Type Isomorphisms for Multiplicative-Additive Linear Logic

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Introduction

(Type) Isomorphisms relate types/formulas/objects which are "the same"

$$\begin{tikzcd} A \arrow[loop left, "id_A"] \arrow[rr, bend left, "f"] & & B \arrow[ll, bend left, "g"] \arrow[loop right, "id_B"] \end{tikzcd}$$

Instantiation in λ-calculus, logics,...

Equational theory for $\begin{cases} \lambda\text{-calculus with products and unit type} \\ \text{cartesian closed categories} \end{cases}$ [Sol83]

$\times$	$A \times (B \times C) = (A \times B) \times C$		$A \times B = B \times A$
$\times$ and $\rightarrow$	$(A \times B) \rightarrow C = A \rightarrow (B \rightarrow C)$		$A \rightarrow (B \times C) = (A \rightarrow B) \times (A \rightarrow C)$
1	$A \times 1 = A$	$1 \rightarrow A = A$	$A \rightarrow 1 = 1$

Introduction

(Type) Isomorphisms relate types/formulas/objects which are "the same"

$$\begin{tikzcd} A \arrow[loop left, "id_A"] \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \arrow[loop right, "id_B"] \end{tikzcd}$$

Instantiation in λ-calculus, logics,...

Equational theory for $\begin{cases} \text{Multiplicative Linear Logic} \\ \star\text{-autonomous categories} \end{cases}$ [BDC99]

Associativity	$A \otimes (B \otimes C) = (A \otimes B) \otimes C$	$A \parr (B \parr C) = (A \parr B) \parr C$
Commutativity	$A \otimes B = B \otimes A$	$A \parr B = B \parr A$
Neutrality	$A \otimes 1 = A$	$A \parr \perp = A$

$$(A \otimes B) \multimap C = (A^\perp \parr B^\perp) \parr C = A^\perp \parr (B^\perp \parr C) = A \multimap (B \multimap C)$$

(Type) Isomorphisms relate types/formulas/objects which are “the same”

$$\begin{tikzcd} A \arrow[loop left, "id_A"] \arrow[rr, bend left, "f"] & & B \arrow[ll, bend left, "g"] \arrow[loop right, "id_B"] \end{tikzcd}$$

Instantiation in λ-calculus, logics,...

Equational theory for $\begin{cases} \text{Multiplicative-Additive Linear Logic} \\ \star\text{-autonomous categories with finite products} \end{cases}$

Associativity	$A \otimes (B \otimes C) = (A \otimes B) \otimes C$		$A \parr (B \parr C) = (A \parr B) \parr C$	
	$A \oplus (B \oplus C) = (A \oplus B) \oplus C$		$A \mathbin{\&} (B \mathbin{\&} C) = (A \mathbin{\&} B) \mathbin{\&} C$	
Commutativity	$A \otimes B = B \otimes A$	$A \parr B = B \parr A$	$A \oplus B = B \oplus A$	$A \mathbin{\&} B = B \mathbin{\&} A$
Neutrality	$A \otimes 1 = A$	$A \parr \bot = A$	$A \oplus 0 = A$	$A \mathbin{\&} \top = A$
Distributivity	$A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \mathbin{\&} C) = (A \parr B) \mathbin{\&} (A \parr C)$	
Annihilation	$A \otimes 0 = 0$	$A \parr \top = \top$		

Plan

1. **Definitions**
   - Multiplicative-Additive Linear Logic
   - Type Isomorphisms
   - Proof-Nets
2. **Isomorphisms of Multiplicative-Additive Linear Logic**
   - Simplifications
   - Unit-free case
   - Full case

Multiplicative-Additive Linear Logic

Formulas

$$A, B \ ::= \ \underbrace{X \mid \overset{not}{X^\perp}}_{atoms} \mid \underbrace{A \overset{and}{\otimes} B \mid A \overset{or}{⅋} B \mid \overset{true}{1} \mid \overset{false}{\perp}}_{multiplicative} \mid \underbrace{A \overset{and}{\&} B \mid A \overset{or}{\oplus} B \mid \overset{true}{\top} \mid \overset{false}{0}}_{additive}$$

Multiplicative-Additive Linear Logic

Formulas

$$A, B \ ::= \ \underbrace{X \mid \overset{\textit{not}}{X^\perp}}_{\textit{atoms}} \mid \underbrace{A \overset{\textit{and}}{\otimes} B \mid A \overset{\textit{or}}{\parr} B \mid \overset{\textit{true}}{1} \mid \overset{\textit{false}}{\perp}}_{\textit{multiplicative}} \mid \underbrace{A \overset{\textit{and}}{\&} B \mid A \overset{\textit{or}}{\oplus} B \mid \overset{\textit{true}}{\top} \mid \overset{\textit{false}}{0}}_{\textit{additive}}$$

Rules

$$\frac{}{\vdash A^\perp, A}\ ax \qquad \frac{\vdash A, \Gamma \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\ cut$$

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\ \otimes \qquad \frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\ \parr \qquad \frac{}{\vdash 1}\ 1 \qquad \frac{\vdash \Gamma}{\vdash \perp, \Gamma}\ \perp$$

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Gamma}{\vdash A \& B, \Gamma}\ \& \qquad \frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma}\ \oplus_1 \qquad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma}\ \oplus_2 \qquad \frac{}{\vdash \top, \Gamma}\ \top$$

Multiplicative-Additive Linear Logic

Formulas

$$A, B \;::=\; \underbrace{X \mid \overset{not}{X^\perp}}_{atoms} \mid \underbrace{A \overset{and}{\otimes} B \mid A \overset{or}{\parr} B \mid \overset{true}{1} \mid \overset{false}{\perp}}_{multiplicative} \mid \underbrace{A \overset{and}{\&} B \mid A \overset{or}{\oplus} B \mid \overset{true}{\top} \mid \overset{false}{0}}_{additive}$$

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$$\frac{\vdash A, \color{red}{\Gamma} \qquad \vdash B, \color{red}{\Gamma}}{\vdash A \& B, \color{red}{\Gamma}}\,\& \qquad \frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma}\,\oplus_1 \qquad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma}\,\oplus_2 \qquad \frac{}{\vdash \top, \Gamma}\,\top$$

