Un calcul pour interpréter les séquents classiques

A calculus for interpreting the classical sequents

Pierre Lescanne, LIP, ENS de Lyon

La mathématique c’est l’art de donner le même nom à des choses différentes.

Henri Poincaré

This talk

I will try to show you

- that logical propositions and types of programs. are the same
- and that therefore should deserve the same name

and then I will tell you about the Curry Howard correspondence.

This talk

I will speak about

- A model of deduction,
- A model of computation,
- The link between those models

This talk

I will speak about

- A model of deduction, the essence of logic,
- A model of computation, the essence of computer science,
- The link between those models a deep connection between logic and computer science.

The model of deduction

The model of deduction is called sequent calculus.

Its main features are judgments, that are entities of the form $\Gamma \vdash \Delta$

where Γ and Δ are two sets of propositions.

The model of deduction

Γ should be understood as a conjunction of propositions.

Δ should be understood as a disjunction of propositions.

$\Gamma \vdash \Delta$ should be read from Γ on can deduce Δ.

For instance, $p_1, p_2, p_3 \vdash q_1, q_2$ should be read as

from p_1 and p_2 and p_3 one deduces q_1 or q_2

The model of deduction

The model of deduction is called the sequent calculus.

with only introduction rules, *i.e.*, $\dfrac{\Gamma, A \vdash \Delta \quad \ldots \quad \ldots}{\Gamma \vdash A \circ A', \Delta}$

The model of computation

It is a simple model for exception handling based on pairs of caller-callee.

Such a pair will be called a capsule.

The model of deduction:
the sequent calculus

What is a sequent ?

A sequent is a judgment of the form $\Gamma \vdash \Delta$

where Γ and Δ are sets of propositions.

$\Gamma \vdash \Delta$ should be read as

from the conjunction of propositions Γ

one can deduce the disjunction of propositions in Δ.

What is a sequent ?

A **sequent** is a judgment of the form $\Gamma \vdash \Delta$

where Γ and Δ are sets of propositions.

$\Gamma \vdash \Delta$ should be read as

from the conjunction of propositions Γ

one can deduce the disjunction of propositions in Δ*.*

If one can prove a sequent of the form $\vdash A$ one has a proof of the proposition A.

Classical logic

In classical logic one can prove sequents of the form

- $\vdash A, A \to B$,
- $\vdash A \to B, B \to A$,
- $\vdash ((A \to \bot) \to \bot) \to A$, i.e., $\vdash \neg\neg A \to A$

or the typical Peirce law: $\vdash ((A \to B) \to A) \to A$.

which contains only implications.

The model of computation : the sequent calculus

One needs rules to build proofs.

$$\frac{antecedent_1 \quad \ldots \quad antecedent_2}{consequent}\,(foo)$$

A proof of $\Gamma \vdash \Delta$ is a tree built with rules whose root is $\Gamma \vdash \Delta$.

$$\dfrac{\dfrac{\dfrac{\cdots \qquad \cdots}{\cdots}\,(foo_1) \qquad \dfrac{\cdots}{\cdots}\,(bar)}{\cdots}\,(foo_2) \qquad \cdots}{\Gamma \vdash \Delta}\,(foo_1)$$

The implicative sequent calculus

Propositions are made only

- of propositional variables
- and of the implication operator.

The implicative sequent calculus (the rules)

$$\frac{}{\Gamma, A \vdash \Delta, A}\,(ax)$$

The implicative sequent calculus (the rules)

$$\frac{}{\Gamma, A \vdash \Delta, A}\,(ax)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\,(\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\,(\to R)$$

The implicative sequent calculus (the rules)

$$\frac{}{\Gamma, A \vdash \Delta, A}\,(ax)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\,(\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\,(\to R)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\,(cut)$$

A proof of the Peirce law

$$\vdash ((A \to B) \to A) \to A$$

A proof of the Peirce law

$$\dfrac{(A \to B) \to A \vdash A}{\vdash ((A \to B) \to A) \to A}\;(\to R)$$

A proof of the Peirce law

$$\dfrac{\dfrac{\vdash A \to B, A \qquad A \vdash A}{(A \to B) \to A \vdash A}\,(\to L)}{\vdash ((A \to B) \to A) \to A}\,(\to R)$$

A proof of the Peirce law

Easy

$$\dfrac{\dfrac{\vdash A \to B, A \qquad A \vdash A}{(A \to B) \to A \vdash A}\,(\to L)}{\vdash ((A \to B) \to A) \to A}\,(\to R)$$

