A NATURAL COUNTING OF LAMBDA TERMS

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Abstract. We study the sequences of numbers corresponding to lambda terms of given sizes, where the size is this of lambda terms with de Bruijn indices in a very natural model where all the operators have size 1. For plain lambda terms, the sequence corresponds to two families of binary trees for which we exhibit bijections. We study also the distribution of normal forms, head normal forms and strongly normalizing terms. In particular we show that strongly normalizing terms are of density 0 among plain terms.

Keywords: lambda calculus, combinatorics, functional programming, test, random generator, ranking, unranking

1. Introduction

In this paper we consider a natural way of counting the size of λ-terms, namely λ-terms presented by de Bruijn indices[1] in which all the operators are counted with size 1. This means that abstractions, applications, successors and zeros have all size 1. Formally

$$\begin{aligned} |\lambda M| &= |M| + 1 \\ |M_1\, M_2| &= |M_1| + |M_2| + 1 \\ |Sn| &= |n| + 1 \\ |0| &= 1. \end{aligned}$$

For instance the term for K which is written traditionally $\lambda x.\lambda y.x$ in the lambda calculus is written $\lambda\lambda S0$ using de Bruijn indices and we have:

$$|\lambda\lambda S0| = 4.$$

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[1]Readers not familiar with de Bruijn indices are invited to read Appendix A.

since there are two λ abstractions, one successor S and one $\mathtt{0}$. The term for S (which should not be confused with the successor symbol) is written $\lambda x.\lambda y.\lambda z.(xz)(yz)$ which is written $\lambda\lambda\lambda(((SS\mathtt{0})\mathtt{0})((S\mathtt{0})\mathtt{0}))$ using de Bruijn indices and its size is:

$$|\lambda\lambda\lambda(((SS\mathtt{0})\mathtt{0})((S\mathtt{0})\mathtt{0}))| = 13.$$

since there are three λ abstractions, three applications, three successors S's, and four $\mathtt{0}$'s. The term $\lambda x.xx$ which corresponds to the term $\lambda(\mathtt{0}\,\mathtt{0})$ has size 4 and the term $(\lambda x.xx)(\lambda x.xx)$ which corresponds to the term ω is written $(\lambda(\mathtt{0}\,\mathtt{0}))\,\lambda(\mathtt{0}\,\mathtt{0})$ and has size 9. The term $\lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$ which corresponds to the fixpoint Y is written $\lambda((\lambda((S\mathtt{0})\,(\mathtt{0}\,\mathtt{0})))\ \lambda((S\mathtt{0})\,(\mathtt{0}\,\mathtt{0})))$ and has size 16.

2. Lambda terms

2.1. Counting plain terms with a natural size: L_∞. Since the terms are either applications, abstractions or de Bruijn indices, the set $\mathcal{L}_\infty$ of lambda terms is solution of the functional equation:

$$\mathcal{L}_\infty = \mathcal{L}_\infty\,\mathcal{L}_\infty \ \oplus\ \lambda\mathcal{L}_\infty \ \oplus\ \mathcal{D}$$

where $\mathcal{D}$ is the set of de Bruijn indices which is solution of

$$\mathcal{D} = S\mathcal{D} \oplus \mathtt{0}$$

Let us call L_∞ the generating function for counting the number of the plain terms. It is solution of the functional equation:

$$L_\infty \quad = \quad zL_\infty^2 + zL_\infty + \frac{z}{1-z},$$

which yields the equation:

$$zL_\infty^2 - (1-z)L_\infty + \frac{z}{1-z} = 0 \tag{1}$$

which has discriminant

$$\begin{aligned} \Delta_{L_\infty} &= (1-z)^2 - 4\frac{z^2}{1-z} \\ &= \frac{(1-z)^3 - 4z^2}{1-z} \\ &= \frac{1-3z-z^2-z^3}{1-z} \end{aligned}$$

This gives the solution

$$\begin{aligned} L_\infty &= \frac{(1-z) - \sqrt{\Delta_{L_\infty}}}{2z} \\ &= \frac{(1-z)^{3/2} - \sqrt{1-3z-z^2-z^3}}{2z\sqrt{1-z}} \end{aligned}$$

which has $\rho_{L_\infty} \doteq 0.29559774252208393$ as pole closest to 0. Since $1/\rho_{L_\infty} \doteq 3.382975767906247$, the number of λ-terms of size n grows like $3.3829...^n$. See Theorem 1 for a better approximation.

The 18 first values of $[z^n]L_\infty$ are:

0, 1, 2, 4, 9, 22, 57, 154, 429, 1223, 3550, 10455, 31160, 93802, 284789, 871008, 2681019

This sequence is **A105633** in the *Online Encyclopedia of Integer Sequences.*

Theorem 1.

$$[z^{n+1}]K = [z^n]L_\infty \sim \left(\frac{1}{\rho_{L_\infty}}\right)^n \frac{C}{n^{\frac{3}{2}}}$$

with $C \doteq 0.60676...$ *and* $\rho_{L_\infty} \doteq 0.29559....$

Proof. The proof mimics this of Theorem 1 in [3]. Let us write L_∞ as

$$\begin{aligned} L_\infty &= \frac{(1-z) - \sqrt{\frac{1-3z-z^2-z^3}{1-z}}}{2z} \\ &= \frac{(1-z) - \sqrt{\rho_{L_\infty}(1-\frac{z}{\rho_{L_\infty}})\frac{Q(z)}{1-z}}}{2z} \end{aligned}$$

where

$$\begin{aligned} R(z) &= z^3 + z^2 + 3z - 1 \\ Q(z) &= \frac{R(z)}{\rho_{L_\infty} - z} \end{aligned}$$

Applying Theorem VI.1 of [2], we get:

$$[z^n]L_\infty \sim \left(\frac{1}{\rho_{L_\infty}}\right)^n \cdot \frac{n^{-3/2}}{\Gamma(-\frac{1}{2})}\,\widetilde{C}$$

with

$$\widetilde{C} = \frac{-\sqrt{\rho_{L_\infty}\frac{Q(\rho_{L_\infty})}{1-\rho_{L_\infty}}}}{2\rho_{L_\infty}}$$

Notice that $Q(\rho_{L_\infty}) = R'(\rho_{L_\infty}) = 3\rho_{L_\infty}^2 + 2\rho_{L_\infty} + 3$. From this we get

$$C = \frac{\widetilde{C}}{\Gamma(-\frac{1}{2})} \doteq 0.60676...$$

□

Figure 1 shows approximations of $[x^n]L_\infty$.

