Non-constructivity of the Cantor-Bernstein theorem

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Constructivity (1/2)

Theorem

\( \pi + e \) is transcendental or \( e \cdot \pi \) is transcendental (or both are).
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- we do not know whether \( \pi + e \) is transcendental or not...
- nor do we know that for \( e \cdot \pi \)
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- nor do we know that for \(e \cdot \pi\)

Morality

\(\sim\) Not all mathematical arguments are equally informative.
Constructivity (2/2)

In broad strokes
Reject excluded middle and reductio ad absurdum.

\[ A \lor \neg A \quad \neg \neg A \Rightarrow A \]

- Interesting for a variety of reasons, non-philosophical or otherwise
- Large amounts of mathematics can still be formalized
  abstract nonsense, finitary combinatorics, \((\mathbb{Q}, <)\)

Some things that break down easily

- decidability of equality for \(\mathbb{R}\) or \(2^\mathbb{N}\)
  \[ \forall x, y \in 2^\mathbb{N}. x = y \lor x \neq y \]
- infinitary combinatorics
- ordinal theory

- Some taboos: \(\mathbb{R}_{\text{Cauchy}} \cong \mathbb{R}_{\text{Dedekind}}\) (as fields), \(2^\mathbb{N} \cong \mathbb{N}^\mathbb{N}\) (as sets)
The CB theorem
If there exists injection $f : A \to B$ and $g : B \to A$, then there exists a bijection $h : A \cong B$. 
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→ excluded middle used to show that we have a partition
What (can’t) we do constructively?

- We can ask for the successor of a node in $f \cup g^{-1} \ldots$
- ...but not predecessor

Taboo: “am I in the range of $f$?”

Even if we could, that would not be enough!

Taboo: “do I have finitely many predecessors?”

Folklore

Cantor-Bernstein fails for models of intuitionistic set theory.

- For the gros topos, $2^\mathbb{N} \not\cong \mathbb{N}^\mathbb{N}$
- In Kleene realizability, easy recursion-theoretic counterexamples.

$\mathbb{N}^\mathbb{N} \rightarrow 2^\mathbb{N}$ constructively as usual

e.g. $\mathbb{N}$ vs $\mathbb{N} + \text{Halt}$
As non-constructive as can be

**Theorem**

*Over intuitionistic set theory (IZF), the Cantor-Bernstein theorem implies excluded middle.*

Plan:

- Proof of a slightly weaker statement (due to Banaschewski and Brümmer)
- Introduce $\mathbb{N}_\infty$ and its effective searchability (due to Escardó)
- Conclude
Quick preliminaries

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Let $\bullet$ be such that $\bullet \notin \mathbb{N}, 2^\mathbb{N}$. Then excluded middle is equivalent to

$$\forall A \subseteq \{\bullet\}. \ A = \emptyset \lor \exists x \in A$$

- 2 is the two-element set
- *cannot* be identified with truth-values $\mathcal{P}(\{\bullet\})$
- we will mostly play around with a singleton set $\{\bullet\}, \mathbb{N}$ and $2^\mathbb{N}$. 
Quick preliminaries

**Remark**

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- *cannot* be identified with truth-values/\( \mathcal{P}(\{\bullet\}) \)
- we will mostly play around with a singleton set \( \{\bullet\}, \mathbb{N} \) and \( 2^{\mathbb{N}} \).

**For the sequel**

Assume \( \bullet \notin \mathbb{N} \cup 2^{\mathbb{N}} \) to be distinguishable from elements of \( \mathbb{N} \) and \( 2^{\mathbb{N}} \)

\[
\forall x \in \{\bullet\} \cup \mathbb{N} \cup 2^{\mathbb{N}}. x \in \mathbb{N} \lor x \in 2^{\mathbb{N}} \lor x = \bullet
\]
Banaschewski and Brümmer’s reversal

A strengthening of Cantor-Bernstein (CBBB)

If there exists injection $f : A \to B$ and $g : B \to A$, then there exists $h : A \cong B$ with $h \subseteq f \cup g^{-1}$

**Theorem (Banaschewski and Brümmer 1986)**

Over IZF, CBBB implies excluded middle.

Fix $A \subseteq \{\bullet\}$ and build maps $f : \mathbb{N} \to A \cup \mathbb{N}$ and $g : A \cup \mathbb{N} \to \mathbb{N}$

$f(n) := n \quad g(\bullet) := 0 \quad g(n) := n + 1$

Is $A$ inhabited or not?

$\to$ is $h(0) = \bullet \text{ or } 0$?
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Fix \( A \subseteq \{\bullet\} \) and build maps \( f : \mathbb{N} \rightarrow A \cup \mathbb{N} \) and \( g : A \cup \mathbb{N} \rightarrow \mathbb{N} \)

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Is \( A \) inhabited or not?

