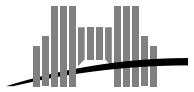
Explicit Substitutions and Intersection Types

Pierre Lescanne, LIP, ENS de Lyon

Le calcul $\lambda\mathbf{x}_{gc}$

One uses explicit names,

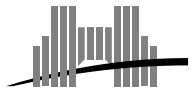
and one makes an intensive use of Barendregt convention on variables.



One uses explicit names,

and one makes an intensive use of Barendregt convention on variables.

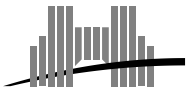
In the same context, a variable is never both free and bound.



The syntax of $\lambda\mathbf{x}_{gc}$

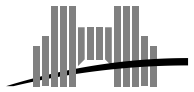
The syntax

$$M, N \quad := \quad x \mid \lambda x.M \mid M\ N \mid M\langle x = N\rangle$$



The rules of $\lambda\mathbf{x}_{gc}$

(B)	$(\lambda x M)\, P$	$\longrightarrow$	$M\langle x = P\rangle$	
(App)	$(M\,N)\langle x = A\rangle$	$\longrightarrow$	$M\langle x = A\rangle\, N\langle x = A\rangle$	
(Abs)	$(\lambda y M)\langle x = A\rangle$	$\longrightarrow$	$\lambda y M\langle x = A\rangle$	
(VarI)	$x\langle x = A\rangle$	$\longrightarrow$	A	
(VarK)	$y\langle x = A\rangle$	$\longrightarrow$	y	
(gc)	$M\langle x = A\rangle$	$\longrightarrow$	M	if $x \notin M$



The sets

$$
\begin{aligned}
\mathrm{x} &= \{\mathsf{App}, \mathsf{Abs}, \mathsf{VarI}, \mathsf{VarK}\} \\
\lambda\mathrm{x} &= \mathrm{x} \cup \{\mathsf{B}\} \\
\mathrm{x}_{gc} &= \{\mathsf{App}, \mathsf{Abs}, \mathsf{VarI}, \mathsf{gc}\} \\
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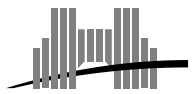
\end{aligned}
$$



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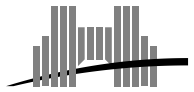
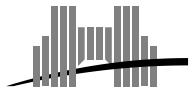
$$

$\mathcal{SN}$ = The set of strongly normalizing terms for $\lambda\mathrm{x}$.

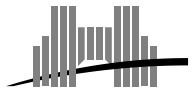
$\mathcal{SN}_{gc}$ = The set of strongly normalizing terms for $\lambda\mathrm{x}_{gc}$.



A counterexample to preservation of strong normalization



We will show that if we add a kind of composition, we do not preserve strong normalization.



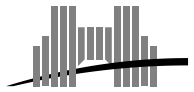
A kind of composition

We consider multisubstitutions $\langle \vec{x} = \vec{B} \rangle$ which we can compose.

$$M\langle \vec{x} = \vec{A} \rangle \langle \vec{y} = \vec{B} \rangle \quad \longrightarrow \quad M\langle \vec{x}, \vec{y} = A_1\langle \vec{y} = \vec{B} \rangle, \ldots, A_m\langle \vec{y} = \vec{B} \rangle, \vec{B} \rangle$$

In what follows we will use a simpler rule

$$M\langle x = A \rangle \langle y = B \rangle \quad \longrightarrow \quad M\langle x = A\langle y = B \rangle \rangle \quad \text{si } y \notin FV(M)$$



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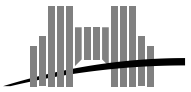
$$M\langle x = A \rangle \langle y = B \rangle \quad \longrightarrow \quad M\langle x = A\langle y = B \rangle \rangle \quad \text{si } y \notin FV(M)$$

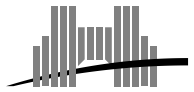
instead of

$$M\langle x = A \rangle \langle y = B \rangle \quad \longrightarrow \quad M\langle x, y = A\langle y = B \rangle, B \rangle$$



