_j^m \not\in K$. Hence by (C5), $(\exists j)(\exists\eta).\ j > i.\ \eta \in \Lambda.\ \Theta_j^m \in (\eta/\Theta_1^m + \dots + \Theta_{j-1}^m)$. Let

$$j = \min\{j \mid j > i.\ (\exists\eta).\ \eta \in \Lambda.\ \Theta_j^m \in (\eta/\Theta_1^m+\dots+\Theta_{j-1}^m)\},$$

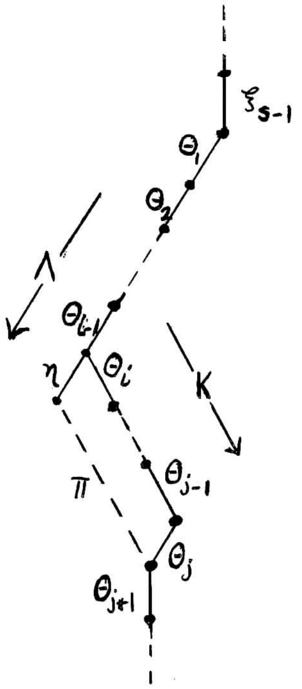
$$\eta = f\{\eta \mid \eta \in \Lambda.\ \Theta_j^m \in (\eta/\Theta_1^m+\dots+\Theta_{j-1}^m)\},$$

$$\begin{aligned}\chi = \{\zeta \mid \zeta \in (\Phi/\xi_1+\dots+\xi_{s-1} + \Theta_1^m+\dots+\Theta_{i-1}^m).\ (\exists k).\\ i \leq k < j-1.\ \Theta_k^m \in (\zeta/\Theta_i^m+\dots+\Theta_{k-1}^m)\},\end{aligned}$$

$$\chi' = (\chi/\eta),$$

$$\pi = f\chi' + f(\chi'/f\chi') + f(\chi'/f\chi' + f(\chi'/f\chi')) + \dots .$$

(We suppress the superscript 'm' in the diagram below.)



Since $\chi' \varepsilon \mathrm{Coinit}_\cap \mathrm{Finite} . \chi' \varepsilon K$, it follows that $\mathrm{Nc}\,\chi' >$ $\mathrm{Nc}(\chi'/f\chi') > \mathrm{Nc}(\chi'/f\chi' + f(\chi'/f\chi')) > \ldots$, so that $\#\pi < \infty$, whence $\pi \,\varepsilon\, \mathrm{Redn}$. Take

$$\omega_{m+1} = \Theta_1{}^m + \Theta_2{}^m + \ldots + \Theta_{i-1}^m + \eta + \pi + \Theta_{j+1}^m + \Theta_{j+2}^m + \ldots$$

We now prove that $\eta + \pi = \Theta_i{}^m + \ldots + \Theta_j{}^m$. First, π comdevelops χ' , whence $(\eta + \pi)$ comdevelops $\chi \cup \{\eta\}$. Second, consider any $\zeta \,\varepsilon\, \chi$. Clearly by definition of χ , $(\Theta_i{}^m + \ldots + \Theta_{j-1}^m)$ develops χ. Since also $(\exists k). i \leq k < j-1 . \Theta_k{}^m \,\varepsilon\, (\zeta/\Theta_i{}^m+\ldots+\Theta_{k-1}^m)$, we have (fixing k) that (since $\chi \varepsilon K$) $\mathrm{Nc}(\zeta/\Theta_i{}^m+\ldots+\Theta_{k-1}^m) \leq 1$ whence $\{\Theta_k{}^m\} = (\zeta/\Theta_i{}^m+\ldots+\Theta_{k-1}^m)$. So $(\zeta/\Theta_i{}^m+\ldots+\Theta_{k-1}^m + \Theta_k{}^m) = \emptyset$, whence $(\zeta/\Theta_i{}^m+\ldots+\Theta_{j-1}^m) = \emptyset$. So $(\Theta_i{}^m+\ldots+\Theta_{j-1}^m)$ comdevelops χ. Since similarly $\{\Theta_j{}^m\} = (\eta/\Theta_1{}^m+\ldots+\Theta_{j-1}^m)$, it follows that $(\Theta_i{}^m+\ldots+\Theta_{j-1}^m + \Theta_j{}^m)$ comdevelops $\chi \cup \{\eta\}$.

