Retractions for Multiplicative Linear Logic

Rémi Di Guardia

ENS Lyon (LIP), France

Bath 2024, 1 March

Isomorphisms

Isomorphisms relate types/formulas/objects A and B which are "the same"

$$A \simeq B$$

$$\begin{tikzcd} A \arrow[loop left, "id_A"] \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \arrow[loop right, "id_B"] \end{tikzcd}$$

Instantiation in λ-calculus, logics,...

Isomorphisms

Isomorphisms relate types/formulas/objects A and B which are “the same”

$$A \simeq B$$

$$\begin{tikzcd} A \arrow[loop left, "id_A"] \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \arrow[loop right, "id_B"] \end{tikzcd}$$

Equational theory for λ-calculus with products and unit / cartesian closed categories [Sol83]

$\times$	$A \times (B \times C) \simeq (A \times B) \times C$		$A \times B \simeq B \times A$
$\times$ and $\to$	$(A \times B) \to C \simeq A \to (B \to C)$		$A \to (B \times C) \simeq (A \to B) \times (A \to C)$
1	$A \times 1 \simeq A$	$1 \to A \simeq A$	$A \to 1 \simeq 1$

Isomorphisms

Isomorphisms relate types/formulas/objects A and B which are "the same"

$$A \simeq B$$

$$\begin{tikzcd} id_A \quad A \arrow[loop left] \arrow[rr, bend left, "f"] & & B \arrow[ll, bend left, "g"] \arrow[loop right] \quad id_B \end{tikzcd}$$

Equational theory for Multiplicative Linear Logic / $\star$-autonomous categories [BDC99]

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$	$A \parr (B \parr C) \simeq (A \parr B) \parr C$
Commutativity	$A \otimes B \simeq B \otimes A$	$A \parr B \simeq B \parr A$
Neutrality	$A \otimes 1 \simeq A$	$A \parr \bot \simeq A$

$$(A \otimes B) \multimap C = (A^\perp \parr B^\perp) \parr C \simeq A^\perp \parr (B^\perp \parr C) = A \multimap (B \multimap C)$$

Isomorphisms

Isomorphisms relate types/formulas/objects A and B which are "the same"

$$A \simeq B$$

$$\begin{tikzcd} A \arrow[loop left, "id_A"] \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \arrow[loop right, "id_B"] \end{tikzcd}$$

Equational theory for Multiplicative-Additive Linear Logic / $\star$-autonomous categories with finite products [DGL23]

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$		$A \parr (B \parr C) \simeq (A \parr B) \parr C$ $A \,\&\, (B \,\&\, C) \simeq (A \,\&\, B) \,\&\, C$	
Commutativity	$A \otimes B \simeq B \otimes A$	$A \parr B \simeq B \parr A$	$A \oplus B \simeq B \oplus A$	$A \,\&\, B \simeq B \,\&\, A$
Neutrality	$A \otimes 1 \simeq A$	$A \parr \bot \simeq A$	$A \oplus 0 \simeq A$	$A \,\&\, \top \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \,\&\, C) \simeq (A \parr B) \,\&\, (A \parr C)$	
Annihilation	$A \otimes 0 \simeq 0$		$A \parr \top \simeq \top$	

Retractions

Retractions relate A and B when A is a “sub-type” of B

$$A \trianglelefteq B$$

$$\begin{tikzcd} A \arrow[loop left, "id_A"] \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \arrow[loop right, dashed, "id_B"] \end{tikzcd}$$

Instantiation in λ-calculus, logics,...

bool $\trianglelefteq$ nat with $f(\text{false}) = 0$, $f(\text{true}) = 1$ and $g(n) = n$ is equal to 1

Definition

Cantor-Bernstein property: if $A \trianglelefteq B$ and $B \trianglelefteq A$ then $A \simeq B$.