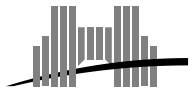
And the system of rules

$$
\begin{array}{lrcll}
(\mathsf{B}) & (\lambda x B)\, A & \longrightarrow & B\langle x = A\rangle & \\
(\mathsf{App}) & (MN)\langle x = A\rangle & \longrightarrow & M\langle x = A\rangle\, N\langle x = A\rangle & \\
(\mathsf{Abs}) & (\lambda y M)\langle x = A\rangle & \longrightarrow & \lambda y M\langle x = A\rangle & \\
(\mathsf{VarI}) & x\langle x = A\rangle & \longrightarrow & A & \\
(\mathsf{VarK}) & y\langle x = A\rangle & \longrightarrow & y & \\
(Com) & M\langle x = A\rangle\langle y = B\rangle & \longrightarrow & M\langle x = A\langle y = B\rangle\rangle & \text{si } y \notin FV(M)
\end{array}
$$



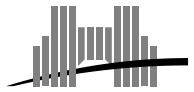
Instead of

$$
\begin{array}{lrcl}
(\text{B}) & (\lambda x B)\, A & \longrightarrow & B\langle x = A\rangle \\
(\text{App}) & (M\,N)\langle \vec{x} = \vec{A}\rangle & \longrightarrow & M\langle \vec{x} = \vec{A}\rangle\, N\langle \vec{x} = \vec{A}\rangle \\
(\text{Abs}) & (\lambda y M)\langle \vec{x} = \vec{A}\rangle & \longrightarrow & \lambda y M\langle \vec{x} = \vec{A}\rangle \\
(\text{VarI}) & x\langle x, \vec{y} = A, \vec{B}\rangle & \longrightarrow & A \\
(\text{VarK}) & y\langle \vec{x} = \vec{A}\rangle & \longrightarrow & y \qquad \text{si } y \notin \vec{x} \\
(Com) & M\langle \vec{x} = \vec{A}\rangle\langle \vec{y} = \vec{B}\rangle & \longrightarrow & M\langle \vec{x}, \vec{y} = \vec{A}\langle \vec{y} = \vec{B}\rangle, \vec{B}\rangle
\end{array}
$$



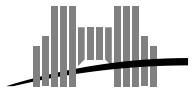
Bloo and Geuver's example

$$\begin{aligned} \alpha_0 &= \langle y = (\lambda y \cdot M) P \rangle \\ \alpha_{m+1} &= \langle y = P \alpha_m \rangle \end{aligned}$$



Bloo and Geuver's example (next)

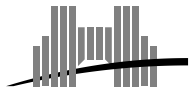
$$
\begin{aligned}
&((\lambda y \cdot \lambda y' \cdot x)((\lambda y \cdot M)P))\,((\lambda y \cdot M)P) \\
&\quad \longrightarrow\!\!\!\!\!\rightarrow \; x\langle y' = (\lambda y \cdot M)P\rangle\langle y = (\lambda y \cdot M)P\rangle \\
&\quad \longrightarrow \; x\langle y' = ((\lambda y \cdot M)P)\langle y = (\lambda y \cdot M)P\rangle\rangle \\
&\quad \longrightarrow\!\!\!\!\!\rightarrow \; x\langle y' = (\lambda y \cdot M\langle y = (\lambda y \cdot M)P\rangle)(P\langle y = (\lambda y \cdot M)P\rangle)\rangle \\
&\quad \equiv\!\!\!\equiv \; x\langle y' = (\lambda y \cdot M\alpha_0)(P\alpha_0)\rangle \\
&\quad \longrightarrow \; x\langle y' = M\alpha_0\langle y = P\alpha_0\rangle\rangle \\
&\quad \equiv\!\!\!\equiv \; x\langle y' = M\alpha_0\alpha_1\rangle
\end{aligned}
$$