We have shown first that $(\eta + \pi)$ comdevelops $\chi \cup \{\eta\}$, and second that $(\Theta_i^m + \ldots + \Theta_j^m)$ comdevelops $\chi \cup \{\eta\}$. So $\eta + \pi \doteq \Theta_i^m + \ldots + \Theta_j^m$, by (N), which holds since OUCh(C, δ). Hence by the definition of ω_{m+1} each of (1) - (4) (for ω_m) implies its corresponding form for ω_{m+1}. This completes the inductive definition of the sequence $\omega_0, \omega_1, \omega_2, \ldots$.

It follows that $(\Theta_1^1 + \Theta_2^2 + \ldots + \Theta_i^i + \ldots)$ develops $(\Phi/\xi_1 + \ldots + \xi_{s-1})_\cap \Lambda$. Since $\deg(\Phi/\xi_1 + \ldots + \xi_{s-1})_\cap \Lambda \leq \deg \Phi = k_0 + 1$ (induction number of the proof), Case 1 applies, giving the desired contradiction (for Case 3), completing the induction step of the proof.

<u>Definition 6.21</u>. We say that $\langle C, \delta \rangle$ has the <u>finitely-many development</u> property (and write ⌜FinDev(C, δ)⌝) iff the following condition is satisfied:

$$(\forall \Phi) : \Phi \in \text{Coinit}_\cap \text{Finite} \;.\supset.\; \{\varphi \mid \varphi \text{ develops } \Phi\} \in \text{Finite} .$$

<u>Theorem 6.22</u>. Choice C . LocFinDesc(C, δ) .⊃. FinDev(C, δ) .

<u>Proof</u>. Assume Choice C . LocFinDesc(C, δ) . Use reductio ad absurdum; so assume also that $\Phi \in$ Finite . $\{\varphi \mid \varphi \text{ develops } \Phi\} \notin$ Finite . Fix f such that f $\in$ Funct : Arg f = $\{\Phi \mid \Phi \in \text{Finite} \;.\; \Phi \neq \emptyset\}$: $(\forall \Phi) : \Phi \in \text{Arg } f \;.\supset.\; f\Phi \in \Phi$. Since $\Phi \in$ Finite . $\{\varphi \mid \varphi \text{ develops } \Phi\} \notin$ Finite , it follows that $(\exists \xi_1) . \xi_1 \in \Phi . \{\xi_1 + \psi \mid \psi \text{ develops } (\Phi/\xi_1)\} \notin$ Finite . Let Φ_1 be the set of all such cells ξ_1; let $\eta_1 = f\Phi_1$. Similarly define Φ_2 and let $\eta_2 = f\Phi_2$. Continuing in this

fashion, we see (by induction) that $(\eta_1 + \eta_2 + \ldots)$ develops Φ . $\#(\eta_1 + \eta_2 + \ldots) = \infty$, contrary to the hypothesis that LocFinDesc(C,δ) .

Corollary 6.22. OUCh(C,δ) . Choice C .$\supset$: $(\forall\Phi)$: Φ ε Finite .$\supset$: $(\exists n)(\forall\varphi)$: φ develops Φ .$\supset$. $\#\varphi \leqslant n$: $(\forall\rho,\sigma)$: ρ,σ comdevelop Φ .$\supset$. $\{\rho,\sigma\}$ ε Coterminal.

Proof. Assume OUCh(C,δ) . Choice C . For Φ ε Finite , let $n = \max\{\#\varphi \mid \varphi \text{ develops } \Phi\}$. This n is as desired.

Note 6.22. Corollary 6.22 is an abstract form of Church-Rosser's Lemma 1, principal lemma in their treatment. (See Church's Monograph.)

Definition 6.23. We write 'WeakAxCh' for $(\forall f)$: f ε Funct . $(\exists N)[\text{Arg } f = \{n \mid n < N\}$. Val f $\subseteq$ (Finite $-$ $\{\emptyset\}$) .$\supset$: $(\exists g)$: g ε Funct : Arg g = Arg f : $(\forall n)$: n ε (Arg g)$\cap$(Arg f) .$\supset$. (gn) ε (fn) .

This states that simultaneous choices can be made from the terms of an arbitrary countable enumeration of finite sets.