2.2. **Counting terms with at most m indices: L_m.**

The set $\mathcal{L}_m$ of terms with free indices $0, .., m-1$ is described as

$$\mathcal{L}_m = \mathcal{L}_m\mathcal{L}_m \oplus \lambda\mathcal{L}_{m+1} \bigoplus_{i=0}^{m-1} S^i(0).$$

The set $\mathcal{L}_0$ is the set of closed lambda terms. If we consider the λ-terms with at most m free indices, we get:

$$L_m = zL_m^2 - zL_{m+1} + \frac{z(1-z^m)}{1-z}$$

which yields:

$$zL_m^2 - L_m + z\left(L_{m+1} + \frac{1-z^m}{1-z}\right) \quad = \quad 0.$$

Let us state

$$\Delta_{L_m} = 1 - 4z^2\left(L_{m+1} + \frac{1-z^m}{1-z}\right)$$

$\mathbf{n}$	$[\mathbf{x^n}]\mathbf{L}_\infty$
10	3550
20	253106837
30	27328990723991
40	3503758934959966001
50	493839291745701673090756
60	73920774614279746859303111580
70	11535317831253359292868402823579507
80	1855899670106913269845444317474927546423
90	305649725186484753579669948042728038245882292
100	51274965000307280025396615989999357497440689837989

$\mathbf{n}$	$\lfloor(\mathbf{1}/\rho_{\mathbf{L}_\infty})^{\mathbf{n}}\mathbf{C}/\mathbf{n}^{\mathbf{3/2}}\rfloor$
10	3767
20	261489930
30	27945182509468
40	3563589864915927683
50	500623883981281516056181
60	74770204056757299054875868847
70	11649230835743409545961872906078995
80	1871967051054756263272240387385909197928
90	308005368563187477433148735955649926279818246
100	51631045600653143846184406311963448514677624135086

FIGURE 1. Approximation of $[x^n]L_\infty$.

we have

$$L_m = \frac{1-\sqrt{\Delta_{L_m}}}{2z}.$$

Notice that L_m is defined using L_{m+1}. The sequences $([z^n]L_m)_{n\in\mathbb{N}}$ do not occur in the *Online Encyclopedia of Integer Sequences*.

2.3. **Counting λ-terms with another notion of size.** Assume we take another notion of size in which $\underline{0}$ has size 0 and applications have size 2, whereas abstraction and succession keep their size 1. In other words:

$$\begin{array}{rcl} |\lambda M| & = & |M|+1 \\ |M_1\, M_2| & = & |M_1|+|M_2|+2 \\ |Sn| & = & |n|+1 \\ |\underline{0}| & = & 0. \end{array}$$

The generating function[2] A_1 fulfills the identity:

$$z^2A_1^2-(1-z)A_1+\frac{1}{1-z}.$$

The reader may check that

$$L_\infty = z\,A_1 \quad \text{and} \quad [z^n]A_1 = [z^{n+1}]L_\infty.$$

[2]We write this function A_1 as a reference to the function $A(x,1)$ described in sequence **A105632** of the *Online Encyclopedia of Integer Sequences*.

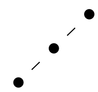
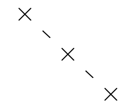
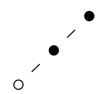
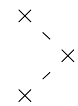
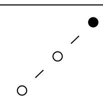
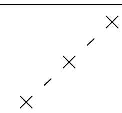
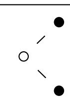
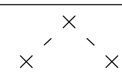
$\lambda-$**terms**	**black-white trees**	**zigzag free trees**	**neutral hnf**
$S^2\mathtt{0}$			$S^3\mathtt{0}$
$\lambda S\mathtt{0}$			$\mathtt{0}\,(S\mathtt{0})$
$\lambda\lambda\mathtt{0}$			$\mathtt{0}\,(\lambda\mathtt{0})$
$\mathtt{0}\,\mathtt{0}$			$(S\mathtt{0})\,\mathtt{0}$

FIGURE 2. Bijection between λ-terms, E_1-free black-white binary trees, zigzag-free trees of size 3 ($L_3 = 4$) and neutral head normal forms of size 4 ($K_4 = 4$).

Hence both notions of size correspond to sequence **A105633**. In Appendix B we consider the case where all the operators (application, abstraction and succession) have size 1 and $\mathtt{0}$ has size 0.