A proof of the Peirce law

$$\dfrac{\dfrac{\vdash A \to B, A \qquad \boxed{\dfrac{}{A \vdash A}\,(ax)}}{(A \to B) \to A \vdash A}\,(\to L)}{\vdash ((A \to B) \to A) \to A}\,(\to R)$$

A proof of the Peirce law

$$\dfrac{\dfrac{\boxed{\dfrac{\dfrac{}{A \vdash B, A}\,(ax)}{\vdash A \to B, A}\,(\to R)} \qquad \dfrac{}{A \vdash A}\,(ax)}{(A \to B) \to A \vdash A}\,(\to L)}{\vdash ((A \to B) \to A) \to A}\,(\to R)$$

A proof of the Peirce law

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash B, A}\,(ax)}{\vdash A \to B, A}\,(\to R) \qquad \dfrac{}{A \vdash A}\,(ax)}{(A \to B) \to A \vdash A}\,(\to L)}{\vdash ((A \to B) \to A) \to A}\,(\to R)$$

The active formula

The active formula is the formula on the lower part of a rule

which is “split” by the rule.

For instance in

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\,(\to L)$$

the active formula is $A \to B$.

The active formula

It make sense to track the active formulae and to suppose that A and B become the new active formulae:

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta} (\rightarrow L)$$

Similarly

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta} (\rightarrow R)$$

We have to prove B using the proposition A and to split B if necessary.

Active formula

But our proof of the Peirce law does not fulfill this statement on active formulae.

$$\dfrac{\dfrac{\dfrac{}{A \vdash B, A}\,(ax)}{\vdash A \to B, A}\,(\to R) \qquad \dfrac{}{A \vdash A}\,(ax)}{\dfrac{(A \to B) \to A \vdash A}{\vdash ((A \to B) \to A) \to A}\,(\to R)}\,(\to L)$$

Active formula

But our proof of the Peirce law does not fulfill this statement on active formulae.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash B, A}\,(ax)}{\vdash A \to B, A}\,(\to R) \qquad \dfrac{}{A \vdash A}\,(ax)}{(A \to B) \to A \vdash A}\,(\to L)}{\vdash ((A \to B) \to A) \to A}\,(\to R)$$

The rules of the implicative sequent calculus with active formulae

$$\frac{}{\Gamma, A \vdash \Delta, A}(L - ax) \qquad \frac{}{\Gamma, A \vdash \Delta, A}(R - ax)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}(\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}(\to R)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}(cut)$$

The rules of the implicative sequent calculus with active formulae

Four comments:

- One needs to introduce two axioms according to the side of the active formula.
- In (cut) the new introduced proposition becomes the active formula.
- The lower sequent of (cut) has no active formula.
- One needs to introduce a new rule that activates a formula and enables a (cut) above that rule.

The rules of the implicative sequent calculus with active formulae

$$\frac{}{\Gamma, A \vdash \Delta, A}\,(L - ax) \qquad \frac{}{\Gamma, A \vdash \Delta, A}\,(R - ax)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\,(\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\,(\to R)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\,(cut) \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A, \Delta}\,(\mu)$$

A new proof of the Peirce law

$$\dfrac{\dfrac{(A \to B) \to A \vdash (A \to B) \to A, A \qquad \dfrac{\mathcal{A}_1 \qquad \mathcal{A}_2}{(A \to B) \to A, (A \to B) \to A \vdash A}}{\dfrac{(A \to B) \to A \vdash A}{(A \to B) \to A \vdash A}\,(\mu)}\,(cut)}{\vdash ((A \to B) \to A) \to A}\,(\to R)$$

where

$$\dfrac{\mathcal{A}_1 \qquad\qquad \mathcal{A}_2}{(A \to B) \to A, (A \to B) \to A \vdash A}$$