Multiplicative-Additive Linear Logic

Formulas

$$A, B \ ::= \ \underbrace{X \mid \overset{not}{X^\perp}}_{atoms} \mid \underbrace{A \overset{and}{\otimes} B \mid A \overset{or}{⅋} B \mid \overset{true}{1} \mid \overset{false}{\perp}}_{multiplicative} \mid \underbrace{A \overset{and}{\&} B \mid A \overset{or}{\oplus} B \mid \overset{true}{\top} \mid \overset{false}{0}}_{additive}$$

Rules

$$\frac{}{\vdash A^\perp, A}\ ax \qquad \frac{\vdash A, \Gamma \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\ cut$$

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$$\frac{\vdash A, \Gamma \qquad \vdash B, \Gamma}{\vdash A \& B, \Gamma}\ \& \qquad \frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma}\ \oplus_1 \qquad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma}\ \oplus_2 \qquad \frac{}{\vdash \top, \Gamma}\ \top$$

- slice

Multiplicative-Additive Linear Logic

Formulas

$$A, B \;::=\; \underbrace{X \mid \overset{\textit{not}}{X^\perp}}_{\textit{atoms}} \mid \underbrace{A \overset{\textit{and}}{\otimes} B \mid A \overset{\textit{or}}{⅋} B \mid \overset{\textit{true}}{1} \mid \overset{\textit{false}}{\perp}}_{\textit{multiplicative}} \mid \underbrace{A \overset{\textit{and}}{\&} B \mid A \overset{\textit{or}}{\oplus} B \mid \overset{\textit{true}}{\top} \mid \overset{\textit{false}}{0}}_{\textit{additive}}$$

Rules

$$\frac{}{\vdash X^\perp, X}\,ax \qquad \frac{\vdash A, \Gamma \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\,cut$$

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\,\otimes \qquad \frac{\vdash A, B, \Gamma}{\vdash A ⅋ B, \Gamma}\,⅋ \qquad \frac{}{\vdash 1}\,1 \qquad \frac{\vdash \Gamma}{\vdash \perp, \Gamma}\,\perp$$

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Gamma}{\vdash A \& B, \Gamma}\,\& \qquad \frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma}\,\oplus_1 \qquad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma}\,\oplus_2 \qquad \frac{}{\vdash \top, \Gamma}\,\top$$

- slice
- axiom-expansion $\xrightarrow{\eta}$

Multiplicative-Additive Linear Logic

Formulas

$$A, B ::= \underbrace{X \mid \overset{not}{X^\perp}}_{atoms} \mid \underbrace{A \overset{and}{\otimes} B \mid A \overset{or}{\parr} B \mid \overset{true}{1} \mid \overset{false}{\perp}}_{multiplicative} \mid \underbrace{A \overset{and}{\&} B \mid A \overset{or}{\oplus} B \mid \overset{true}{\top} \mid \overset{false}{0}}_{additive}$$

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- slice
- axiom-expansion $\xrightarrow{\eta}$
- cut-elimination $\xrightarrow{\beta}$

Unit-free Multiplicative-Additive Linear Logic

Formulas

$$A, B \ ::= \ \underbrace{X \mid \overset{not}{X^\perp}}_{atoms} \mid \underbrace{A \overset{and}{\otimes} B \mid A \overset{or}{⅋} B \mid \overset{true}{1} \mid \overset{false}{\perp}}_{multiplicative} \mid \underbrace{A \overset{and}{\&} B \mid A \overset{or}{\oplus} B \mid \overset{true}{\top} \mid \overset{false}{0}}_{additive}$$

Rules

$$\frac{}{\vdash X^\perp, X}\,ax \qquad \frac{\vdash A, \Gamma \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\,cut$$

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\,\otimes \qquad \frac{\vdash A, B, \Gamma}{\vdash A ⅋ B, \Gamma}\,⅋ \qquad \frac{}{\vdash 1}\,1 \qquad \frac{\vdash \Gamma}{\vdash \perp, \Gamma}\,\perp$$

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Gamma}{\vdash A \& B, \Gamma}\,\& \qquad \frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma}\,\oplus_1 \qquad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma}\,\oplus_2 \qquad \frac{}{\vdash \top, \Gamma}\,\top$$

- slice
- axiom-expansion $\xrightarrow{\eta}$
- cut-elimination $\xrightarrow{\beta}$

Type Isomorphisms

In category theory:

$$\begin{tikzcd} A \arrow[loop left, "id_A"] \arrow[rr, bend left, "f"] & & B \arrow[ll, bend left, "g"] \arrow[loop right, "id_B"] \end{tikzcd}$$

In λ-calculus:

Isomorphism $A \simeq B$

Terms $M : A \to B$ and $N : B \to A$ such that

$$N \circ M =_{\beta\eta} \lambda x^A.x \qquad \text{and} \qquad M \circ N =_{\beta\eta} \lambda y^B.y$$

Type Isomorphisms

In category theory:

$$\begin{tikzcd} A \arrow[loop left, "id_A"] \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \arrow[loop right, "id_B"] \end{tikzcd}$$

In linear logic:

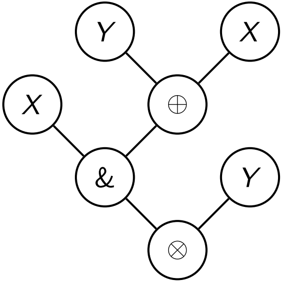
Isomorphism $A \simeq B$

Proofs $\pi \vdash A^\perp, B$ and $\sigma \vdash B^\perp, A$ such that

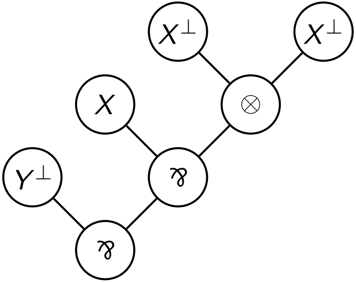
$$\dfrac{\dfrac{\sigma}{\vdash A, B^\perp} \qquad \dfrac{\pi}{\vdash B, A^\perp}}{\vdash A^\perp, A}\,cut =_{\beta\eta} \dfrac{}{\vdash A^\perp, A}\,ax \quad \text{and} \quad \dfrac{\dfrac{\pi}{\vdash B, A^\perp} \qquad \dfrac{\sigma}{\vdash A, B^\perp}}{\vdash B^\perp, B}\,cut =_{\beta\eta} \dfrac{}{\vdash B^\perp, B}\,ax$$