3. E-FREE BLACK-WHITE BINARY TREES

A black-white binary tree is a binary tree with colored nodes using two colors, *black* $\bullet$ and *white* $\circ$. *The root of a black-white binary tree is* $\bullet$, *by convention.* A E-free black-white binary tree is a black-white binary tree in which edges from a set E are forbidden. For instance if the set of forbidden edges is $E_1 = \{ {}_{\bullet}\diagup^{\circ}, {}^{\bullet}\diagdown_{\circ}, {}^{\bullet}\diagdown_{\bullet}, {}^{\circ}\diagdown_{\circ} \}$, this means that only edges in $A_1 = \{ {}_{\circ}\diagup^{\bullet}, {}_{\bullet}\diagup^{\bullet}, {}_{\circ}\diagup^{\circ}, {}^{\circ}\diagdown_{\bullet} \}$ are allowed. The E_1-free black-white binary trees of size 3 and 4 are as many as lambda terms of size 3 and 4. They are listed in Fig. 2 and Fig. 3 second column. For $E_1 = \{ {}_{\bullet}\diagup^{\circ}, {}^{\bullet}\diagdown_{\circ}, {}^{\bullet}\diagdown_{\bullet}, {}^{\circ}\diagdown_{\circ} \}$, like for $E_2 = \{ {}_{\circ}\diagup^{\bullet}, {}_{\bullet}\diagup^{\bullet}, {}_{\circ}\diagup^{\circ}, {}^{\circ}\diagdown_{\bullet} \}$, which is obtained by left-right symmetry, the E-free black-white binary trees are counted by A105633 [4]. In what follows we will consider E_1 and we will rather speak in terms of an allowed set of pattern namely A_1. For simplicity, we will call in this paper *black-white trees*, the binary black-white trees with allowed pattern set A_1.

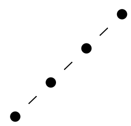
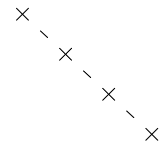
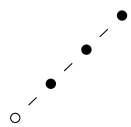
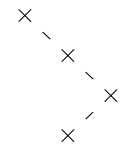
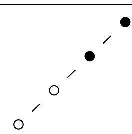
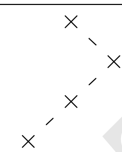
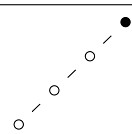
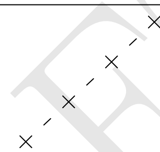
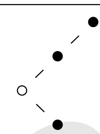
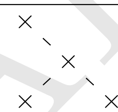
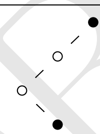
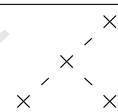
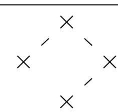
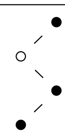
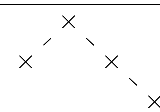
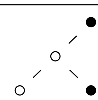
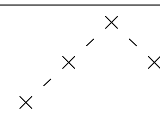
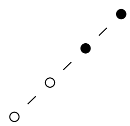
$\lambda -$ terms	black-white trees	zigzag free trees	neutral hnf
$S^3 0$			$S^4 0$
$\lambda S^2 0$			$0\,(S^2 0)$
$\lambda\lambda S 0$			$0\,(\lambda S 0)$
$\lambda\lambda\lambda 0$			$0\,(\lambda\lambda 0)$
$0\,(S 0)$			$(S 0)\,(S 0)$
$0\,(\lambda 0)$			$(S 0)\,(\lambda 0)$
$(\lambda 0)\,0$			$0\,0\,0$
$(S 0)\,0$			$(S^2 0)\,0$
$\lambda(0\,0)$			$0\,(0\,0)$

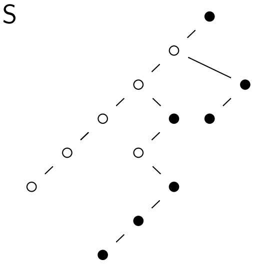
FIGURE 3. Bijection between λ-terms, E_1-free black-white binary trees and zigzag free trees of size 4 ($L_4 = 9$) and neutral head normal forms of size 5 ($K_5 = 9$).

Before giving the bijection, let us give the trees corresponding to $\mathsf{K} = \lambda\lambda S(\mathtt{0})$, to $\mathsf{S} = \lambda\lambda\lambda(SS\mathtt{0}\,\mathtt{0})\,(S\mathtt{0}\,\mathtt{0})$, to $\omega = (\lambda(\mathtt{0}\,\mathtt{0}))\,\lambda(\mathtt{0}\,\mathtt{0})$, and to $Y = \lambda(\lambda(S\,\mathtt{0}\,(\mathtt{0}\,\mathtt{0}))\,\lambda(S\,\mathtt{0}\,(\mathtt{0}\,\mathtt{0})))$:

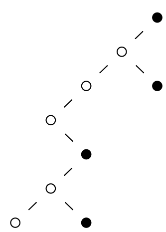
for K



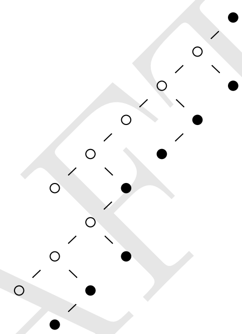
for S



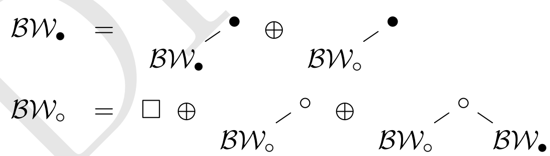
for ω



for Y



3.1. **Recursive description.** Assume $\square$ is the empty tree which is usually not represented in drawing. The E_1-free black-white binary trees are described by the following grammar:



which yields the following equations for the generating functions:

$$\begin{aligned} BW_\bullet &= zBW_\bullet + zBW_\circ \\ BW_\circ &= 1 + zBW_\circ + zBW_\circ\,BW_\bullet \end{aligned}$$

hence

$$BW_\circ \quad = \quad \frac{1-z}{z}\,BW_\bullet$$

and

$$z(1-z)BW_\bullet^2 + (1-z)^2 BW_\bullet + z = 0.$$

which is the same equation up to a multiplication by $1-z$ as (1) namely the equation defining L_∞

3.2. **The bijection.** Let us define the function LtoBw from λ-terms to black-white trees:

$$
\begin{array}{rcl}
\mathsf{LtoBw}(\theta) & = & \bullet \\
\mathsf{LtoBw}(S(n)) & = & \bullet \diagup^{\mathsf{LtoBw}(n)} \\
\mathsf{LtoBw}(\lambda M) & = & \circ \diagup^{\mathsf{LtoBw}(M)} \\
\mathsf{LtoBw}(M_1\, M_2) & = & \circ \begin{array}{l} \diagup^{\mathsf{LtoBw}(M_2)} \\ \diagdown_{\mathsf{LtoBw}(M_1)} \end{array}
\end{array}
$$