Bloo and Geuver's example (next)

$$
\begin{array}{lcl}
M\alpha_0\alpha_{m+1} & \equiv\!\equiv & M\langle y = (\lambda y \cdot M)P\rangle\langle y = P\alpha_m\rangle \\
 & \longrightarrow & M\langle y' = ((\lambda y \cdot M)P)\langle y = P\alpha_m\rangle\rangle \\
 & \twoheadrightarrow & M\langle y' = (\lambda y \cdot M\langle y = P\alpha_m\rangle)\, b\langle y = P\alpha_m\rangle\rangle \\
 & \equiv\!\equiv & M\langle y' = (\lambda y \cdot M\alpha_{m+1})\, P\alpha_{m+1}\rangle \\
 & \longrightarrow & M\langle y' = M\alpha_{m+1}\langle y = P\alpha_{m+1}\rangle\rangle \\
 & \equiv\!\equiv & M\langle y' = M\alpha_{m+1}\alpha_{m+2}\rangle
\end{array}
$$

$$
\begin{array}{lcl}
M\alpha_{p+1}\alpha_{m+1} & \equiv\!\equiv & M\langle y' = P\alpha_p\rangle\langle y = P\alpha_m\rangle \\
 & \longrightarrow & M\langle y' = P\alpha_p\langle y = P\alpha_m\rangle\rangle \\
 & \equiv\!\equiv & M\langle y' = P\alpha_p\alpha_{m+1}\rangle
\end{array}
$$



Bloo and Geuver's example (next)

And we get the infinite derivation

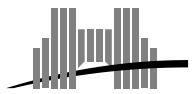
$$((\lambda y \cdot \lambda y' \cdot x)((\lambda y \cdot M)P))\,((\lambda y \cdot M)P)$$

$$\twoheadrightarrow \ldots M\alpha_0\alpha_1 \ldots \twoheadrightarrow \ldots M\alpha_1\alpha_2 \ldots \twoheadrightarrow \ldots M\alpha_0\alpha_2 \ldots$$

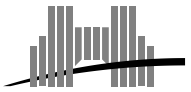
$$\twoheadrightarrow \ldots a\alpha_2\alpha_3 \ldots \twoheadrightarrow \ldots a\alpha_1\alpha_3 \ldots \twoheadrightarrow \ldots a\alpha_0\alpha_3 \ldots$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\twoheadrightarrow \ldots a\alpha_0\alpha_{m+1} \ldots \twoheadrightarrow \ldots a\alpha_{m+1}\alpha_{m+2} \ldots \twoheadrightarrow \ldots a\alpha_0\alpha_{m+2} \ldots$$

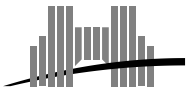


Intersection Types



$$\text{start}\ \frac{}{\Gamma \vdash x : \sigma}\ (x : \sigma) \in \Gamma$$

$$\to \text{I}\ \frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x.M : \sigma \to \tau} \qquad \to \text{E}\ \frac{\Gamma \vdash M : \sigma \to \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash M\ N : \tau}$$



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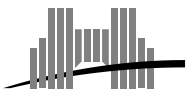
$$\cap\text{-I}\ \frac{\Gamma \vdash M : \tau_1 \quad \Gamma \vdash M : \tau_2}{\Gamma \vdash M : \tau_1 \cap \tau_2} \qquad \cap\text{-E}\ \frac{\Gamma \vdash M : \tau_1 \cap \tau_2}{\Gamma \vdash M : \tau_i}\ i \in \{1, 2\}$$



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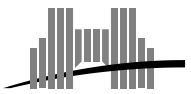
$$\text{cut}\ \frac{\Gamma, x : \sigma \vdash M : \tau \quad \Gamma \vdash A : \sigma}{\Gamma \vdash M\langle x = A\rangle : \tau}$$

$$\cap\text{-I}\ \frac{\Gamma \vdash M : \tau_1 \quad \Gamma \vdash M : \tau_2}{\Gamma \vdash M : \tau_1 \cap \tau_2} \qquad \cap\text{-E}\ \frac{\Gamma \vdash M : \tau_1 \cap \tau_2}{\Gamma \vdash M : \tau_i}\ i \in \{1, 2\}$$



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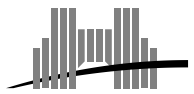
$$\text{cut}\ \frac{\Gamma, x : \sigma \vdash M : \tau \quad \Gamma \vdash A : \sigma}{\Gamma \vdash M\langle x = A\rangle : \tau}$$

$$\text{drop}\ \frac{\Gamma \vdash M : \tau \quad \text{A typable}}{\Gamma \vdash M\langle x = A\rangle : \tau}\ x \notin AV(M)$$

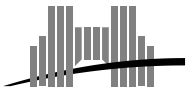
$$\cap\text{-I}\ \frac{\Gamma \vdash M : \tau_1 \quad \Gamma \vdash M : \tau_2}{\Gamma \vdash M : \tau_1 \cap \tau_2} \qquad \cap\text{-E}\ \frac{\Gamma \vdash M : \tau_1 \cap \tau_2}{\Gamma \vdash M : \tau_i}\ i \in \{1, 2\}$$



Availability: a new concept

The notion of free variables should be generalized with care

We speak of available variables.