Remark 6.23. We have ignored stratification in formulating this definition. For use in a typewise restricted set theory, replace '$(\forall n)$: n ε(Arg g)$\cap$(Arg f) .$\supset$. (gn) (fn)' by 'Val g $\subseteq$ USC V : $(\forall n)$: n ε(Arg g)$\cap$(Arg f) .$\supset$. (gn) $\subseteq$ (fn)' . See Rosser's Textbook.

<u>Theorem</u> <u>6.24(a)</u>. $(\forall C,\delta)[\mathrm{LocFinDesc}(C,\delta) \;.\supset.\; \mathrm{FinDev}(C,\delta)] \;.\supset.\; \mathrm{WeakAxCh}$.

<u>Proof</u>. Assume the antecedent. By logical manipulations we then have $(\forall C,\delta):.\ \mathrm{DerivCx}(C,\delta)\ .\ (\exists\Phi).\Phi \varepsilon \mathrm{Coinit}_\cap\mathrm{Finite}\ .\ \{\varphi \mid \varphi \text{ develops } \Phi\} \not\varepsilon \mathrm{Finite} :\supset.\ (\exists\Phi)(\exists\varphi).\Phi \varepsilon \mathrm{Coinit}_\cap\mathrm{Finite}\ .\ \varphi \text{ develops } \Phi\ .\ \#\varphi = \infty$. Define N by the condition that $\mathrm{Arg}\ f = \{n \mid n < N\}$. If $N < \infty$, the theorem follows by induction. So let $f \varepsilon \mathrm{Funct}\ .\ \mathrm{Arg}\ f = \mathrm{Nn}\ .\ \mathrm{Val}\ f \subseteq (\mathrm{Finite} - \{\emptyset\})$. To prove $(\exists g).\ g \varepsilon \mathrm{Funct}\ .\ \mathrm{Arg}\ g = \mathrm{Nn}\ .\ (\forall n).\ (gn) \varepsilon (fn)$. Let $V = \{h \mid h \varepsilon \mathrm{Funct} : (\exists n): n \varepsilon \mathrm{Nn} : \mathrm{Arg}\ h = \{m \mid m \varepsilon \mathrm{Nn}\ .\ m < n\} : (\forall m): m \varepsilon \mathrm{Nn}\ .\ m < n\ .\supset.\ hm \varepsilon fm\}$. [In the notation and terminology of Part II Section 1, $V = \{h \mid h \varepsilon \mathrm{Expr} : (\forall m): m < \mathrm{Length}\ h\ .\supset.\ hm \varepsilon fm\}$.] The empty function $\emptyset \varepsilon V$. Let $\Sigma = \{\langle h,x\rangle \mid h \varepsilon V\ .\ x \varepsilon f(1 + \max(\mathrm{Arg}\ h \cup \{-1\}))\}$ [$x \varepsilon f(\mathrm{Length}\ h)$, in the notation and terminology of II.1]; for $\langle h,x\rangle \varepsilon \Sigma$ define $\perp(h,x) = h\ .\ \&\ .\ \mathrm{T}(h,x) = h \cup \{\langle 1 + \max(\mathrm{Arg}\ h \cup \{-1\}),x\rangle\}$ [$= h \nleftrightarrow \langle\langle x\rangle\rangle$, a la II.1]. Where $\perp\xi \neq \perp\eta$, define $(\xi/\eta) = \emptyset$; define $(\xi/\xi) = \emptyset$; and where $\perp\xi = \perp\eta\ .\ \xi \neq \eta$, define $(\xi/\eta) = \{\zeta \mid \perp\zeta = \mathrm{T}\eta\}$. With $\delta(\xi,\eta) = (\xi/\eta)$ we have $\mathrm{DerivCx}(C,\delta)$. This complex has a greatest vertex $\emptyset$ with finitely-many cells (one for each element of $f0$) descending therefrom; the endpoint of each cell has finitely-many cells (one for each element of $f1$) descending therefrom, etc. (And by the definition of derivate, every descent is a development.) Let $\Phi = \{\xi \mid \perp\xi = \emptyset\}$. Then $\Phi \varepsilon \mathrm{Coinit}_\cap\mathrm{Finite}$. Also, $(\forall h): h \varepsilon V$.