In other words a new node is added on the leftmost node of the tree. from black-white trees to λ-terms Let us now define the function BwtoL

$$
\begin{array}{rcl}
\mathsf{BwtoL}(\bullet) & = & \theta \\
\mathsf{BwtoL}\left(\bullet \diagup^{T} \right) & = & S(\mathsf{BwtoL}(T)) \\
\mathsf{BwtoL}\left(\circ \diagup^{T} \right) & = & \lambda\, \mathsf{BwtoL}(T) \\
\mathsf{BwtoL}\left(\circ \begin{array}{l} \diagup^{T_2} \\ \diagdown_{T_1} \end{array} \right) & = & \mathsf{BwtoL}(T_1)\, \mathsf{BwtoL}(T_2)
\end{array}
$$

In other words, to decompose a binary tree which is not the node $\bullet$, we look for the left most node.

- If the leftmost node is $\bullet$, then the λ-term is a de Bruijn index. Actually there are only $\bullet$'s (indeed $\bullet \diagup^{\circ}$ is forbidden) and the tree is linear. If this linear tree has n $\bullet$'s it represents $S^{n-1}(\theta)$.
- If the leftmost node is $\circ$ and has no child, then the λ-term is an abstraction of the bijection of the rest.
- If the leftmost node is $\circ$ and has a right child, then the λ-term is an application of the bijection of the right subtree on the bijection of the above tree .

Proposition 1. $\mathsf{LtoBw} \circ \mathsf{BwtoL} = id_\Lambda$ *and* $\mathsf{BwtoL} \circ \mathsf{LtoBw} = id_{\mathcal{BW}_\bullet}$

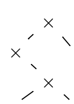
3.3. **The bijection in Haskell.** In this section we describe Haskell programs for the bijections. First we define black-white trees. We consider three kinds of trees: leafs (of arity zero and size zero) corresponding to $\Box$ and not represented in drawing.

```
  data LTerm = Zero | S LTerm | DBAbs LTerm | DBApp LTerm LTerm

  data BTree = Leaf | Black BTree BTree | White BTree BTree

-- insert a tree t at the leftmost  node
insertLeftmost :: BwTree -> BwTree -> BwTree
insertLeftmost t BwLeaf = t
insertLeftmost t (Black t1 t2) = Black (insertLeftmost t t1) t2
insertLeftmost t (White t1 t2) = White (insertLeftmost t t1) t2
```

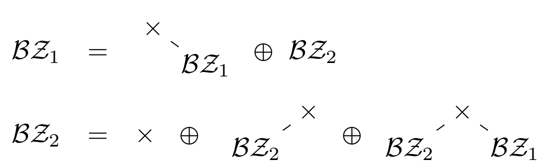
```
-- bijection from lambda terms to E-free black-white binary trees
lToBw :: LTerm -> BwTree
lToBw Zero = Black BwLeaf BwLeaf
lToBw (S n) = Black (lToBw n) BwLeaf
lToBw (DBAbs t) = insertLeftmost (White BwLeaf BwLeaf)  (lToBw t)
lToBw (DBApp t1 t2) =
    insertLeftmost (White BwLeaf (lToBw t1)) (lToBw t2)

-- bijection from E-free 2-binary trees to lambda terms,
-- True means "black", False means "white"
btToL :: BwTree -> LTerm
btToL (Black BwLeaf BwLeaf) = Zero
btToL t  = let (b,t2,t1) = removeLeftmost t
           in if b then let n = btToL t2
                        in (S n)
                   else case t1 of
                         BwLeaf -> DBAbs (btToL t2)
                         Black _ _ -> DBApp (btToL t1) (btToL t2)

-- Take a bw-tree and returns 1. a boolean (black or white?)
-- 2. the pruned tree 3. the tree pending on the leftmost
removeLeftmost :: BwTree -> (Bool,BwTree,BwTree)
removeLeftmost (Black BwLeaf BwLeaf) = (True, BwLeaf, BwLeaf)
removeLeftmost (White BwLeaf t) = (False, BwLeaf, t)
removeLeftmost (Black t1 BwLeaf) =
    let (b,t’,t3) = removeLeftmost t1 in (b,Black t’ BwLeaf,t3)
removeLeftmost (White t1 t2) =
     let (b,t’,t3) = removeLeftmost t1 in (b,White t’ t2,t3)
```


4. Binary trees without zigzags

4.1. **Non empty zigzag free binary trees.** Consider $\mathcal{BZ}_1$ the set of binary trees with no zigzag i.e., with no subtree like



$\mathcal{BZ}_1$ is described by



Like L_∞ and B, BZ_1 is solution of the equation:

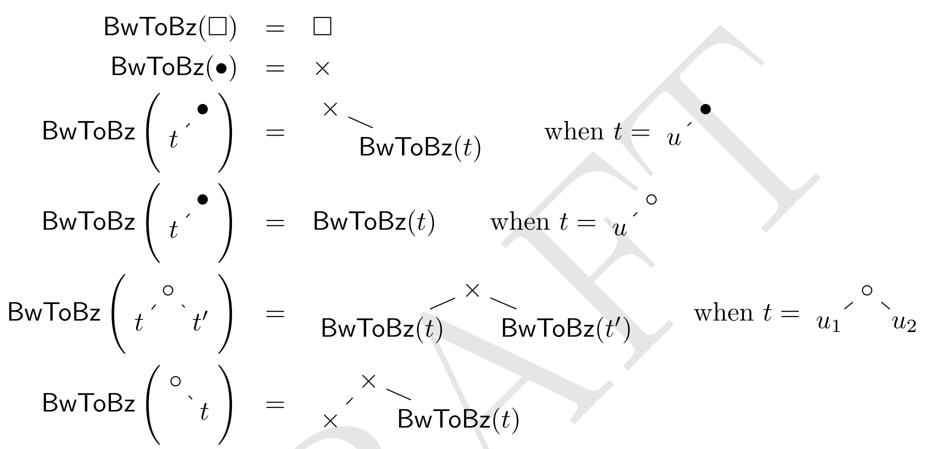
$$z(1-z)BZ_1^2 + (1-z)^2 BZ_1 + z = 0.$$