Proof-Nets of Hughes & Van Glabbeek [HvG05]

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash X^\perp, X$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash X^\perp, X$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash X, X, X^\perp \otimes X^\perp$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash X^\perp, X$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash X^\perp, X$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash X, X, X^\perp \otimes X^\perp$}
\RightLabel{$\oplus_2$}
\UnaryInfC{$\vdash Y \oplus X, X^\perp \otimes X^\perp$}
\RightLabel{$\&$}
\BinaryInfC{$\vdash X \,\&\, (Y \oplus X), X, X^\perp \otimes X^\perp$}
\RightLabel{$\parr$}
\UnaryInfC{$\vdash X \,\&\, (Y \oplus X), X \parr (X^\perp \otimes X^\perp)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash Y^\perp, Y$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash (X \,\&\, (Y \oplus X)) \otimes Y, Y^\perp, X \parr (X^\perp \otimes X^\perp)$}
\RightLabel{$\parr$}
\UnaryInfC{$\vdash (X \,\&\, (Y \oplus X)) \otimes Y, Y^\perp \parr (X \parr (X^\perp \otimes X^\perp))$}
\end{prooftree}
$$



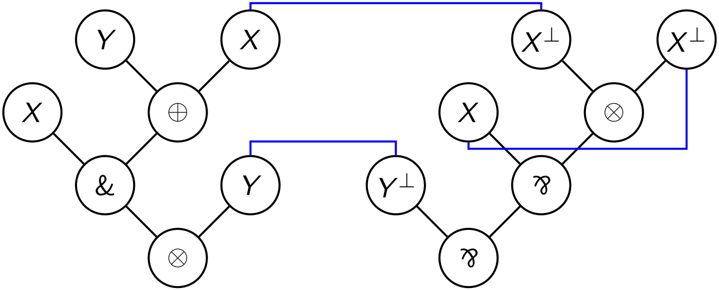
$(X \,\&\, (Y \oplus X)) \otimes Y$



$Y^\perp \parr (X \parr (X^\perp \otimes X^\perp))$

Proof-Nets of Hughes & Van Glabbeek [HvG05]

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash X^\perp, X}\,ax \qquad \dfrac{}{\vdash X^\perp, X}\,ax}{\vdash X, X, X^\perp \otimes X^\perp}\,\otimes \qquad \dfrac{\dfrac{\dfrac{}{\vdash X^\perp, X}\,ax \qquad \dfrac{}{\vdash X^\perp, X}\,ax}{\vdash X, X, X^\perp \otimes X^\perp}\,\otimes}{\vdash Y \oplus X, X, X^\perp \otimes X^\perp}\,\oplus_2}{\vdash X \,\&\, (Y \oplus X), X, X^\perp \otimes X^\perp}\,\&}{\vdash X \,\&\, (Y \oplus X), X \parr (X^\perp \otimes X^\perp)}\,\parr \qquad \dfrac{}{\vdash Y^\perp, Y}\,ax}{\dfrac{\vdash (X \,\&\, (Y \oplus X)) \otimes Y, Y^\perp, X \parr (X^\perp \otimes X^\perp)}{\vdash (X \,\&\, (Y \oplus X)) \otimes Y, Y^\perp \parr (X \parr (X^\perp \otimes X^\perp))}\,\parr}\,\otimes$$



$(X \,\&\, (Y \oplus X)) \otimes Y \quad Y^\perp \parr (X \parr (X^\perp \otimes X^\perp))$

Proof-Nets of Hughes & Van Glabbeek [HvG05]

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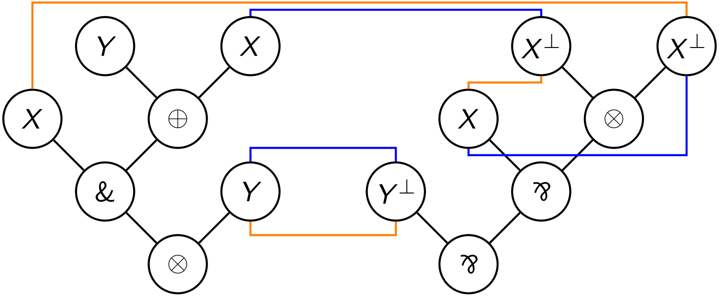
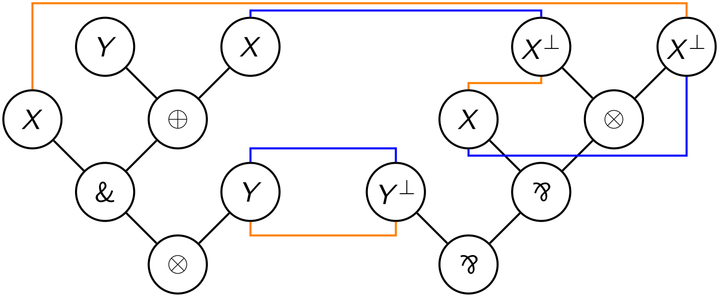
\RightLabel{$\parr$}
\UnaryInfC{$\vdash (X \,\&\, (Y \oplus X)) \otimes Y, Y^\perp \parr (X \parr (X^\perp \otimes X^\perp))$}
\end{prooftree}
$$



$(X \,\&\, (Y \oplus X)) \otimes Y \quad Y^\perp \parr (X \parr (X^\perp \otimes X^\perp))$

Proof-Nets of Hughes & Van Glabbeek [HvG05]