Retractions

Retractions relate A and B when A is a “sub-type” of B

$$A \trianglelefteq B$$

$$\begin{tikzcd} A \arrow[loop left, "id_A"] \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \arrow[loop right, dashed, "id_B"] \end{tikzcd}$$

Equational theory for simply typed affine λ-calculus [RU02]

$\simeq$	$A \to B \to C \simeq B \to A \to C$
$\lhd \ (= \trianglelefteq \setminus \simeq)$	$A \lhd B \to A$ $A \lhd (A \to X) \to X$ if A is $Y_1 \to Y_2 \to \cdots \to X$

Retractions

Retractions relate A and B when A is a “sub-type” of B

$$A \trianglelefteq B$$

$$\begin{tikzcd} A \arrow[loop left, "id_A"] \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \arrow[loop right, dashed, "id_B"] \end{tikzcd}$$

Equational theory for Multiplicative Linear Logic [UNKNOWN]

$\simeq$	associativity and commutativity of $\otimes$ and $⅋$, neutrality of 1 and $\perp$
$\lhd \quad (= \trianglelefteq \setminus \simeq)$	???

Plan

1. Retractions in (fragments of) Linear Logic
2. Definitions
   - Proof Net
   - Retraction
3. Good properties of retractions in MLL – or why it should be easy
4. Retractions of the shape $X \trianglelefteq \cdot$ (universal super-types)
   - Looking for a pattern
   - Quasi-Beffara
   - Beffara $X \lhd X \otimes (X^\perp \parr X)$
5. Difficulties for $A \trianglelefteq B$

Linear Logic

Formulas

$$
\begin{aligned}
A, B \; \coloneqq \; & \mid X \mid X^{\perp} && \text{(atom)} \\
& \mid A \parr B \mid A \otimes B \mid \perp \mid 1 && \text{(multiplicative)} \\
& \mid A \,\&\, B \mid A \oplus B \mid \top \mid 0 && \text{(additive)} \\
& \mid ?A \mid !A && \text{(exponential)}
\end{aligned}
$$

Linear Logic

Formulas

$$
\begin{aligned}
A, B \;\; \coloneqq \; & \mid X \mid X^{\perp} && \text{(atom)} \\
& \mid A \parr B \mid A \otimes B \mid \perp \mid 1 && \text{(multiplicative)} \\
& \mid A \mathbin{\&} B \mid A \oplus B \mid \top \mid 0 && \text{(additive)} \\
& \mid ?A \mid !A && \text{(exponential)}
\end{aligned}
$$

Fragment = subset of formulas keeping atoms and the :

- additive $\rightarrow$ Additive Linear Logic (ALL);
- multiplicative and exponential $\rightarrow$ Multiplicative Exponential Linear Logic (MELL);
- ...

Retractions and Provability

Fact

$$!X \trianglelefteq\, !X \otimes !(X \otimes A) \iff A \textit{ is provable}$$
$$X \trianglelefteq X \,\&\, (X \otimes A) \iff A \textit{ is provable}$$
$$A \trianglelefteq A \oplus B \iff B \vdash A \textit{ is provable}$$

Retractions and Provability

Fact

$$!X \trianglelefteq\, !X \otimes !(X \otimes A) \iff A \text{ is provable}$$
$$X \trianglelefteq X \,\&\, (X \otimes A) \iff A \text{ is provable}$$
$$A \trianglelefteq A \oplus B \iff B \vdash A \text{ is provable}$$

Fragment	Provability
LL	Undecidable ☹
MELL	TOWER-hard ☹
	(decidability is open)
MALL	PSPACE-complete ☹
ALL	P-complete

(an overview of these results can be found in [Lin95])

Retractions and Provability

Fact

$$!X \trianglelefteq\, !X \otimes\, !(X \otimes A) \iff A \text{ is provable}$$
$$X \trianglelefteq X \,\&\, (X \otimes A) \iff A \text{ is provable}$$
$$A \trianglelefteq A \oplus B \iff B \vdash A \text{ is provable}$$

Fragment	Provability
LL	Undecidable ☹
MELL	TOWER-hard ☹
	(decidability is open)
MALL	PSPACE-complete ☹
ALL	P-complete

(an overview of these results can be found in [Lin95])

No example found for Multiplicative Linear Logic, which is often the simpler fragment ☺