4.2. A formula. Sapounakis et al. [6] give the formula:

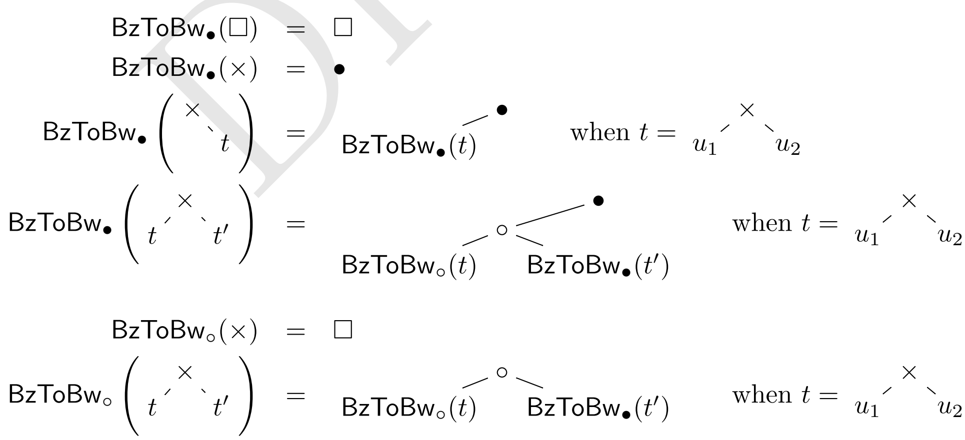
$$[z^n]BZ_1 = [z^n]L_\infty = \sum_{k=0}^{(n-1)\div 2} \frac{(-1)^k}{n-k}\binom{n-k}{k}\binom{2n-3k}{n-2k-1}$$

5. The bijections between black white trees and zigzag free trees

5.1. From black white trees to zigzag free trees. Let us call BwToBz the bijection from black white trees to zigzag free trees. Notice that the fourth equation removes a $\bullet$ and the last equation adds a $\times$.



5.2. From zigzag free trees to black white trees. We use two functions $\mathsf{BzToBw}_\bullet$ and $\mathsf{BzToBw}_\circ$. Notice also that one adds a $\bullet$ and that one removes a $\times$.



Proposition 2. $\mathsf{BzToBw}_\bullet \circ \mathsf{BwToBz} = id_{\mathcal{BW}_\bullet}$ *and* $\mathsf{BwToBz} \circ \mathsf{BzToBw}_\bullet = id_{\mathcal{BZ}}$.

5.3. Haskell code.

```haskell
-- Bijection from zigzag frees to black-white
----- to black-white black rooted
bzToBw_black  :: BzTree -> BwTree
```

```
bzToBw_black  BzLeaf = BwLeaf
bzToBw_black (Node BzLeaf BzLeaf) = Black BwLeaf BwLeaf
bzToBw_black (Node BzLeaf t@(Node _ _)) = Black (bzToBw_black t) BwLeaf
bzToBw_black (Node tLeft@(Node _ _) tRight) =
    Black (White (bzToBw_white tLeft) (bzToBw_black tRight)) BwLeaf -- one adds a black

----- to black-white white rooted
bzToBw_white :: BzTree -> BwTree
bzToBw_white (Node tLeft@(Node _ _) tRight) =
    White (bzToBw_white tLeft) (bzToBw_black tRight)
bzToBw_white (Node BzLeaf BzLeaf) = BwLeaf -- one removes a node
```

6. The bijections between lambda terms and zigzag free trees

6.1. From lambda terms to zigzag free trees. Lest us call LToBz this bijection. It is described in Figure 4

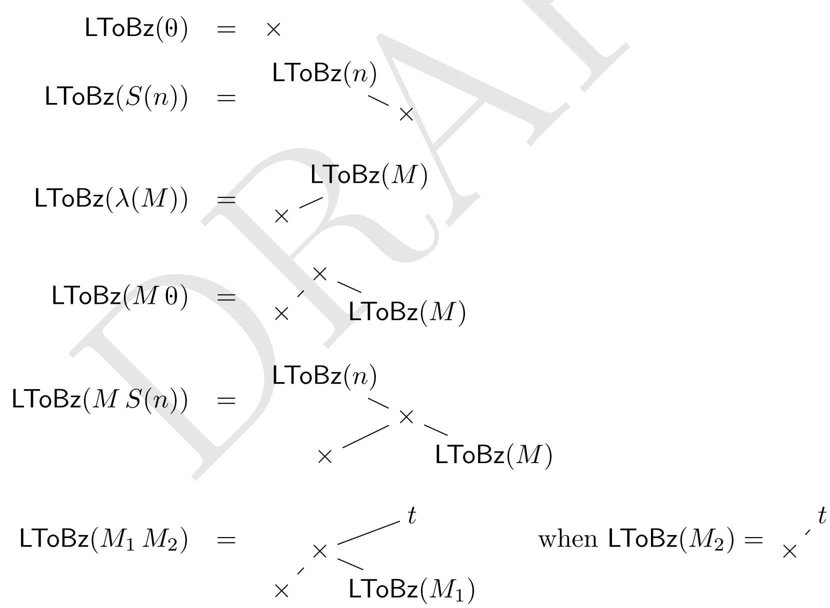


FIGURE 4. The bijection LToBz from lambda terms to zigzag free trees

6.2. From zigzag free terms to lambda terms. The bijection called BzToL is defined in Figure 5.

Proposition 3. $\mathsf{LToBz} \circ \mathsf{BzToL} = id_{\mathcal{BZ}}$ *and* $\mathsf{BzToL} \circ \mathsf{LToBz} = id_{\Lambda}$.