$$=$$

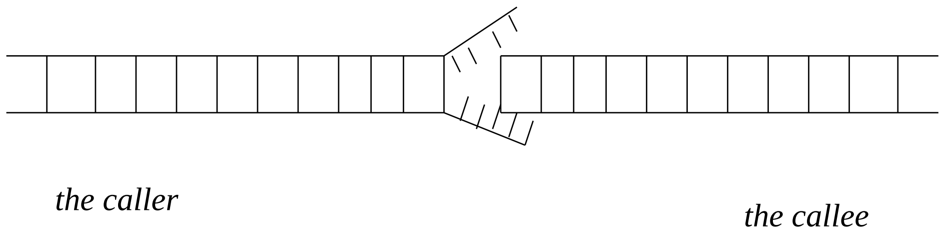
$$\dfrac{\dfrac{\dfrac{\dfrac{(A \to B) \to A, A \vdash A, B, A \qquad (A \to B) \to A, A, A \vdash A, B}{(A \to B) \to A, A \vdash B, A}\,(cut)}{(A \to B) \to A, A \vdash B, A}\,(\mu)}{(A \to B) \to A \vdash A \to B, A}\,(\to R) \qquad (A \to B) \to A, A \vdash A}{(A \to B) \to A, (A \to B) \to A \vdash A}\,(\to L)$$

The model of computation:

Herbelin’s calculus

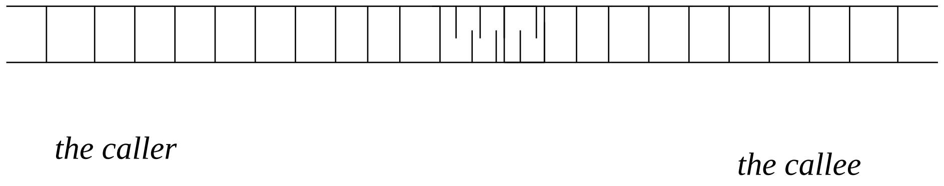
A model of computation

The model of computation relies on capsules



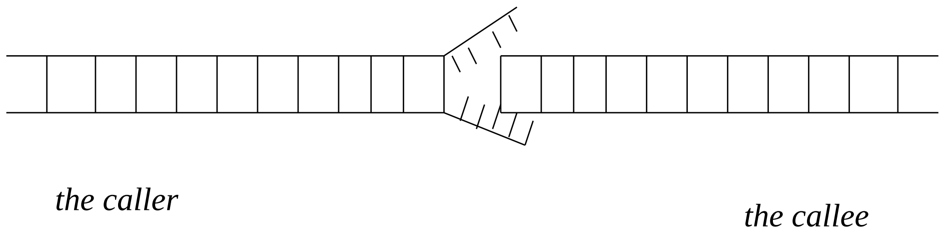
A model of computation

The model of computation relies on capsules



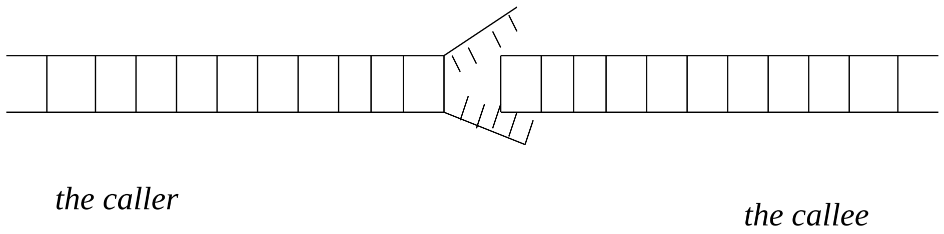
A model of computation

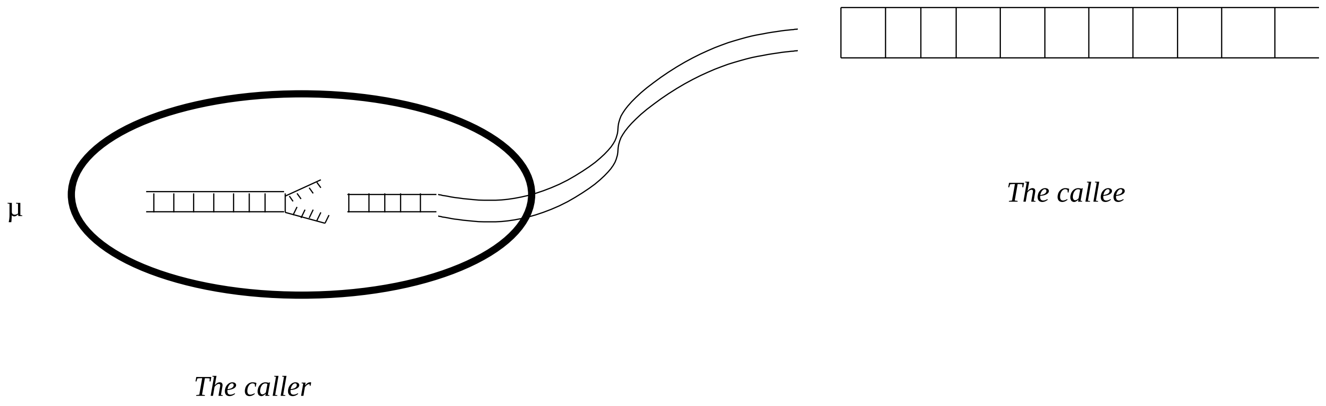
The model of computation relies on capsules