\( \rightarrow \) is \( h(0) = \bullet \) or 0?

No!
For general Cantor-Bernstein

- $h(0)$ might be uninformative
- But asking "Is $\bullet \in h(\mathbb{N})$" would be enough
- Reduction to a weaker instance of excluded middle
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**Idea**

Replace $\mathbb{N}$ with another domain $\mathbb{N}_\infty$ for which we can ask our question

“For any $h : \mathbb{N}_\infty \to A \cup \mathbb{N}_\infty$, is $\bullet \in h(\mathbb{N}_\infty)$?”
**Meet** $\mathbb{N}_\infty$

### Definition

$$\mathbb{N}_\infty := \{ p \in 2^\mathbb{N} | \exists n \leq 1 \in \mathbb{N}. p(n) = 1 \}$$

- Alternative definition: final coalgebra for $X \mapsto 1 + X$
- Call $\infty$ the sequence $n \mapsto 0$
- Embedding $\mathbb{N} \to \mathbb{N}_\infty$: let’s write it $n \mapsto n$. streams of ⊲ that might halt the infinite stream
**Definition**

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- Alternative definition: final coalgebra for \( X \mapsto 1 + X \), streams of \( \bullet \) that might halt
- Call \( \infty \) the sequence \( n \mapsto 0 \), the infinite stream
- Embedding \( \mathbb{N} \to \mathbb{N}_\infty \): let’s write it \( n \mapsto n \).
- **Classically**, \( \mathbb{N}_\infty = \mathbb{N} \cup \{ \infty \} \), equivalent to \( \Sigma_1^0 \)-excluded middle
- Can constructively define addition, but not subtraction or an equality map \( \mathbb{N}_\infty^2 \to 2 \)
\( \mathbb{N}_{\infty} \) is searchable

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\[ \mathbb{N}_{\infty} := \{ p \in 2^{\mathbb{N}} \mid \exists n \leq 1 \in \mathbb{N}. p(n) = 1 \} \quad (\simeq \nu X. 1 + X) \]

**Theorem (Escardó 2013)**

There is a map \( \varepsilon : 2^{\mathbb{N}_{\infty}} \to \mathbb{N}_{\infty} \) that picks witnesses

\[ \forall p \in 2^{\mathbb{N}_{\infty}}. (\exists n \in \mathbb{N}_{\infty}. p(n) = 1) \implies p(\varepsilon(p)) = 1 \]

provably in constructive set theory

(nice to compare and contrast with \( 2^{\mathbb{N}} \)...)

11/14
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\[ \varepsilon(p) = \begin{cases} 0 & \text{if } p(0) = 1 \\ \text{Succ}(\varepsilon(p \circ \text{Succ})) & \text{otherwise} \end{cases} \]

(nice to compare and contrast with \( 2^{\mathbb{N}} \ldots \))

\[ \begin{array}{ccc} \mathbb{N} & \xrightarrow{\rightarrow} & \mathbb{N} \\ \downarrow & & \downarrow \\ \mathbb{N}_\infty & \xrightarrow{\text{Succ}} & \mathbb{N}_\infty \end{array} \]

where

\[ \begin{array}{ccc} \mathbb{N} & \xrightarrow{n \mapsto n+1} & \mathbb{N} \\ \downarrow & & \downarrow \\ \mathbb{N}_\infty & \xrightarrow{\text{Succ}} & \mathbb{N}_\infty \end{array} \]
Recursive version in Haskell

```haskell

type Ninfty = Int -> Bool

ofInt :: Int -> Ninfty
ofInt n i = n == i

epsilon :: (Ninfty -> Bool) -> Ninfty
epsilon p k = not exSmallerWitness && p (ofInt k)
  where exSmallerWitness = any (p . ofInt) [0..k-1]
```
Proof

**Theorem (Escardó 2013)**

There is a map $\varepsilon : 2^{\mathbb{N}_\infty} \to \mathbb{N}_\infty$ that picks witnesses

$$\forall p \in 2^{\mathbb{N}_\infty}. \ (\exists n \in \mathbb{N}_\infty. \ p(n) = 1) \implies p(\varepsilon(p)) = 1$$

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- Define $p \in 2^{\mathbb{N}_\infty}$ by $p(n) := "h(n) = $•"
- Conclude using $p(\varepsilon(p)) = 1 \iff \bullet \in A$
Further thoughts

Remarks

- Trick very much unlike the folklore examples
- Does not give concrete counterexamples in 2-valued models
- Requires the axiom of infinity

Consider $\mathcal{C}^{\text{op}} \to \text{Finset}$ for finite $\mathcal{C}$

Extensions?

- Restriction to e.g., sets with discrete equalities?
- Any relation to investigations of the CB property in more general categories?
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Thanks for listening! Questions?