Plan

Formula & Sequent

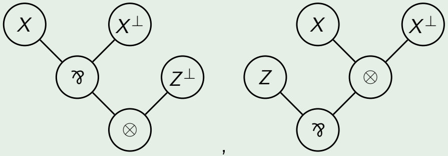
Formulas

$$A, B \ ::= \ X \mid \overset{\textit{not}}{X^{\perp}} \mid A \overset{\textit{and}}{\otimes} B \mid A \overset{\textit{or}}{\parr} B$$

Duality

$$(X^{\perp})^{\perp} = X$$
$$(A \otimes B)^{\perp} = B^{\perp} \parr A^{\perp}$$
$$(A \parr B)^{\perp} = B^{\perp} \otimes A^{\perp}$$

Examples

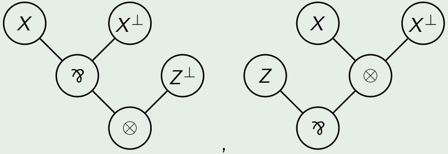


Formula & Sequent

Formulas

$$A, B \ ::= \ X \mid \overset{not}{X^\perp} \mid A \overset{and}{\otimes} B \mid A \overset{or}{\parr} B$$

Examples



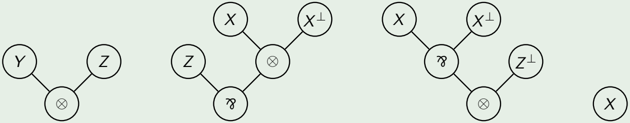
Duality

$$(X^\perp)^\perp = X$$
$$(A \otimes B)^\perp = B^\perp \parr A^\perp$$
$$(A \parr B)^\perp = B^\perp \otimes A^\perp$$

Sequent

$$\vdash A_1, \ldots, A_n$$

Example



What about the units?

Result from [BDC99]

Let A and B be two formulas without sub-formulas of the shape $- \otimes 1$, $1 \otimes -$, $\perp \parr -$ nor $- \parr \perp$. Take π and π' cut-free proofs respectively of $\vdash A^\perp, B$ and $\vdash B^\perp, A$. Then all 1 and $\perp$-rules in π and π' belongs to the following pattern:

$$\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash \perp, 1}\,(\perp)$$

So can replace the units by atoms, up to isomorphism.

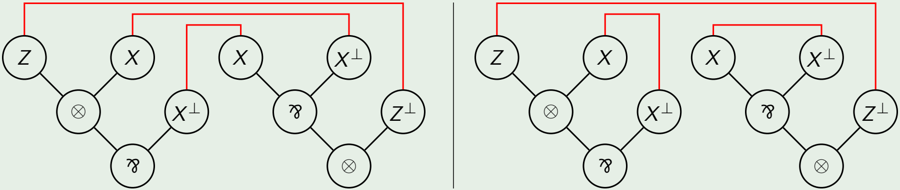
(Also easy to check the mix-rules do not matter, for the identity has none.)

Proof Structure

Proof Structure

Sequent with edges between dual leaves (some X and $X^\perp$), these edges partitioning the leaves of the sequent.

Examples



Correctness & Proof Net

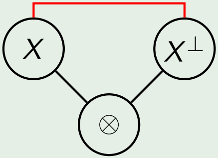
Correctness Graph

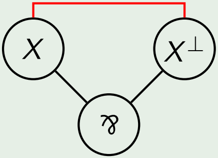
In a proof structure, keep only one premise of each $⅋$-node.

Danos-Regnier Correctness Criterion

A proof structure is *correct*, and called a *proof net*, if all its correctness graphs are acyclic and connected (i.e. are trees).

Toy examples





Correctness & Proof Net

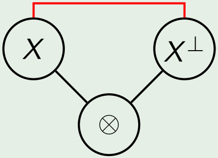
Correctness Graph

In a proof structure, keep only one premise of each $\parr$-node.

Danos-Regnier Correctness Criterion

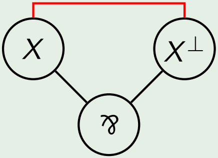
A proof structure is *correct*, and called a *proof net*, if all its correctness graphs are acyclic and connected (i.e. are trees).