A model of computation

The model of computation relies on capsules





A model of computation

The model of computation relies on capsules $\langle r \parallel e \rangle$ that contain two constituents:

- a caller r
- and a callee e .

with the syntax

$$\begin{array}{rcl} c & ::= & \langle r \parallel e \rangle \\ r & ::= & x \mid \lambda x.r \mid \mu\alpha.c \\ e & ::= & \alpha \mid r \bullet e \end{array}$$

Callers

A caller is

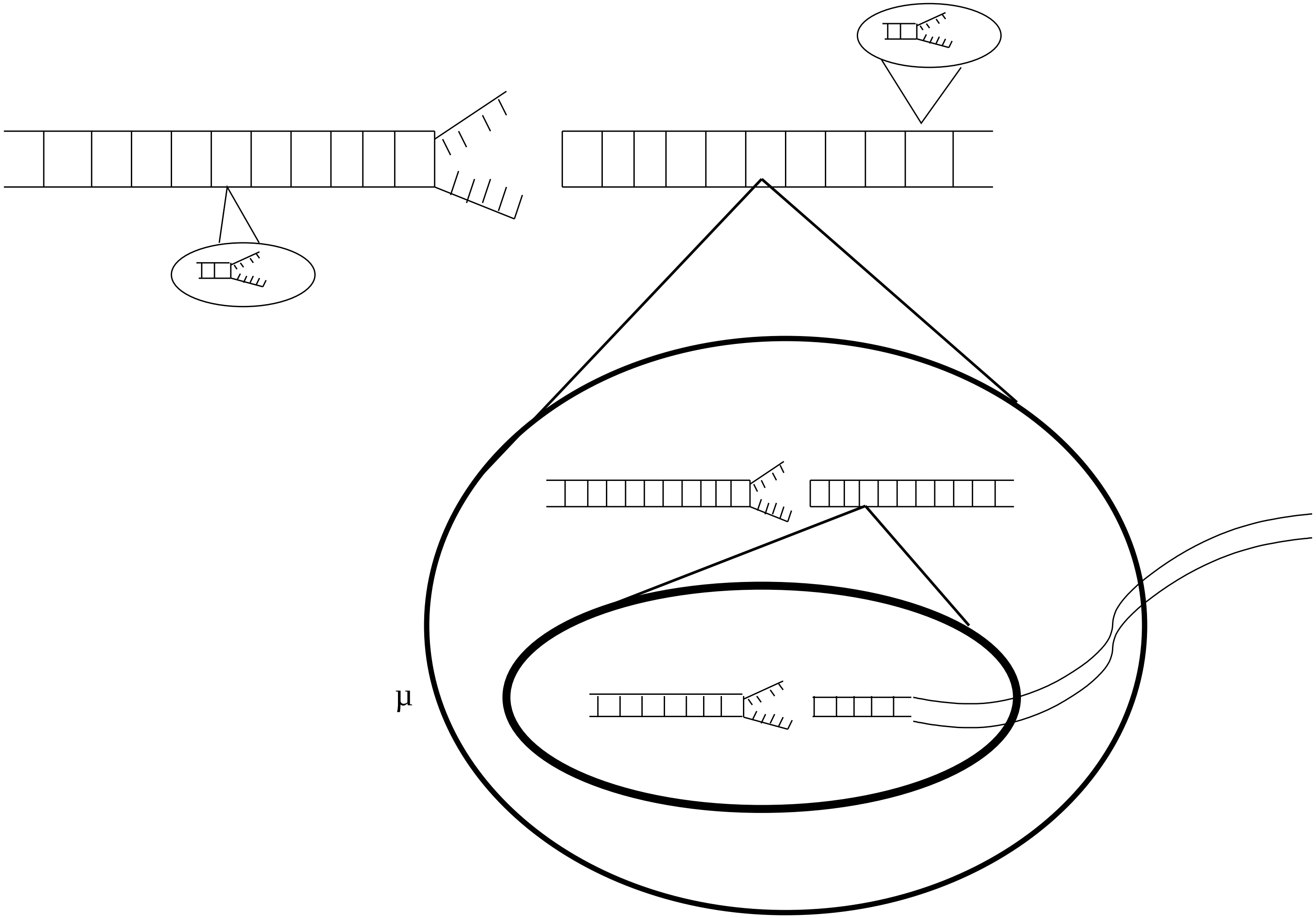
- either a variable x,
- or a λ-abstraction $\lambda x.r$ which expects a value to take the place of x in r,
- or a μ-abstraction $\mu\alpha.c$ which expects a callee to take place of α in c producing a new capsule.

