Proof theory and linear logic

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Introduction

In Mathematics / Theoretical computer science:

- pose definitions
- write proofs

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- X the smallest positive integer not definable in under sixty letters [57 letters]
- ? the smallest positive integer not definable in under twenty letters [58 letters]

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Proof theory: study proofs and their properties

Why studying proofs?

An absolutely true result

$$-1 = 1$$

Proof.

$$-1 = (-1)^{\frac{2}{2}} = ((-1)^2)^{\frac{1}{2}} = 1^{\frac{1}{2}} = 1$$

$\square$

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Continuum hypothesis

There is no set whose cardinal is strictly between that of the integers and the real numbers.

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Continuum hypothesis

There is no set whose cardinal is strictly between that of the integers and the real numbers.

This hypothesis is not provable. But its negation neither is!

A Formal Proof

Lemma

For all integer n, there exists an integer k such that n is equal to $k + 1$.

Proof.

Any n is equal to $(n - 1) + 1$. $\square$

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$$\forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n = k + 1$$

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Lemma

$$\forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n = k + 1$$

Proof.

We prove $\forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n = k + 1$.
It suffices to prove $\exists k \in \mathbb{Z}, n = k + 1$ for arbitrary $n \in \mathbb{Z}$.
Instanciate $k = n - 1 \in \mathbb{Z}$. It holds that $n = (n - 1) + 1$. $\square$

A Formal Proof

Lemma

$$\forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n = k + 1$$

Proof.

$$\dfrac{\dfrac{\dfrac{}{n = (n-1) + 1}\ (eq)}{\exists k \in \mathbb{Z}, n = k + 1}\ (\exists)}{\forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n = k + 1}\ (\forall)$$

□

Classical Logic

$$\frac{\Gamma \vdash A[y/x], \Sigma}{\Gamma \vdash \exists x\ A, \Sigma}\ (\exists) \qquad \frac{\Gamma \vdash A, \Sigma}{\Gamma \vdash \forall x\ A, \Sigma}\ (\forall)$$

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$$\frac{\Gamma \vdash A, \Sigma \quad \Delta \vdash B, \Theta}{\Gamma, \Delta \vdash A \wedge B, \Sigma, \Theta}\ (\wedge) \qquad \frac{\Gamma \vdash A, B, \Sigma}{\Gamma \vdash A \vee B, \Sigma}\ (\vee)$$

(and more rules)

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(and more rules)

Very symmetric but bad properties: many trees for the same "proof"

Intuitionistic Logic

Cauchy-Lipschitz theorem: unique solution to some differential problems.
Engineer point of view: still no answer :(

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Intuitionistic Logic by changing the rules from Classical Logic
Constructive: from a proof of $\exists x\ A$ can recover an algorithm computing x

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Intuitionistic Logic by changing the rules from Classical Logic
Constructive: from a proof of $\exists x\ A$ can recover an algorithm computing x

But *weaker* logic (no excluded middle)

Linear Logic

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(and more rules)

Linear Logic

$$\frac{\Gamma \vdash A, \Sigma \qquad \Gamma \vdash B, \Sigma}{\Gamma \vdash A \mathbin{\&} B, \Sigma}\ (\&) \qquad \frac{\Gamma \vdash A, \Sigma}{\Gamma \vdash A \oplus B, \Sigma}\ (\oplus) \qquad \frac{\Gamma \vdash B, \Sigma}{\Gamma \vdash A \oplus B, \Sigma}\ (\oplus)$$

$$\frac{\Gamma \vdash A, \Sigma \qquad \Delta \vdash B, \Theta}{\Gamma, \Delta \vdash A \otimes B, \Sigma, \Theta}\ (\otimes) \qquad \frac{\Gamma \vdash A, B, \Sigma}{\Gamma \vdash A \parr B, \Sigma}\ (\parr)$$

(and **even** more rules)

Linear Logic

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(and **even** more rules)

- Good properties
- Generalizes both classical and intuitionistic logics
- Linear use of hypotheses: A implies B means A consumed to prove B

Restaurant Menu

Menu	35€
Entree	Quiche or Salmon
Plat	Pasta or Duck
Dessert	Fruit (Banana or Apple according to season) or Cake (Flan or Chocolate according to Chief's mood)
Sides	Water at will

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$\multimap$ linear implication, consume its premise (or as $A \implies B = \neg A \vee B$)

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$\&$ and where we (the client) choose between two options

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- $\otimes$ and where we get both options

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$$35€ \multimap [(Q \,\&\, S) \otimes (P \,\&\, D) \otimes ((B \oplus A) \,\&\, (F \oplus C))$$

- $\multimap$ linear implication, consume its premise (or as $A \implies B = \neg A \vee B$)
- $\&$ and where we (the client) choose between two options
- $\otimes$ and where we get both options
- $\oplus$ or where we (the client) do not choose between two options