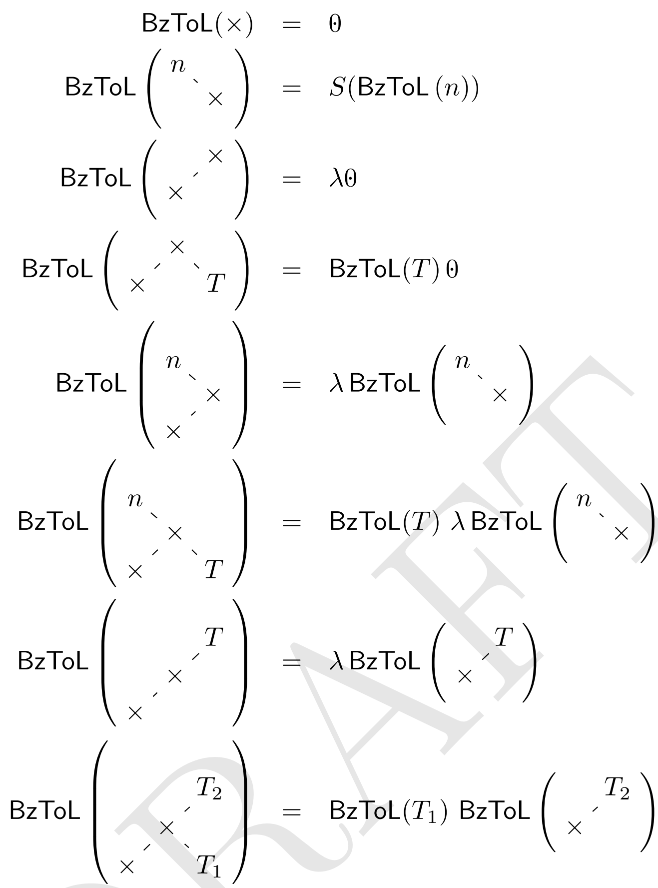
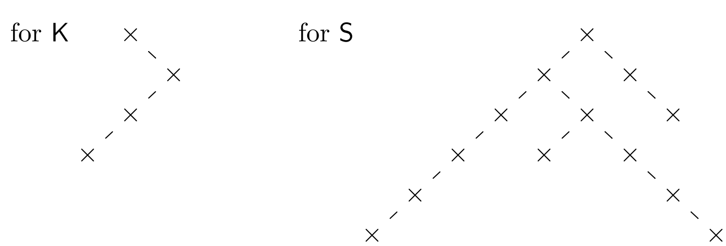
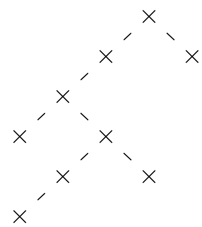


FIGURE 5. The bijection BzToL

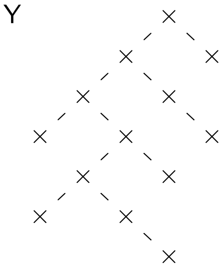
6.3. **Examples.** Let us look at the bijection on classical examples, namely K, S, ω and Y:



for ω



for Y



7. Normal forms

We are now interested in normal forms, that are terms irreducible by β reduction that are also terms which do not have subterms of the form $(\lambda \mathsf{M})\,\mathsf{N}$.

There are three associated classes: $\mathcal{N}$ (the normal forms), $\mathcal{M}$ (the neutral terms, which are the normal forms without head abstractions) and $\mathcal{D}$ (the de Bruijn indices) :

$$\begin{array}{rcl} \mathcal{N} & = & \mathcal{M} + \lambda\mathcal{N} \\ \mathcal{M} & = & \mathcal{M}\mathcal{N} + \mathcal{D} \\ \mathcal{D} & = & S\mathcal{D} + \mathtt{0}. \end{array}$$

Let us call N the generating function of $\mathcal{N}$, M the generating function for $\mathcal{M}$ and D the generating function for $\mathcal{D}$. The above equations yield the equations for the generating functions:

$$\begin{array}{rcl} N & = & M + zN \\ M & = & zMN + D \\ D & = & zD + z \end{array}$$

Clearly

$$\begin{array}{rcl} D & = & \dfrac{z}{1-z} \\ M & = & \dfrac{D}{1-zN} \\ N & = & \dfrac{D}{1-zN} + zN \end{array}$$

from which one gets

$$\begin{array}{rcl} z(1-z)N^2 - (1-z)N + D & = & 0 \\ z(1-z)^2N^2 - (1-z)^2N + z & = & 0 \end{array}$$

with discriminant

$$\begin{array}{rcl} \Delta_N & = & (1-z)^4 - 4z^2(1-z)^2 \\ & = & (1-z)^2(1-2z-3z^2). \end{array}$$

Hence the pole of Δ_N are 1, -1 and $1/3$ and the smallest pole $\rho_n = 1/3$ and normal forms grow like 3^n. Therefore the set of normal forms is of density zero in the set of terms.