Note: values and callers are the same.

Callees

A callee is basically a list of values, more precisely it is

- either a variable α,
- or a pair $r \bullet e$ of a value (caller) r and a callee e,
- or an $\widetilde{\mu}$-abstration $\widetilde{\mu}x.c$


μ

The reductions

$$\begin{array}{lrcl}
(\lambda) & \langle \lambda x \cdot r \parallel r' \bullet e \rangle & \longrightarrow & \langle r[x \leftarrow r'] \parallel e \rangle \\
(\mu) & \langle \mu\alpha \cdot c \parallel e \rangle & \longrightarrow & c[\alpha \leftarrow e] \\
(\widetilde{\mu}) & \langle r \parallel \widetilde{\mu} x \cdot c \rangle & \longrightarrow & c[x \leftarrow r]
\end{array}$$

The reductions

$$\begin{array}{llcl}
(\lambda) & \langle \lambda x \cdot r \parallel r' \bullet e \rangle & \longrightarrow & \langle r[x \leftarrow r'] \parallel e \rangle \\
(\mu) & \langle \mu\alpha \cdot c \parallel e \rangle & \longrightarrow & c[\alpha \leftarrow e] \\
(\widetilde{\mu}) & \langle r \parallel \widetilde{\mu}x \cdot c \rangle & \longrightarrow & c[x \leftarrow r]
\end{array}$$

Can we type capsules, callers and callees?

- to prove that nothing wrong can happen, i.e., capsules reduces always to capsules,
- to prove termination, *i.e.*, a typed capsule always reduces to a normal form whatever strategy we adopt.

The link between the sequent calculus and Herbelin’s calculus

The type judgments

Thanks to colors, I will considers three types of judgments

They can be seen as annotations of sequent calculus judgments;

Judgments for capsules

$$c : x_1 : A_1, \ldots, x_p : A_p \vdash \alpha_1 : B_1, \ldots, \alpha_q : B_q$$

or in short $c : \Gamma \vdash \Delta$.

Judgments for capsules

In $c : x_1 : A_1, \ldots, x_p : A_p \vdash \alpha_1 : B_1, \ldots, \alpha_q : B_q$,

one says that

- c takes the x_i as arguments with type A_i
- c returns α_j as results with type B_j.

Judgments for callers

$$x_1 : A_1, \ldots, x_p : A_p \vdash r : A, \alpha_1 : B_1, \ldots, \alpha_q : B_q$$

or in short $\Gamma \vdash r : A, \Delta$.

Judgments for callees

$$x_1 : A_1, \ldots, x_p : A_p, e : A \vdash \alpha_1 : B_1, \ldots, \alpha_q : B_q$$

or in short $\Gamma, e : A \vdash \Delta$.

The type system

$$\frac{}{\Gamma, \alpha : A \vdash \alpha : A, \Delta}\,(L-ax) \qquad \frac{}{\Gamma, x : A \vdash x : A, \Delta}\,(R-ax)$$

$$\frac{\Gamma \vdash r : A, \Delta \qquad \Gamma, e : B \vdash \Delta}{\Gamma, r \bullet e : A \to B \vdash \Delta}\,(\to L) \qquad \frac{\Gamma, x : A \vdash r : B, \Delta}{\Gamma \vdash \lambda x.r : A \to B, \Delta}\,(\to R)$$

$$\frac{\Gamma \vdash r : A, \Delta \qquad \Gamma, e : A \vdash \Delta}{\langle r \parallel e \rangle : (\Gamma \vdash \Delta)}\,(cut) \qquad \frac{c : (\Gamma \vdash \beta : B, \Delta)}{\Gamma, \mu\beta.c : B \vdash \Delta}\,(\mu)$$

$$\dfrac{}{\Gamma, A \vdash \Delta, A}\,(L - ax) \qquad \dfrac{}{\Gamma, A \vdash \Delta, A}\,(R - ax)$$