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash X^\perp, X}\,ax \qquad \dfrac{}{\vdash X^\perp, X}\,ax}{\vdash X, X, X^\perp \otimes X^\perp}\,\otimes \qquad \dfrac{\dfrac{\dfrac{}{\vdash X^\perp, X}\,ax \qquad \dfrac{}{\vdash X^\perp, X}\,ax}{\vdash X, X, X^\perp \otimes X^\perp}\,\otimes}{\vdash Y \oplus X, X, X^\perp \otimes X^\perp}\,\oplus_2}{\vdash X \,\&\, (Y \oplus X), X, X^\perp \otimes X^\perp}\,\&}{\vdash X \,\&\, (Y \oplus X), X \parr (X^\perp \otimes X^\perp)}\,\parr \qquad \dfrac{}{\vdash Y^\perp, Y}\,ax}{\dfrac{\vdash (X \,\&\, (Y \oplus X)) \otimes Y, Y^\perp, X \parr (X^\perp \otimes X^\perp)}{\vdash (X \,\&\, (Y \oplus X)) \otimes Y, Y^\perp \parr (X \parr (X^\perp \otimes X^\perp))}\,\parr}\,\otimes$$



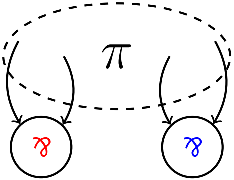
$(X \,\&\, (Y \oplus X)) \otimes Y \quad Y^\perp \parr (X \parr (X^\perp \otimes X^\perp))$

with a complex **correctness criterion**, forbidding some kind of *cycles*

Properties of Proof-Nets

Pros: identify proofs up to rule commutations [HvG16], thus up to $=_{\beta\eta}$

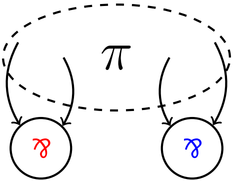
$$
\cfrac{\cfrac{\cfrac{\pi}{\vdash A_1, A_2, B_1, B_2, \Gamma}}{\vdash A_1 \parr A_2, B_1, B_2, \Gamma}\,\parr}{\vdash A_1 \parr A_2, B_1 \parr B_2, \Gamma}\,\parr
\quad =_c \quad
\cfrac{\cfrac{\cfrac{\pi}{\vdash A_1, A_2, B_1, B_2, \Gamma}}{\vdash A_1, A_2, B_1 \parr B_2, \Gamma}\,\parr}{\vdash A_1 \parr A_2, B_1 \parr B_2, \Gamma}\,\parr
$$



Properties of Proof-Nets

Pros: identify proofs up to rule commutations [HvG16], thus up to $=_{\beta\eta}$

$$\dfrac{\dfrac{\dfrac{\pi}{\vdash A_1, A_2, B_1, B_2, \Gamma}}{\vdash A_1 \parr A_2, B_1, B_2, \Gamma}\,\parr}{\vdash A_1 \parr A_2, B_1 \parr B_2, \Gamma}\,\parr \qquad =_c \qquad \dfrac{\dfrac{\dfrac{\pi}{\vdash A_1, A_2, B_1, B_2, \Gamma}}{\vdash A_1, A_2, B_1 \parr B_2, \Gamma}\,\parr}{\vdash A_1 \parr A_2, B_1 \parr B_2, \Gamma}\,\parr$$



Cons: does not work with

- non-expanded axioms
- <u>units</u>

Plan

1. Definitions
   - Multiplicative-Additive Linear Logic
   - Type Isomorphisms
   - Proof-Nets

2. **Isomorphisms of Multiplicative-Additive Linear Logic**
   - Simplifications
   - Unit-free case
   - Full case

Associativity	$A \otimes (B \otimes C) = (A \otimes B) \otimes C$		$A \parr (B \parr C) = (A \parr B) \parr C$	
	$A \oplus (B \oplus C) = (A \oplus B) \oplus C$		$A \& (B \& C) = (A \& B) \& C$	
Commutativity	$A \otimes B = B \otimes A$	$A \parr B = B \parr A$	$A \oplus B = B \oplus A$	$A \& B = B \& A$
Neutrality	$A \otimes 1 = A$	$A \parr \bot = A$	$A \oplus 0 = A$	$A \& \top = A$
Distributivity	$A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \& C) = (A \parr B) \& (A \parr C)$	
Annihilation	$A \otimes 0 = 0$	$A \parr \top = \top$		

Axiom-expanded proofs

Reduction to axiom-expanded proofs

$$\begin{tikzcd} \pi \arrow[r, phantom, "=_{\beta\eta}"] \arrow[d, "\eta^*"] & \sigma \arrow[d, "\eta^*"] \\ \pi' \arrow[r, phantom, "=_\beta"] & \sigma' \end{tikzcd}$$

with in $\pi' =_\beta \sigma'$ only axiom-expanded proofs

Proof.

Simple study of axiom-expansion $\xrightarrow{\eta}$ and cut-elimination $\xrightarrow{\beta}$. □

Remove one obstacle to the use of proof-nets!

Distributivity

Distributed Formula

$$
\begin{array}{rcl|rcl}
A \otimes (B \oplus C) & \to & (A \otimes B) \oplus (A \otimes C) & (C \mathbin{\&} B) \parr A & \to & (C \parr A) \mathbin{\&} (B \parr A) \\
(A \oplus B) \otimes C & \to & (A \otimes C) \oplus (B \otimes C) & C \parr (B \mathbin{\&} A) & \to & (C \parr B) \mathbin{\&} (C \parr A)
\end{array}
$$

$$
\begin{array}{rcl|rcl|rcl|rcl}
A \otimes 1 & \to & A & 1 \otimes A & \to & A & A \parr \bot & \to & A & \bot \parr A & \to & A \\
A \oplus 0 & \to & A & 0 \oplus A & \to & A & A \mathbin{\&} \top & \to & A & \top \mathbin{\&} A & \to & A \\
A \otimes 0 & \to & 0 & 0 \otimes A & \to & 0 & A \parr \top & \to & \top & \top \parr A & \to & \top
\end{array}
$$

Distributivity

Distributed Formula

$A \otimes (B \oplus C)$	$\to$	$(A \otimes B) \oplus (A \otimes C)$	$(C \mathbin{\&} B) \parr A$	$\to$	$(C \parr A) \mathbin{\&} (B \parr A)$
$(A \oplus B) \otimes C$	$\to$	$(A \otimes C) \oplus (B \otimes C)$	$C \parr (B \mathbin{\&} A)$	$\to$	$(C \parr B) \mathbin{\&} (C \parr A)$