8. Head normal forms

We are now interested in the set of head normal forms

$$\begin{aligned} \mathcal{H} &= \mathcal{K} + \lambda\mathcal{H} \\ \mathcal{K} &= \mathcal{K}\mathcal{L}_\infty + \mathcal{D} \end{aligned}$$

which yields the equations

$$\begin{aligned} H &= K + zH \\ K &= zKL_\infty + D \end{aligned}$$

and

$$\begin{aligned} K &= \frac{D}{1 - zL_\infty} \\ H &= \frac{K}{1 - z} \end{aligned}$$

From which we draw

$$K = z + zL_\infty.$$

This can be explained by the following bijection (see Figure 2 and Figure 3):

Proposition 4. *If P is a neutral head normal form, it is of the form:*

- $P = 0\, N_1 N_2 \ldots N_p$ *with* $p \geq 1$ *(of size* $k+1$*) then it is in bijection with* $(\lambda\, N_1)N_2 \ldots N_p$ *(of size* k*),*
- $P = (Sn)\, N_1 \ldots N_p$ *(of size* $k+1$*) then it is in bijection with* $n\, N_1 \ldots N_p$ *(of size* k*),*
- $P = 0$ *(of size* 1*), treated by the case* z*.*

From Theorem 1 we get:

Proposition 5.

$$[z^{n+1}]K \sim \left(\frac{1}{\rho_{L_\infty}}\right)^n \frac{C}{n^{\frac{3}{2}}}$$

with $C \doteq 0.60676...$ *and* $\rho_{L_\infty} \doteq 0.29559....$

The density of a set $\mathcal{A}$ in a set $\mathcal{B}$ is

$$\lim_{n\to\infty} \frac{A_n}{B_n}$$

where A_n (respectively B_n) are the numbers of elements of $\mathcal{A}$ (respectively of $\mathcal{B}$) of size n. For instance the density of $\mathcal{K}$ in $\mathcal{L}_\infty$ is

$$\lim_{n\to\infty} \frac{[z^n]K}{[z^n]L_\infty};$$

Hence the proposition.

Proposition 6. *The density of* $\mathcal{K}$ *in* $\mathcal{L}_\infty$ *(i.e., the density of neutral head normal forms among plain terms) is* ρ_{L_∞}.

Proposition 7.

$$[z^n]H \sim \left(\frac{1}{\rho_{L_\infty}}\right)^n \frac{C_H}{n^{\frac{3}{2}}}$$

with $C_H \doteq 0.254625911836762946...$

Proof. The proof is like this of Theorem 1 with

$$C_H = \frac{-\sqrt{\rho_{L_\infty} \frac{Q(\rho_{L_\infty})}{1-\rho_{L_\infty}}}}{2(1-\rho_{L_\infty})\Gamma(-\frac{1}{2})} \doteq 0.254625911836762946...$$

□

Figure 6 compares the coefficients of H with its approximation.

$\mathbf{n}$	$[\mathbf{x^n}]\mathbf{H}$
10	1902
20	118768916
30	12338289374047
40	1552505356757052270
50	216408050593408223194666
60	32156818736630052190010494575
70	4992016749940033843389032870415375
80	800041142163881275363093897487465240590
90	131362728872240507612558556757894820073668254
100	21984069003048322712483528437236630547685953755064
$\mathbf{n}$	$\lfloor (\mathbf{1}/\rho_{\mathbf{L}_\infty})^{\mathbf{n}}\mathbf{C_H}/\mathbf{n^{3/2}} \rfloor$
10	1581
20	109732518
30	11727010776119
40	1495436887319673848
50	210083497584679365571791
60	31376820974748144171493861802
70	4888522574435898663355075650509052
80	785558576073780985739070920824898277393
90	129252413184969184232722751628403772087829182
100	21666626365243195881127917362969390314273901016408)

Figure 6. Approximation of $[x^n]H$.

Proposition 8. *The density of $\mathcal{H}$ in $\mathcal{L}_\infty$ (i.e., the density of head normal forms among plain terms) is $\rho_{L_\infty}/(1-\rho_{L_\infty}) \doteq 0.41964337760707887...$*

Proof. Actually $\frac{C_H}{C} = \frac{\rho_{L_\infty}}{(1-\rho_{L_\infty})}$. □

9. Terms containing specific subterms

Consider a term M of size p and the set $\mathcal{T}$ of terms that contain M as subterm.

$$\mathcal{T} \quad = \quad t + \lambda\mathcal{T} + \mathcal{T}\mathcal{L}_\infty + \mathcal{L}_\infty\mathcal{T} - \mathcal{T}\mathcal{T}$$

which yields

$$T = z^p + zT + 2zTL_\infty - zT^2$$

and

$$zT^2 + (1 - 2zL_\infty - z)T - z^p \quad = \quad 0.$$

Notice that

$$1 - 2zL_\infty - z \quad = \quad \sqrt{\Delta_{L_\infty}}$$

Then the discriminant is

$$\begin{aligned} \Delta_T &= \Delta_{L_\infty} + 4z^{p+1} \\ (1-z)\Delta_T &= (1-z)\Delta_{L_\infty} + 4z^{p+1}(1-z). \end{aligned}$$

In the interval $(0,1)$, Δ_∞ is decreasing (its derivative is negative) and $(1-z)\Delta_T > (1-z)\Delta_{L_\infty}$. Hence the root ρ_T of Δ_T is larger than the root ρ_{L_∞} of Δ_∞, that is $\rho_T > \rho_{L_\infty}$. Beside:

$$T = \frac{\sqrt{\Delta_T} - \sqrt{\Delta_{L_\infty}}}{2z}.$$

Hence the number of terms that do not have M as subterm is given by

$$L_\infty - T = \frac{(1-z) - \sqrt{\Delta_T}}{2z}.$$

Theorem 2. *The density of terms that do not have* M *as subterm is* 0*.*

Proof. Indeed the smallest pole of $L_\infty - T$ is ρ_T and the smallest pole of L_∞ is ρ_{L_∞}. Therefore,

$$\begin{aligned} [z^n](L_\infty - T) &\bowtie \left(\frac{1}{\rho_T}\right)^n \\ [z^n]L_\infty &\bowtie \left(\frac{1}{\rho_{L_\infty}}\right)^n \end{aligned}$$