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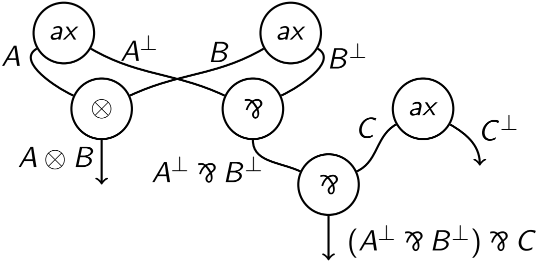
$$35€ \multimap [(Q \,\&\, S) \otimes (P \,\&\, D) \otimes ((B \oplus A) \,\&\, (F \oplus C)) \otimes !W]$$

- $\multimap$ linear implication, consume its premise (or as $A \implies B = \neg A \vee B$)
- $\&$ and where we (the client) choose between two options
- $\otimes$ and where we get both options
- $\oplus$ or where we (the client) do not choose between two options
- $!$ unlimited resource

Proof Nets: graphs as proofs



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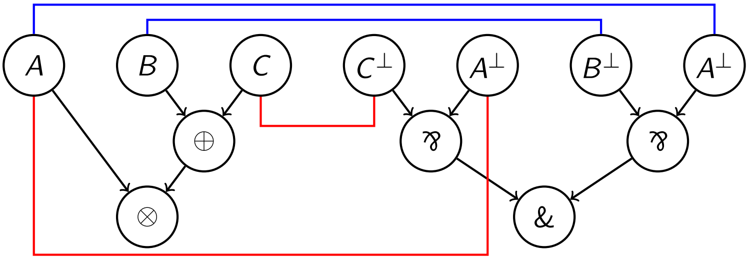

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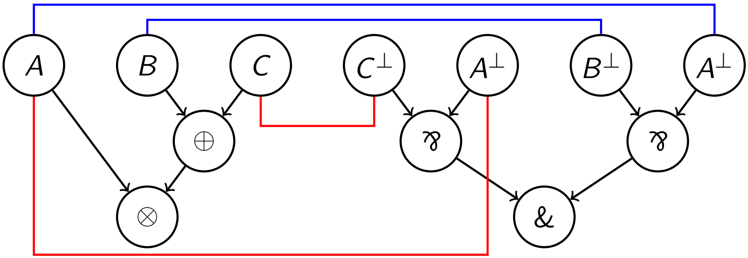

Even better properties: one graph for one “proof”!
But does not work for the full logic.

My thesis

- Use proof nets to find results, e.g. isomorphisms

$$\begin{tikzcd} A \arrow[loop left, "id_A"] \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \arrow[loop right, "id_B"] \end{tikzcd}$$

$$(A \times B) \to C \simeq A \to (B \to C)$$

Associativity	$A \otimes (B \otimes C) = (A \otimes B) \otimes C$		$A \parr (B \parr C) = (A \parr B) \parr C$	
	$A \oplus (B \oplus C) = (A \oplus B) \oplus C$		$A \& (B \& C) = (A \& B) \& C$	
Commutativity	$A \otimes B = B \otimes A$	$A \parr B = B \parr A$	$A \oplus B = B \oplus A$	$A \& B = B \& A$
Neutrality	$A \otimes 1 = A$	$A \parr \bot = A$	$A \oplus 0 = A$	$A \& \top = A$
Distributivity	$A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \& C) = (A \parr B) \& (A \parr C)$	
Annihilation	$A \otimes 0 = 0$	$A \parr \top = \top$		

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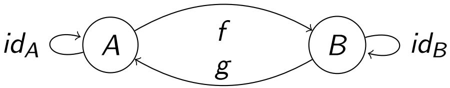

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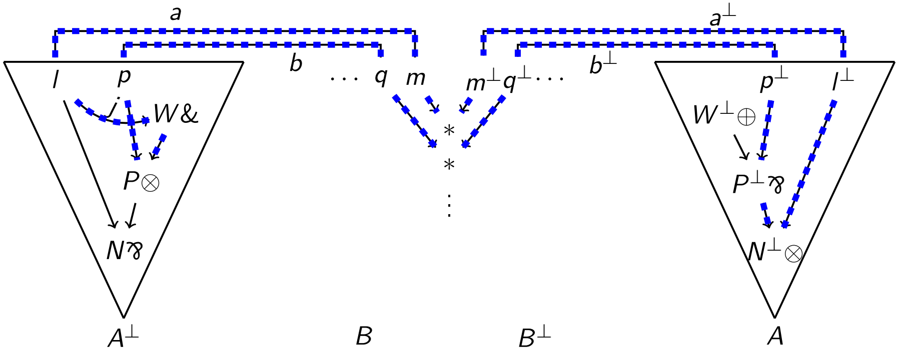

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- Proof nets on more parts of the logic

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- Proof nets on more parts of the logic
- Formalization in Coq

Thank you!