$A \otimes 1$	$\to$	A	$1 \otimes A$	$\to$	A	$A \parr \bot$	$\to$	A	$\bot \parr A$	$\to$	A
$A \oplus 0$	$\to$	A	$0 \oplus A$	$\to$	A	$A \mathbin{\&} \top$	$\to$	A	$\top \mathbin{\&} A$	$\to$	A
$A \otimes 0$	$\to$	0	$0 \otimes A$	$\to$	0	$A \parr \top$	$\to$	$\top$	$\top \parr A$	$\to$	$\top$

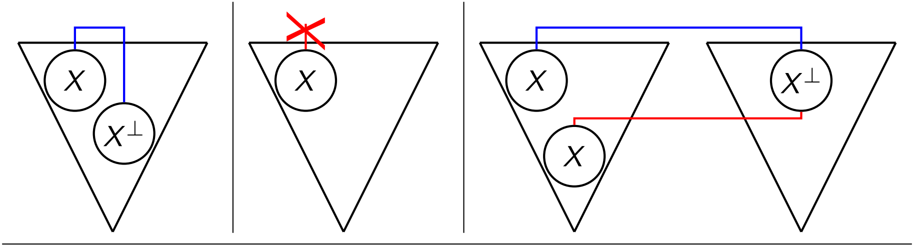
Proposition

*If $\mathcal{E}$ is complete for **distributed** formulas, then $\mathcal{E}$ + neutrality + distributivity + annihilation is complete for **arbitrary** formulas.*

Associativity	$A \otimes (B \otimes C) = (A \otimes B) \otimes C$		$A \parr (B \parr C) = (A \parr B) \parr C$	
	$A \oplus (B \oplus C) = (A \oplus B) \oplus C$		$A \mathbin{\&} (B \mathbin{\&} C) = (A \mathbin{\&} B) \mathbin{\&} C$	
Commutativity	$A \otimes B = B \otimes A$	$A \parr B = B \parr A$	$A \oplus B = B \oplus A$	$A \mathbin{\&} B = B \mathbin{\&} A$
Neutrality	$A \otimes 1 = A$	$A \parr \bot = A$	$A \oplus 0 = A$	$A \mathbin{\&} \top = A$
Distributivity	$A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \mathbin{\&} C) = (A \parr B) \mathbin{\&} (A \parr C)$	
Annihilation	$A \otimes 0 = 0$	$A \parr \top = \top$		

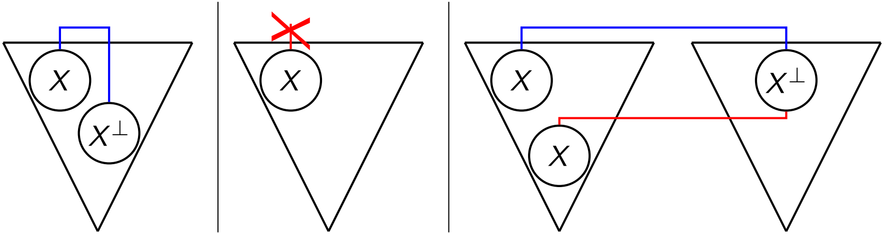
Shape of distributed isomorphisms

Forbidden configurations in distributed isomorphisms:

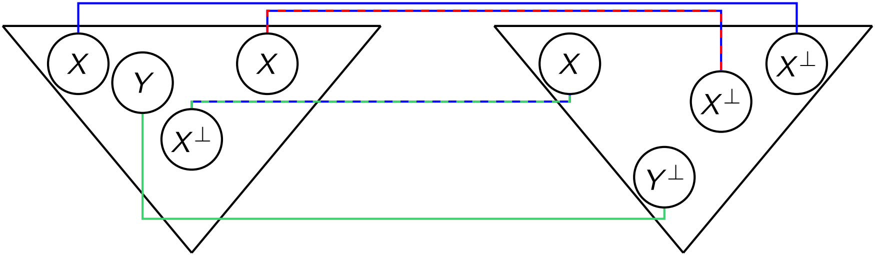


Shape of distributed isomorphisms

Forbidden configurations in distributed isomorphisms:

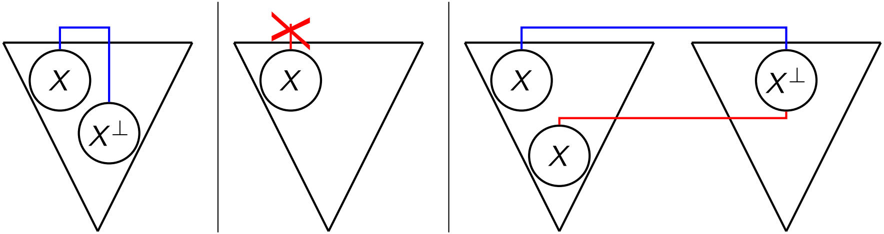


General shape:

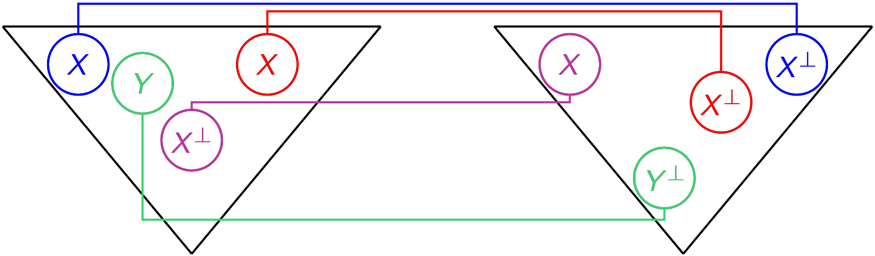


Shape of distributed isomorphisms

Forbidden configurations in distributed isomorphisms:



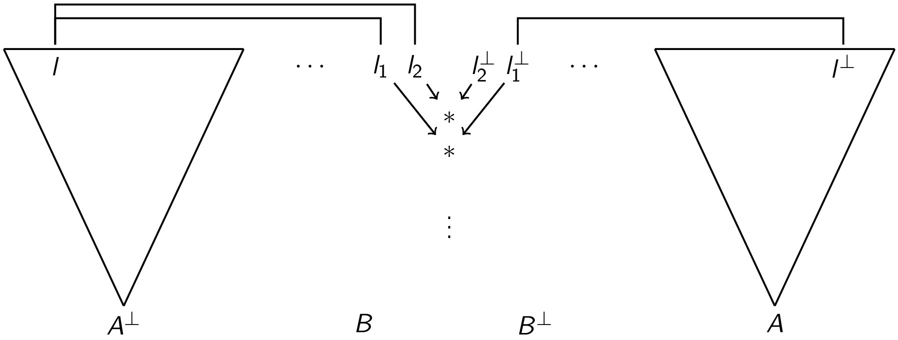
General shape:



Why this shape?

$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$ **not** of this shape

Correctness criterion to get this "local" shape from "global" *distributivity*



1. Forbidden configuration

Why this shape?

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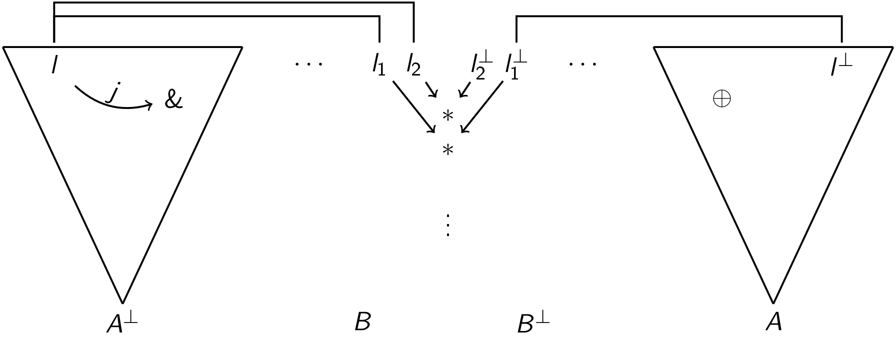

1. Forbidden configuration
2. Dependence on a &

Why this shape?

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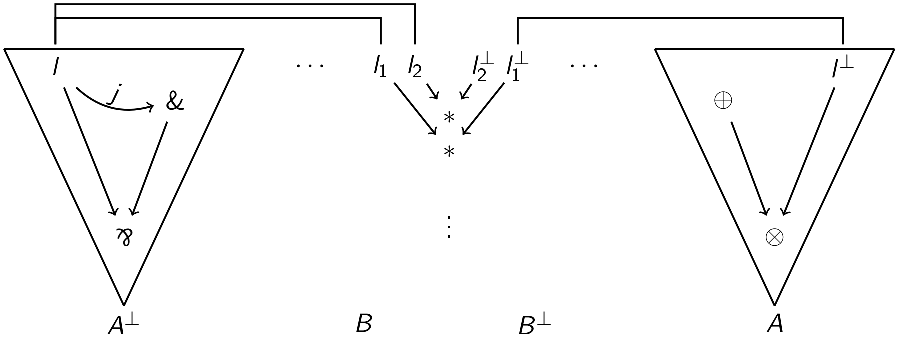
Correctness criterion to get this "local" shape from "global" *distributivity*

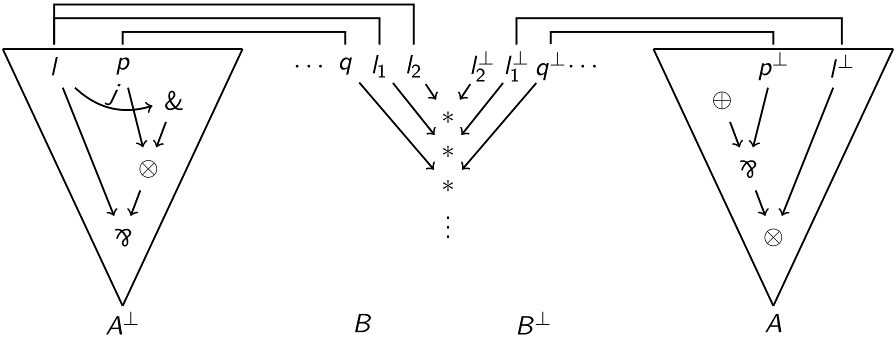


1. Forbidden configuration
2. Dependence on a &
3. ⅋ below

Why this shape?

$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$ **not** of this shape

Correctness criterion to get this "local" shape from "global" *distributivity*



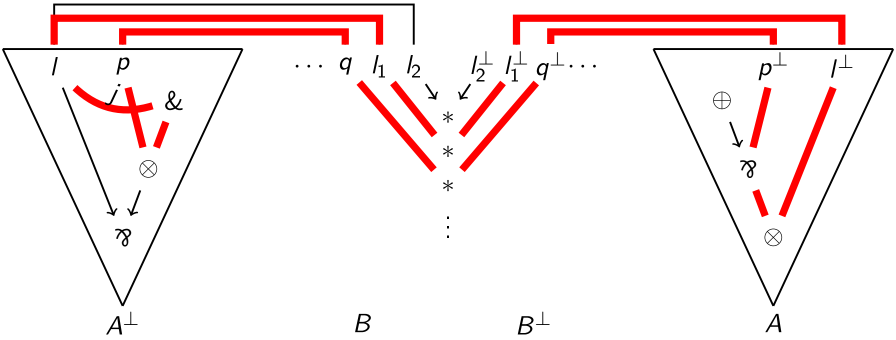
1. Forbidden configuration
2. Dependence on a $\&$
3. $⅋$ below

4. Distributivity

Why this shape?