Hence, since $\rho_T > \rho_{L_\infty}$

$$\lim_{n\to\infty} \frac{[z^n](L_\infty - T)}{[z^n]L_\infty} = \left(\frac{\rho_{L_\infty}}{\rho_T}\right)^n = 0.$$

□

For instance if $|t| = 9$, that is for instance if $t = \omega = (\lambda(\mathbf{0}\,\mathbf{0}))\,\lambda(\mathbf{0}\,\mathbf{0})$, then

$$\rho_T \doteq 0.2956014673597697$$

and

$$\frac{\rho_{L_\infty}}{\rho_T} \doteq 0.9999873991231537.$$

Corollary 1. *The density of terms that contain* M *as subterm is* 1*.*

Corollary 2. *Asymptotically almost no* λ*-term is strongly normalizing.*

Proof. In other words, *the density of strongly normalizing terms is* 0*.* Indeed, the density of terms that contain $(\lambda(\mathbf{0}\,\mathbf{0}))\,\lambda(\mathbf{0}\,\mathbf{0})$ is 1. Hence the density of non strongly normalizing terms is 1. Hence the density of strongly normalizing terms is 0. □

10. Conclusion

Figure 7 summarizes what we obtained on densities of terms.

Moreover, this research opens many issues, among others about generating random terms and random normal forms using Boltzmann samplers [5].

nf / sn	nhdnf	hdnf	terms with M / $\overline{\mathsf{sn}}$
0	0.295...	0.419...	1

nf = normal forms
nhdnf = neutral head normal forms hdnf = head normal forms
terms with M = terms containing subterm M
sn = strongly normalizing terms $\overline{\mathsf{sn}}$ = non strongly normalizing terms

FIGURE 7. Summary of densities

References

[1] Nicolas Bourbaki. *Theory of Sets*. Elements of Mathematics. Springer-Verlag, 2004.

[2] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics*. Cambridge University Press, 2008.

[3] Katarzyna Grygiel and Pierre Lescanne. Counting terms in the binary lambda calculus. *CoRR*, abs/1401.0379, 2014. Published in the Proceedings of *25th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms* 2014, `https://hal.inria.fr/hal-01077251`.

[4] Nancy S. S. Gu, Nelson Y. Li, and Toufik Mansour. 2-binary trees: Bijections and related issues. *Discrete Mathematics*, 308(7):1209–1221, 2008.

[5] Pierre Lescanne. Boltzmann samplers for random generation of lambda terms. Technical report, ENS de Lyon, 2014. `http://hal-ens-lyon.archives-ouvertes.fr/docs/00/97/90/74/PDF/boltzmann.pdf`.

[6] Aristidis Sapounakis, Ioannis Tasoulas, and Panagiotis Tsikouras. Ordered trees and the in-order traversal. *Discrete Mathematics*, 306(15):1732–1741, 2006.

Appendix A. De Bruijn notations

De Bruijn indices are a system of notations for bound variables due to Nikolaas de Bruijn and somewhat connected to those proposed by Bourbaki [1]. The goal is to replace bound variables by placeholders and to link each bound variable to its binder. For instance (see Figure 8) Bourbaki ([1] p. 20) proposes to represent placeholders by boxes $\square$ and to represent binds by drawn lines. This requires a two dimensional notation. For example, he considers the formula:

$$(\tau x)\, \neg(x \in A') \vee x \in A''$$

Notice that we use an infix notation whereas he uses a prefix notation which gives $\tau \vee \neg \in xA' \in xA$. The formula contains the binder τ (a binder that Bourbaki introduces) and two occurrences of the bound variable x, this involves two $\square$'s and two drawn lines from τ, namely to the first $\square$ and to the second $\square$. De Bruijn proposes to represent the placeholders (in other words the variables) by natural numbers which represent the length of the link, that is the number of binders crossed when reaching the actual binder of the variables. In our proposal, we write natural numbers using the functions *zero* $\mathtt{0}$ and *successor* S. For instance, 3 is written $SS S\mathtt{0}$. With de Bruijn notations, Bourbaki's formula is written:

$$\tau\, (\neg \mathtt{0} \in A') \vee \mathtt{0} \in A''$$

and the lambda terms $\lambda x.\lambda y.\lambda z.(xz)(yz)$ is written $\lambda\lambda\lambda(((SS\mathtt{0})\mathtt{0})((S\mathtt{0})\mathtt{0}))$ which would correspond to the drawing of Figure 9 in Bourbaki style.

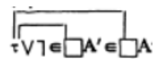


FIGURE 8. Bourbaki's notations for formula $\tau \vee \neg \in xA' \in xA$.

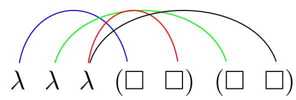


FIGURE 9. S in Bourbaki style

APPENDIX B. ANOTHER NATURAL COUNTING OF LAMBDA TERMS

Another natural counting is a counting where:

$$\begin{array}{rcl} |\lambda M| & = & |M| + 1 \\ |M_1\, M_2| & = & |M_1| + |M_2| + 1 \\ |Sn| & = & |n| + 1 \\ |\theta| & = & 0. \end{array}$$

The generating function is solution of

$$zM_\infty^2 - (1-z)M_\infty + \frac{1}{1-z} = 0$$

with discriminant

$$\begin{array}{rcl} \Delta_{M_\infty} & = & (1-z)^2 - 4\dfrac{z}{1-z} \\ & = & \dfrac{(1-z)^3 - 4z}{1-z} \\ & = & \dfrac{1 - 7z + 3z^2 - z^3}{1-z} \end{array}$$

and with root closest to 0: $\rho_{M_\infty} \doteq 0.152292401860433$ and $1/\rho_{M_\infty} = 6.5663157700831193$. The first values are:

1, 3, 10, 40, 181, 884, 4539, 24142, 131821, 734577, 4160626

The sequence grows significantly faster than **A105633** and is unknown in the *On-line Encyclopedia of Integer Sequences*.

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