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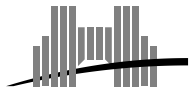
A variable is available if it does occur in a closure which will eventually disappear.

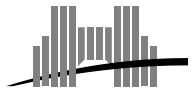


Availability: a new concept

The notion of free variables should be generalized with care

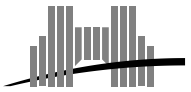
We speak of available variables.

$$
\begin{cases}
AV(x) & := & \{x\} & \\
AV(\lambda x.M) & := & AV(M)\backslash\{x\} & \\
AV(M\ N) & := & AV(M) \cup AV(N) & \\
AV(M\langle x = N\rangle) & := & (AV(M)\backslash\{x\}) \cup AV(N) & \text{if } x \in AV(M) \\
AV(M\langle x = N\rangle) & := & AV(M) & \text{if } x \notin AV(M)
\end{cases}
$$



here is how we define free variables.

$$\begin{cases} FV(x) & := & \{x\} \\ FV(\lambda x.M) & := & FV(M)\backslash\{x\} \\ FV(M\ N) & := & FV(M) \cup FV(N) \\ FV(M\langle x = N\rangle) & := & (FV(M)\backslash\{x\}) \cup FV(N) \end{cases}$$



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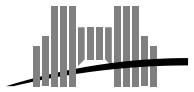
AV(M\langle x = N\rangle) & := & AV(M) & \text{if } x \notin AV(M)
\end{cases}
$$

$$
\begin{cases}
FV(x) & := & \{x\} \\
FV(\lambda x.M) & := & FV(M)\backslash\{x\} \\
FV(M\ N) & := & FV(M) \cup FV(N) \\
FV(M\langle x = N\rangle) & := & (FV(M)\backslash\{x\}) \cup FV(N)
\end{cases}
$$

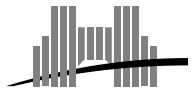


The result

A term is strongly normalisable

if and only if

it is tybable.



The result

A term is strongly normalisable

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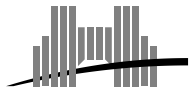
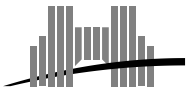
it is tybable.

Consequence : $\mathcal{SN} = \mathcal{SN}_{gc}$.



Basic Lemmas



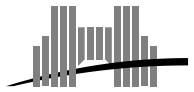
Composition Lemma

If

$$M\langle y = Q\rangle\langle x = P\langle y = Q\rangle\rangle \mathbf{T} \in \mathcal{SN}_{\mathsf{gc}}$$

then

$$M\langle x = P\rangle\langle y = Q\rangle \mathbf{T} \in \mathcal{SN}_{\mathsf{gc}}.$$



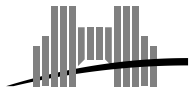
Closure Lemma

If

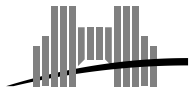
$$M\langle \boldsymbol{z} = \boldsymbol{S}\rangle \boldsymbol{T} \in \mathcal{SN}_{\mathsf{gc}} \text{ and } A \in \mathcal{SN}_{\mathsf{gc}}$$

then

$$M\langle x = A\rangle\langle \boldsymbol{z} = \boldsymbol{S}\rangle \boldsymbol{T} \in \mathcal{SN}_{\mathsf{gc}}.$$



Conclusion and Perspectives



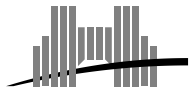
Conclusion

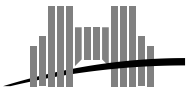
We has a direct proof whcih does not rely on λ-calculus.

Explicit Substitution calculi lye in front of λ-calculus,

as a more general theory

of which λ-calculus is only a specific case.



Perspectives

To study other systmes and their applications.