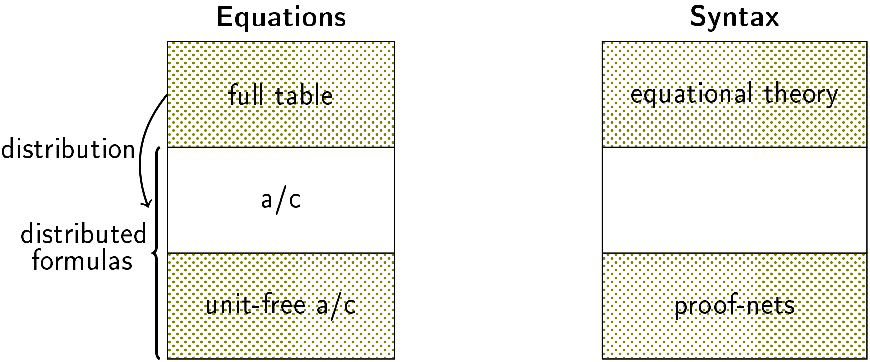
$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$ **not** of this shape

Correctness criterion to get this "local" shape from "global" *distributivity*



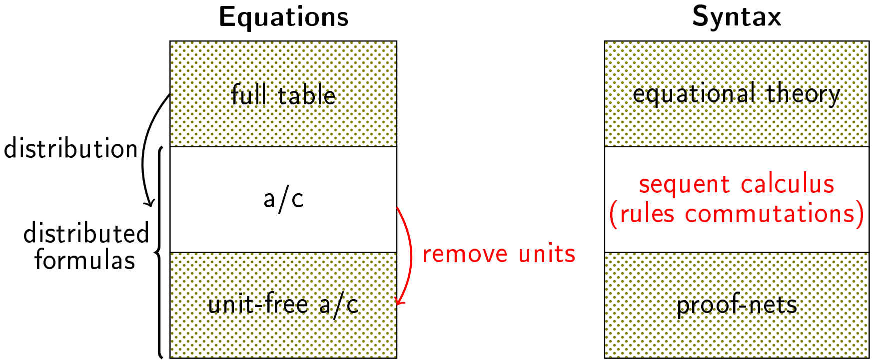
1. Forbidden configuration
2. Dependence on a &
3. ⅋ below
4. Distributivity
5. Contradictory cycle

Where are we?



Equations
full table
distribution
a/c
distributed formulas
unit-free a/c
Syntax
equational theory
proof-nets

Where are we?


Equations
Syntax
full table
distribution
a/c
distributed formulas
unit-free a/c
remove units
equational theory
sequent calculus
(rules commutations)
proof-nets

Confluence in sequent caculus

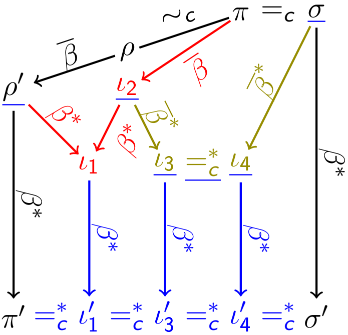
Confluence up to rule commutations

$$\begin{tikzcd} \pi \arrow[r, phantom, "=_\beta"] \arrow[d, "\beta^*"] & \sigma \arrow[d, "\beta^*"] \\ \pi' \arrow[r, phantom, "=_c^*"] & \sigma' \end{tikzcd}$$

Confluence in sequent caculus

Confluence up to rule commutations

$$\begin{tikzcd}
\pi \arrow[r, phantom, "=_\beta"] \arrow[d, "\overline{\beta}^*"] & \sigma \arrow[d, "\overline{\beta}^*"] \\
\pi' \arrow[r, phantom, "=_c^*"] & \sigma'
\end{tikzcd}$$



Handling the Units 0, $\top$, 1, $\perp$

In isomorphisms of **distributed** formulas:

ⓐ

$$\dfrac{}{\vdash \top, 0}\;\top \qquad\qquad \dfrac{\dfrac{\dfrac{}{\vdash 1}\;1}{\vdash F}\;\oplus_i}{\vdash \perp, F}\;\perp$$

Handling the Units 0, $\top$, 1, $\bot$

In isomorphisms of **distributed** formulas:

ⓐ
$$\dfrac{}{\vdash \top, 0}\,\top \qquad\qquad \dfrac{\dfrac{\dfrac{}{\vdash 1}\,1}{\vdash F}\,\oplus_i}{\vdash \bot, F}\,\bot$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

ⓑ
$$\dfrac{}{\vdash \top, 0}\,\top \qquad\qquad \dfrac{\dfrac{\dfrac{}{\vdash 1}\,1}{\vdash \bot, 1}\,\bot}{\vdash \bot, F}\,\oplus_i$$

Handling the Units 0, $\top$, 1, $\bot$

In isomorphisms of **distributed** formulas:

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$$\dfrac{}{\vdash \top, 0}\,\top \qquad\qquad \dfrac{\dfrac{\dfrac{}{\vdash 1}\,1}{\vdash F}\,\oplus_i}{\vdash \bot, F}\,\bot$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

ⓑ
$$\dfrac{}{\vdash \top, 0}\,\top \qquad\qquad \dfrac{\dfrac{\dfrac{}{\vdash 1}\,1}{\vdash \bot, 1}\,\bot}{\vdash \bot, F}\,\oplus_i$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

ⓒ
$$\dfrac{}{\vdash X_0^{\bot}, X_0}\,ax \qquad\qquad \dfrac{\dfrac{}{\vdash X_1^{\bot}, X_1}\,ax}{\vdash X_1^{\bot}, F[X_1/1]}\,\oplus_i$$

Results

Theorem

Isomorphisms of Multiplicative-Additive Linear Logic:

Associativity	$A \otimes (B \otimes C) = (A \otimes B) \otimes C$		$A \parr (B \parr C) = (A \parr B) \parr C$	
	$A \oplus (B \oplus C) = (A \oplus B) \oplus C$		$A \,\&\, (B \,\&\, C) = (A \,\&\, B) \,\&\, C$	
Commutativity	$A \otimes B = B \otimes A$	$A \parr B = B \parr A$	$A \oplus B = B \oplus A$	$A \,\&\, B = B \,\&\, A$
Neutrality	$A \otimes 1 = A$	$A \parr \perp = A$	$A \oplus 0 = A$	$A \,\&\, \top = A$
Distributivity	$A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \,\&\, C) = (A \parr B) \,\&\, (A \parr C)$	
Annihilation	$A \otimes 0 = 0$	$A \parr \top = \top$		