$$\dfrac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\,(\to L) \qquad \dfrac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\,(\to R)$$

$$\dfrac{\Gamma \vdash A, \Delta \quad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\,(cut) \qquad \dfrac{\Gamma \vdash A, \Delta}{\Gamma \vdash A, \Delta}\,(\mu)$$

$$\dfrac{}{\Gamma, \alpha : A \vdash \alpha : A, \Delta}\,(L - ax) \qquad \dfrac{}{\Gamma, x : A \vdash x : A, \Delta}\,(R - ax)$$

$$\dfrac{\Gamma \vdash r : A, \Delta \qquad \Gamma, e : B \vdash \Delta}{\Gamma, r \bullet e : A \to B \vdash \Delta}\,(\to L) \qquad \dfrac{\Gamma, x : A \vdash r : B, \Delta}{\Gamma \vdash \lambda x.r : A \to B, \Delta}\,(\to R)$$

$$\dfrac{\Gamma \vdash r : A, \Delta \qquad \Gamma, e : A \vdash \Delta}{\langle r \parallel e \rangle : (\Gamma \vdash \Delta)}\,(cut) \qquad \dfrac{c : (\Gamma \vdash \beta : B, \Delta)}{\Gamma, \mu\beta.c : B \vdash \Delta}\,(\mu)$$

Subject reduction

One can prove that

- if $c : \Gamma \vdash \Delta$ and $c \longrightarrow c'$ then $c' : \Gamma \vdash \Delta$,
- if $\Gamma \vdash r : A, \Delta$ and $r \longrightarrow r'$ then $\Gamma \vdash r' : A, \Delta$,
- if $\Gamma, e : A \vdash \Delta$ and $e \longrightarrow e'$ then $\Gamma, e' : A \vdash \Delta$,

Curry-Howard correspondence

One gets a Curry-Howard correspondence, namely

- terms are proofs,
- types are propositions,
- term reductions are proof simplifications .

Peirce law again

Let T be $(A \to B) \to A$.

$$x : T, y : A \vdash y : A, \gamma : B, \beta : A \qquad x : T, y : A, \beta : A \vdash \beta : A, \gamma : B$$

$$\overline{}\;(cut)$$

$$\dfrac{\dfrac{\dfrac{\langle y \parallel \beta \rangle : (x : T, y : A \vdash \gamma : B, \beta : A)}{x : T, y : A \vdash \mu\gamma.\langle y \parallel \beta \rangle : B, \beta : A}\;(\mu)}{x : T \vdash \lambda y.\mu\gamma.\langle y \parallel \beta \rangle : A \to B, \beta : A}\;(\to R) \qquad x : T, \beta : A \vdash \beta : A}{x : T, (\lambda y.\mu\gamma.\langle y \parallel \beta \rangle) \bullet \beta : T \vdash \beta : A}\;(\to L$$

The tree for typing Peirce law is

$$\dfrac{\dfrac{\dfrac{x : T \vdash x : T, \beta : A \qquad \mathcal{A}}{\langle x \mid (\lambda y.\mu\gamma.\langle y \parallel \beta\rangle) \bullet \beta\rangle : (x : T \vdash \beta : A)}\,(cut)}{x : T \vdash \mu\beta.\langle x \parallel (\lambda y.\mu\gamma.\langle y \parallel \beta\rangle) \bullet \beta\rangle : A,}\,(\mu)}{\vdash \lambda x.\mu\beta.\langle x \parallel (\lambda y.\mu\gamma.\langle y \parallel \beta\rangle) \bullet \beta\rangle : ((A \to B) \to A) \to A,}\,(\to L)$$

Reductions as simplifications of proofs

Reductions are simplifications (normalizations) of proofs

Let us look at

$$(\lambda) \quad \langle \lambda x \cdot r \parallel r' \bullet e \rangle \quad \longrightarrow \quad \langle r[x \leftarrow r'] \parallel e \rangle$$

It corresponds to

$$\dfrac{\dfrac{\dfrac{\mathcal{D}}{\Gamma, x : A \vdash r : B, \Delta}}{\Gamma \vdash \lambda x.r : A \to B, \Delta}\,(\to R) \qquad \dfrac{\Gamma \vdash r' : A, \Delta \quad \Gamma, e : B \vdash \Delta}{\Gamma, r' \bullet e : A \to B \vdash \Delta}\,(\to L)}{\langle \lambda x \cdot r \parallel r' \bullet e \rangle : \Gamma \vdash \Delta}\,(cut)$$

and

$$\dfrac{\dfrac{\mathcal{D}[x \leftarrow r']}{\Gamma, \vdash r[x \leftarrow r'] : B, \Delta} \qquad \Gamma, e : B \vdash \Delta}{\langle r[x \leftarrow r'] \parallel e \rangle : \Gamma \vdash \Delta}\,(cut)$$