$h \neq \emptyset \,.\supset.\, (\exists\rho).\ \rho\ \varepsilon\,(\mathrm{From}\ \emptyset)\cap(\mathrm{To}\ h)$. Since $V \notin$ Finite , it follows that $\{\rho \mid \rho$ develops $\Phi\} \notin$ Finite , whence $\{\varphi \mid \varphi$ develops $\Phi\} \notin$ Finite . Hence $(\exists\Psi)(\exists\psi)\colon \Psi\ \varepsilon$ Coinit $\cap$ Finite . ψ develops Ψ . $\#\psi = \infty$. Fix Ψ, ψ. Let $\{h\} = \perp``\Psi$. Then $(\exists\rho).\ \rho\ \varepsilon\,(\mathrm{From}\ \emptyset)\cap(\mathrm{To}\ h)$. Let $\omega = \rho + \psi$; use the notation $\omega = \xi_0 + \xi_1 + \ldots$. Then $\perp\xi_1 = \emptyset$. $\mathrm{T}\xi_1\ \varepsilon$ f1 . $\mathrm{T}\xi_2\ \varepsilon$ f2 , etc. Take g to be the function with Arg g = Nn such that $(\forall n).\ gn = \mathrm{T}\xi_n$.

Note 6.24. The germ of the proof was communicated in conversation by Paul J. Cohen.

Theorem 6.24(b). WeakAxCh $.\supset.$ $(\forall C,\delta)\ [\mathrm{LocFinDesc}(C,\delta)\ .\supset.\ \mathrm{FinDev}(C,\delta)]$.

Proof. Obviously, WeakAxCh $.\supset\colon (\forall C).$ Choice C . Apply Theorem 6.22.

Corollary 6.24. WeakAxCh $.\equiv.$ $(\forall C,\delta)\ [\mathrm{LocFinDesc}(C,\delta)\ .\supset.\ \mathrm{FinDev}(C,\delta)]$.

Proof. Use Theorems 6.24(a) - (b).

Definition 6.25. We define the set of C-maximal descents (for which we write ⌜CMaxDesc⌝) as follows:

$$\mathrm{CMaxDesc} = \{\pi \mid (\exists z).\ \pi\ \varepsilon\ \mathrm{To}\ z\ .\ z\ \varepsilon\ \mathrm{NormVx}\}\ .$$

(See Definition 1.30.)

Definition 6.26. We say that C is a unique normal vertex complex (and write ⌜UniqueNormVxCx C⌝) iff the following conditon is satisfied:

$(\forall \pi, \tau)$: $\{\pi, \tau\}$ ε Coinitial . π, τ ε CMaxDesc .⊃. $\{\pi, \tau\}$ ε Coterminal .

Remark 6.26. ChR C .⊃. UniqueNormVxCx C .

Definition 6.27. We say that C is a random descent complex (and write ⌜RanDesc C⌝) iff

UniqueNormVxCx C

and the following condition is satisfied:

$(\forall v)(\forall \varphi)$: v ε PrenormVx . φ ε From v .⊃. $\# \varphi < \infty$.

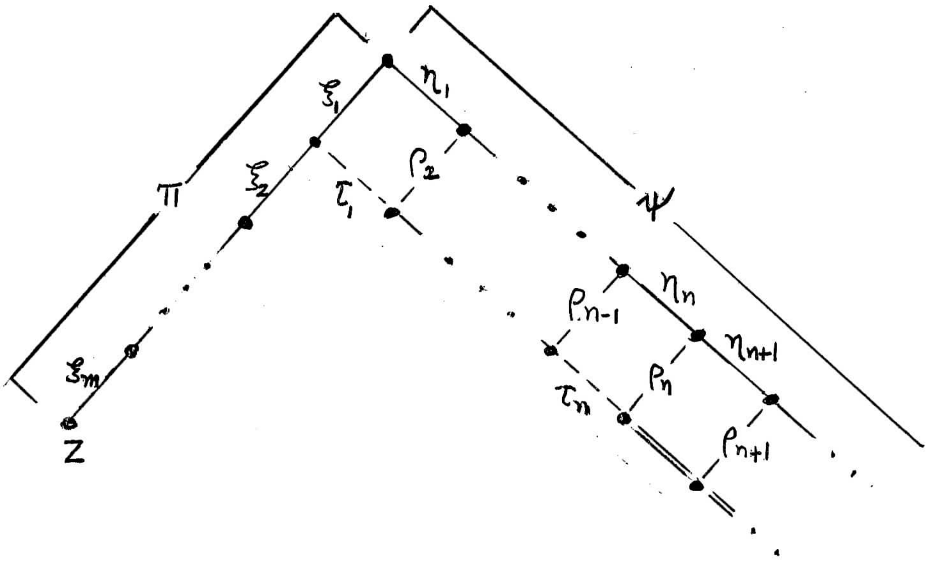
(See Definition 1.31, page 15.)