Results

Theorem

Isomorphisms of $\star$-autonomous categories with finite products:

Associativity	$A \otimes (B \otimes C) = (A \otimes B) \otimes C$		$A \parr (B \parr C) = (A \parr B) \parr C$	
	$A \oplus (B \oplus C) = (A \oplus B) \oplus C$		$A \,\&\, (B \,\&\, C) = (A \,\&\, B) \,\&\, C$	
Commutativity	$A \otimes B = B \otimes A$	$A \parr B = B \parr A$	$A \oplus B = B \oplus A$	$A \,\&\, B = B \,\&\, A$
Neutrality	$A \otimes 1 = A$	$A \parr \bot = A$	$A \oplus 0 = A$	$A \,\&\, \top = A$
Distributivity	$A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \,\&\, C) = (A \parr B) \,\&\, (A \parr C)$	
Annihilation	$A \otimes 0 = 0$	$A \parr \top = \top$		
De Morgan	$A \multimap B = A^\bot \parr B$	$X^{\bot\bot} = X$		
	$(A \otimes B)^\bot = B^\bot \parr A^\bot$	$(A \parr B)^\bot = B^\bot \otimes A^\bot$	$(A \oplus B)^\bot = B^\bot \,\&\, A^\bot$	$(A \,\&\, B)^\bot = B^\bot \oplus A^\bot$
	$1^\bot = \bot$	$\bot^\bot = 1$	$0^\bot = \top$	$\top^\bot = 0$

Results

Theorem

Isomorphisms of $\star$-autonomous categories with finite products:

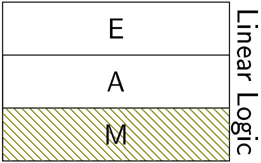
Associativity	$A \otimes (B \otimes C) = (A \otimes B) \otimes C$		$A \parr (B \parr C) = (A \parr B) \parr C$	
	$A \oplus (B \oplus C) = (A \oplus B) \oplus C$		$A \,\&\, (B \,\&\, C) = (A \,\&\, B) \,\&\, C$	
Commutativity	$A \otimes B = B \otimes A$	$A \parr B = B \parr A$	$A \oplus B = B \oplus A$	$A \,\&\, B = B \,\&\, A$
Neutrality	$A \otimes 1 = A$	$A \parr \bot = A$	$A \oplus 0 = A$	$A \,\&\, \top = A$
Distributivity	$A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \,\&\, C) = (A \parr B) \,\&\, (A \parr C)$	
Annihilation	$A \otimes 0 = 0$	$A \parr \top = \top$		
De Morgan	$A \multimap B = A^\perp \parr B$	$X^{\perp\perp} = X$		
	$(A \otimes B)^\perp = B^\perp \parr A^\perp$	$(A \parr B)^\perp = B^\perp \otimes A^\perp$	$(A \oplus B)^\perp = B^\perp \,\&\, A^\perp$	$(A \,\&\, B)^\perp = B^\perp \oplus A^\perp$
	$1^\perp = \bot$	$\bot^\perp = 1$	$0^\perp = \top$	$\top^\perp = 0$

Confluence up to rule commutations in sequent calculus

$$
\begin{tikzcd}
\sigma \arrow[r, "\eta^*"] \arrow[d, phantom, "=_{\beta\eta}"] & \cdot \arrow[r, "\beta^*"] \arrow[d, phantom, "=_{\beta}"] & \sigma' \arrow[d, phantom, "=_{c}^{*}"] \\
\pi \arrow[r, "\eta^*"] & \cdot \arrow[r, "\beta^*"] & \pi'
\end{tikzcd}
$$

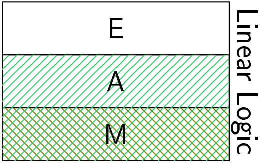
Ongoing and future work

- State of the art: [BDC99]



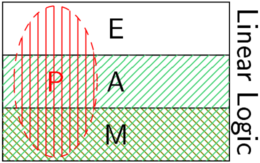
Ongoing and future work

- State of the art: [BDC99], [this talk]



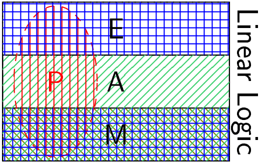
Ongoing and future work

- State of the art: [BDC99], [this talk], [Lau05]

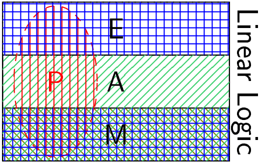


Ongoing and future work

- State of the art: [BDC99], [this talk], [Lau05]
- Isomorphisms for MELL



Ongoing and future work



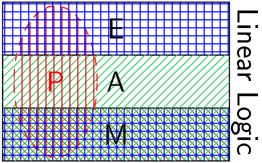
- State of the art: [BDC99], [this talk], [Lau05]
- Isomorphisms for MELL
- Isomorphisms for MALL with quantifiers

$$\forall x, \forall y, A \simeq \forall y, \forall x, A$$
$$\forall x, A \mathbin{\&} B \simeq (\forall x, A) \mathbin{\&} (\forall x, B)$$
$$\forall x, A \parr B \simeq (\forall x, A) \parr B \quad \text{if } x \notin B$$

(and the dual versions)

Ongoing and future work

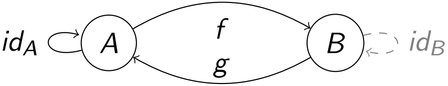
- State of the art: [BDC99], [this talk], [Lau05]
- Isomorphisms for MELL
- Isomorphisms for MALL with quantifiers



$$\forall x, \forall y, A \simeq \forall y, \forall x, A$$
$$\forall x, A \,\&\, B \simeq (\forall x, A) \,\&\, (\forall x, B)$$
$$\forall x, A \parr B \simeq (\forall x, A) \parr B \quad \text{if } x \notin B$$
(and the dual versions)

- Retractions in MLL (subtyping)

$$A \trianglelefteq B$$



$$A \trianglelefteq (A \multimap A) \multimap A \simeq A \parr (A^\perp \otimes A)$$

Thank you!

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