It corresponds to

$$\dfrac{\dfrac{\dfrac{\mathcal{D}}{\Gamma, x : A \vdash r : B, \Delta}}{\Gamma \vdash \lambda x.r : A \to B, \Delta}\,(\to R) \qquad \dfrac{\Gamma \vdash r' : A, \Delta \quad \Gamma, e : B \vdash \Delta}{\Gamma, r' \bullet e : A \to B \vdash \Delta}\,(\to L)}{\langle \lambda x \cdot r \parallel r' \bullet e \rangle : \Gamma \vdash \Delta}\,(cut)$$

and

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{\Gamma \vdash r' : A, \Delta}{} \qquad \dfrac{\Gamma \vdash r' : A, \Delta}{} \\ \mathcal{D}\end{array}}{\Gamma, \vdash r[x \leftarrow r'] : B, \Delta} \qquad \Gamma, e : B \vdash \Delta}{\langle r[x \leftarrow r'] \parallel e \rangle : \Gamma \vdash \Delta}\,(cut)$$

Termination or strong normalization

If c is typeable, then c does not start a non terminating reduction.

Conclusion

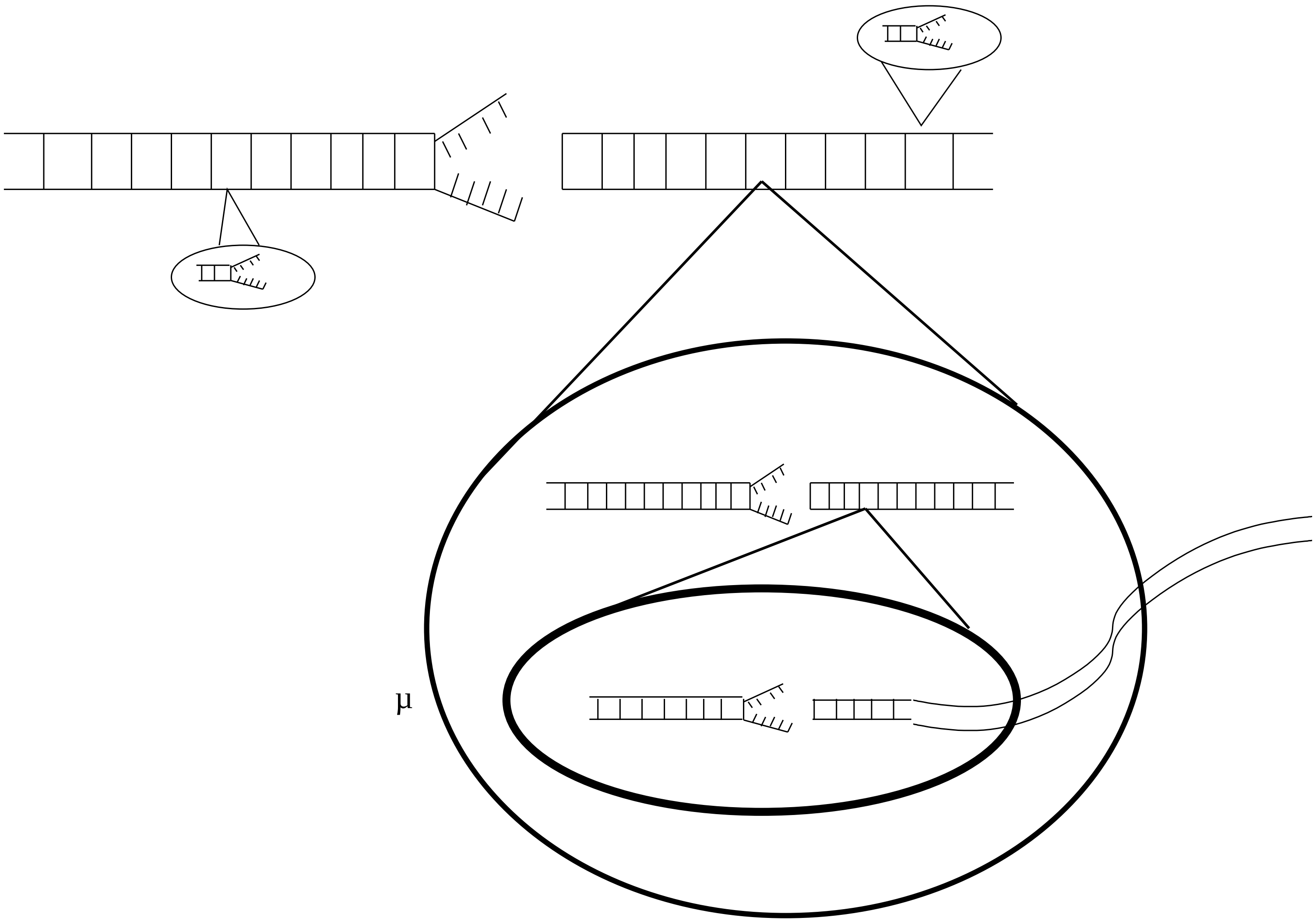
We have proposed a calculus for interpreting classical propositional logic based on capsules of caller-callee.