Toy examples



Not acyclic (but connected)

INCORRECT



Correctness & Proof Net

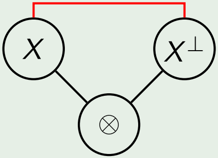
Correctness Graph

In a proof structure, keep only one premise of each ⅋-node.

Danos-Regnier Correctness Criterion

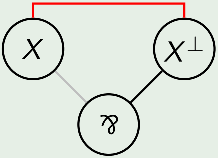
A proof structure is *correct*, and called a *proof net*, if all its correctness graphs are acyclic and connected (i.e. are trees).

Toy examples



Not acyclic (but connected)

INCORRECT



Acyclic and connected

Correctness & Proof Net

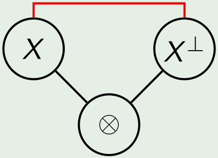
Correctness Graph

In a proof structure, keep only one premise of each $⅋$-node.

Danos-Regnier Correctness Criterion

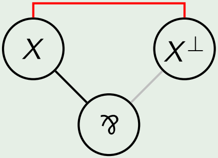
A proof structure is *correct*, and called a *proof net*, if all its correctness graphs are acyclic and connected (i.e. are trees).

Toy examples



Not acyclic (but connected)

INCORRECT



Acyclic and connected

CORRECT

Correctness & Proof Net

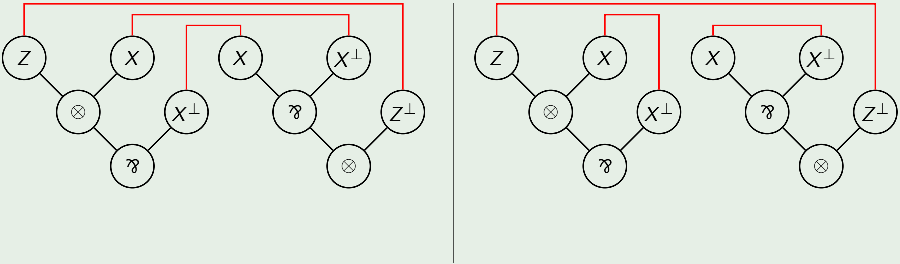
Correctness Graph

In a proof structure, keep only one premise of each $\parr$-node.

Danos-Regnier Correctness Criterion

A proof structure is *correct*, and called a *proof net*, if all its correctness graphs are acyclic and connected (i.e. are trees).

Examples



Correctness & Proof Net

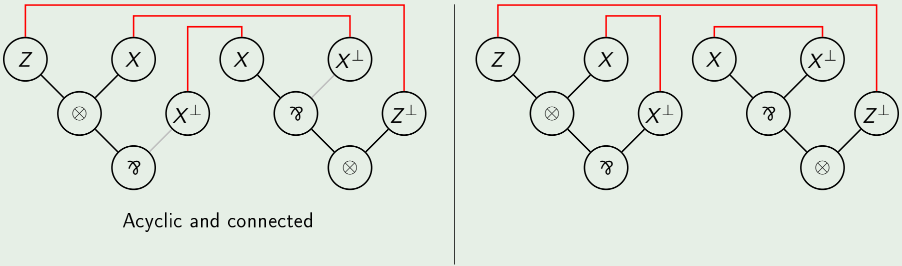
Correctness Graph

In a proof structure, keep only one premise of each $⅋$-node.

Danos-Regnier Correctness Criterion

A proof structure is *correct*, and called a *proof net*, if all its correctness graphs are acyclic and connected (i.e. are trees).

Examples



Acyclic and connected

Correctness & Proof Net

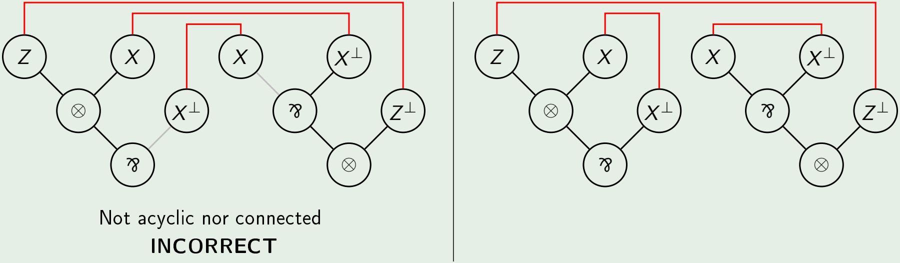
Correctness Graph

In a proof structure, keep only one premise of each $\parr$-node.