Theorem 6.28. ORCh(C,δ) . Choice C .⊃. RanDesc C .

Proof. Assume ORCh(C,δ) . Choice C . Then ChR C by Corollary 6.15(c) , whence UniqueNormVxCx C . Let v ε PrenormVx . ψ ε From v . (To prove $\# \psi < \infty$.) Since Choice C, it follows that $(\exists f)$: f ε Funct : Arg f = $\{\Phi \mid \Phi$ ε Finite . $\Phi \neq \emptyset\}$: $(\forall \Phi)$: Φ ε Arg f .⊃. fΦ ε Φ. Fix f. Since v ε PrenormVx , it follows that $(\exists z)(\exists \pi)$. z ε NormVx . π ε (From v)$\cap$(To z) . Fix z, π . Let $\pi = \xi_1 + \ldots + \xi_m$, $\psi = (\eta_1 + \eta_2 + \ldots)$. Use induction on $m = \# \pi$.

Basis. m = 0 . Then $\pi = \boxtimes$, whence v = z ε NormVx , whence $\psi = \boxtimes$. Then $\# \psi = 0 < \infty$.

<u>Induction Step</u>. $m \geq 1$. Assume $\#\Psi = \infty$. Let $\Phi_{i+1} = (\xi_1/\eta_1 + \ldots + \eta_i)$; note $\Phi_1 = \{\xi_1\}$. Let $\rho_i = f\Phi_i + f(\Phi_i/f\Phi_i) + f(\Phi_i/f\Phi_i + f(\Phi_i/f\Phi_i)) + \ldots$; note $\rho_1 = \{\xi_1\}$. Since LocFinDesc(C,δ), $\#\rho_i < \infty$, whence ρ_i comdevelops Φ_i . Let $\Omega_i = (\eta_i/\rho_i)$, and take $\tau_i = f\Omega_i + f(\Omega_i/f\Omega_i) + \ldots$. Then $\#\tau_i < \infty$, whence τ_i comdevelops Ω_i . By induction



hypothesis, $\#(\tau_1 + \tau_2 + \ldots) < \infty$. Hence $(\exists n)(\forall p)$: $p > n$. $\supset$. $\tau_p = \emptyset$. Fix n. Then $(\forall p)$: $p > n$.$\supset$. $(\eta_{p+1}/\rho_p) = \emptyset$. Since ORCh(C,δ), it follows by Corollary 6.9(a) that $(\forall p)$: $p > n$.$\supset$. $\eta_{p+1} \varepsilon \Phi_p$. It follows by induction on $(p-n)$ that $(\eta_{n+1} + \eta_{n+2} + \ldots)$ develops Φ_n , whence $\#\Psi < \infty$.

Explanation. The condition

$$\begin{cases}(\exists \rho', \sigma').\ \rho' \text{ comdevelops } (\xi/\eta) \ .\ \sigma' \text{ comdevelops } (\eta/\xi)\ . \\ \xi + \sigma \doteq \eta + \rho\ ,\end{cases}$$

(although a local condition) involves several defined concepts, and is inconvenient for use in the verifications at the end of Part II. For this reason we present in Theorem A below a condition — easily seen to be equivalent to this condition — in which most of the defined concepts are eliminated (and in which the notion of a descent does not explicitly occur). The additional fact expressed in Lemma A below is helpful, and its explicit statement should have appeared earlier.

Lemma A. $(\forall \xi, \eta)(\forall \rho', \sigma')$: ρ' develops (ξ/η) . σ' develops (η/ξ) . $\xi + \sigma' \doteq \eta + \rho'$.⊃. ρ' comdevelops (ξ/η) . σ' comdevelops (η/ξ) .

Proof. Assume the antecedent. Then $((\xi/\eta)/\rho') = (\xi/\eta + \rho') = (\xi/\xi + \sigma') = \emptyset$, and similarly $((\eta/\xi)/\sigma') = \emptyset$.