Perspective: a calculus with intersection types

Our goal is to propose a calculus with intersection types to characterize strongly normalizing terms.

After Philip Wadler

Curry-Howard correspondence is as important for information science as Einstein’s physics relativity and Dirac’s quantum mechanics for physics.


μ

Characterizing strongly normalizable terms

$$\frac{\Gamma, x : A, e : C \vdash \Delta}{\Gamma, x : A \cap B, e : C \vdash \Delta} \qquad \frac{\Gamma, x : A \vdash v : C, \Delta}{\Gamma, x : A \cap B \vdash v : C, \Delta}(\cap L_v)$$

$$\frac{\Gamma \vdash v : A, \Delta \qquad \Gamma \vdash v : B, \Delta}{\Gamma \vdash v : A \cap B, \Delta}(\cap R_v)$$

$$\frac{\Gamma, e : A \vdash \Delta \qquad \Gamma, e : B \vdash \Delta}{\Gamma, e : A \cap B \vdash \Delta}(\cap L_e)$$

$$\frac{\Gamma, e : C \vdash \Delta, \alpha : A}{\Gamma, e : C \vdash \Delta, \alpha : A \cap B} \qquad \frac{\Gamma \vdash v : C, \Delta, \alpha : A}{\Gamma \vdash v : C, \Delta, \alpha : A \cap B}(\cap R_e)$$

We have only $\cap$-introduction rules.

We have no $\cap$-introduction rules for variables.

Characterization of strongly normalizing terms

Theorem: c is typeable if and only if c is strongly normalizing.

Characterization of SN terms: the proof

One shows that strongly normalizing terms are typeable:

- normal forms are typeable,
- if a term is typeable, its expansion according to the perpetual strategy is typeable. All strongly normalizing terms are reached for normal forms by perpetual expansions.

One shows that typeable terms are strongly normalizing.

- one uses reducible sets $[\![A]\!]$,
- one proofs that $[\![A]\!] \subseteq \mathcal{SN}$ and that typeable terms are in $[\![A]\!]$.

Perpetual strategy

One define a function perpc which associates with a capsule c the term perpc c obtained by perpetual strategy.

$$\begin{array}{rcl}
\text{perpc } \langle \lambda x \cdot r \parallel r' \bullet e' \rangle & = & \mathbf{if}\ x \in FV(r)\ \mathbf{or}\ \text{nf}(r' \bullet e')\ \mathbf{then}\ \langle \langle r = x \rangle r' \parallel e' \rangle \\
 & & \mathbf{else}\ \langle \lambda x \cdot r \parallel (\text{perpe } r' \bullet e') \rangle \\
\text{perpc } \langle \mu\alpha \cdot c \parallel e \rangle & = & \mathbf{if}\ \alpha \in FV(c)\ \mathbf{or}\ \text{nf}(e)\ \mathbf{then}\ \langle c = \alpha \rangle e \\
 & & \mathbf{else}\ \langle \mu\alpha \cdot c \parallel \text{perpe } e \rangle \\
\text{perpc } \langle x \parallel e \rangle & = & \mathbf{if}\ \text{nf}(e)\ \mathbf{then}\ unit \\
 & & \mathbf{else}\ \langle x \parallel \text{perpe } e \rangle
\end{array}$$

Perpetual strategie

Theorem :

- Si perpc c est typable alors c est typable.
- Si perpv r est typable alors r est typable.
- Si perpe e est typable alors e est typable.