Theorem A. DerivCx(C, δ) $. \supset :. (\forall \xi, \eta) :.$
$(\exists \rho', \sigma')[\rho'$ comdevelops $(\xi/\eta) . \sigma'$ comdevelops (η/ξ) .
$\xi + \sigma' \doteq \eta + \rho'] :$

$$\equiv : \perp\xi = \perp\eta :$$
$$(\exists m, n)(\exists \zeta_1, \ldots, \zeta_m, \Theta_1, \ldots, \Theta_m)(\exists \Phi_0, \ldots, \Phi_m, \Psi_0, \ldots, \Psi_n) :$$
$$\Phi_0 = (\xi/\eta) : (\forall i)[0 \leq i < m . \supset . \zeta_{i+1} \varepsilon \Pi_i . \Pi_{i+1} = (\Pi_i/\zeta_{i+1})] :$$
$$\Psi_0 = (\eta/\xi) : (\forall i)[0 \leq i < n . \supset . \Theta_{i+1} \varepsilon \Upsilon_i . \Upsilon_{i+1} = (\Upsilon_i/\Theta_{i+1})] :$$
$$\top\zeta_m = \top\Theta_n :$$
$$(\forall \kappa)(\forall X_0, \ldots, X_m, \Omega_0, \ldots, \Omega_n) : \kappa \varepsilon \Sigma . \perp\kappa = \perp\xi .$$
$$X_0 = (\kappa/\xi) . (\forall i)[0 \leq i < m . \supset . X_{i+1} = (X_i/\zeta_{i+1})] .$$
$$\Omega_0 = (\kappa/\eta) . (\forall i)[0 \leq i < n . \supset . \Omega_{i+1} = (\Omega_i/\Theta_{i+1})] .$$
$$\supset . X_m = \Omega_n .$$

The phrase '$\kappa \varepsilon \Sigma$' is included since '$\kappa$' is not one of the variables restricted to be cells.

Proof. Given ρ', σ', let $m = \#\rho' . n = \#\sigma'$; fix $\zeta_1, \ldots, \zeta_m, \Theta_1, \ldots, \Theta_n$ such that $\rho' = \zeta_1 + \ldots + \zeta_m . \sigma' = \Theta_1 + \ldots + \Theta_n$ and for relevant i take $\Pi_i = (\xi/\eta + \zeta_1 + \ldots + \zeta_i) . \Psi_i = (\eta/\xi + \Theta_1 + \ldots + \Theta_i)$. Then since $\{\rho', \sigma'\} \varepsilon$ Coinitial$\cap$Coterminal have $\perp\xi = \perp\eta . \top\zeta_m = \top\Theta_n$, and since ρ' develops $(\xi/\eta) . \sigma'$ develops (η/ξ) have for relevant i that $\zeta_{i+1} \varepsilon \Pi_i . \Pi_{i+1} = (\Pi_i/\zeta_{i+1}) . \Theta_{i+1} \varepsilon X_i . X_{i+1} = (X_i/\Theta_{i+1})$. Given also $\kappa, X_0, \ldots, X_n, \Omega_0, \ldots, \Omega_n$ it follows since $\xi + \sigma' \doteq \eta + \rho'$ that $X_m = (\kappa/\xi + \sigma') = (\kappa/\eta + \rho') = \Omega_n$.

Conversely, given m, n, $\zeta_1, \ldots, \zeta_m$, $\Theta_1, \ldots, \Theta_n$, $\Phi_0, \ldots,$ Φ_m, $\Psi_0, \ldots, \Psi_n$ take $\rho' = \zeta_1 + \ldots + \zeta_m$. $\sigma' = \Theta_1 + \ldots + \Theta_n$. Then clearly ρ' develops (ξ/η) . σ' develops (η/ξ) ; since $\perp\xi = \perp\eta$. $T\zeta_m = T\Theta_n$ have $\{\rho', \sigma'\} \varepsilon$ Coinitial$\cap$Coterminal ; and given any $\kappa \varepsilon \Sigma$ either $\perp\kappa \neq \perp\xi$. $(\kappa/\xi + \rho') = \emptyset =$ $(\kappa/\eta + \sigma')$ or $\perp\kappa = \perp\xi$. $(\kappa/\xi + \rho') = X_m = \Omega_n = (\kappa/\eta + \sigma')$. Hence by Lemma A, ρ', σ' are as desired.