Danos-Regnier Correctness Criterion

A proof structure is *correct*, and called a *proof net*, if all its correctness graphs are acyclic and connected (i.e. are trees).

Examples



Not acyclic nor connected

INCORRECT

Correctness & Proof Net

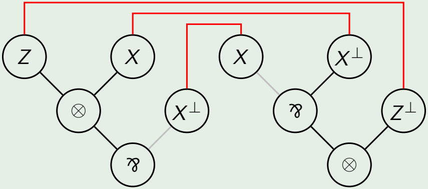
Correctness Graph

In a proof structure, keep only one premise of each $\parr$-node.

Danos-Regnier Correctness Criterion

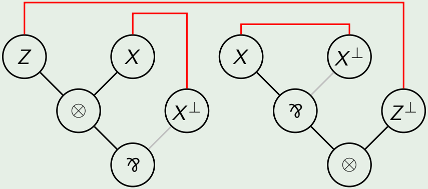
A proof structure is *correct*, and called a *proof net*, if all its correctness graphs are acyclic and connected (i.e. are trees).

Examples



Not acyclic nor connected

INCORRECT



Acyclic and connected

Correctness & Proof Net

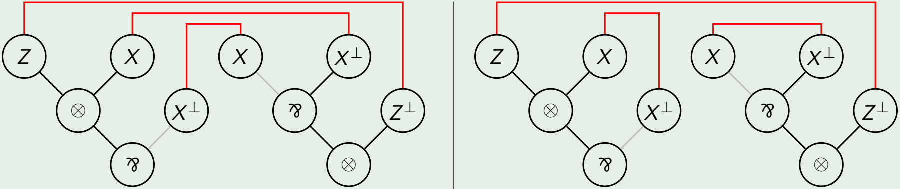
Correctness Graph

In a proof structure, keep only one premise of each $\parr$-node.

Danos-Regnier Correctness Criterion

A proof structure is *correct*, and called a *proof net*, if all its correctness graphs are acyclic and connected (i.e. are trees).

Examples



Not acyclic nor connected

INCORRECT

Acyclic and connected

Correctness & Proof Net

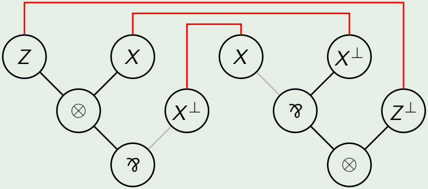
Correctness Graph

In a proof structure, keep only one premise of each $\parr$-node.

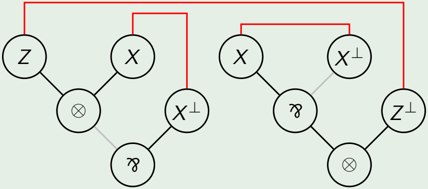
Danos-Regnier Correctness Criterion

A proof structure is *correct*, and called a *proof net*, if all its correctness graphs are acyclic and connected (i.e. are trees).

Examples



Not acyclic nor connected
INCORRECT



Acyclic and connected

Correctness & Proof Net

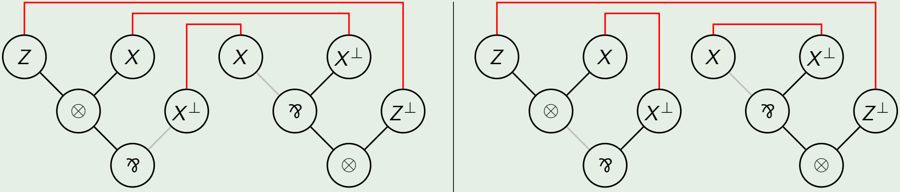
Correctness Graph

In a proof structure, keep only one premise of each ⅋-node.

Danos-Regnier Correctness Criterion

A proof structure is *correct*, and called a *proof net*, if all its correctness graphs are acyclic and connected (i.e. are trees).

Examples



Not acyclic nor connected
INCORRECT

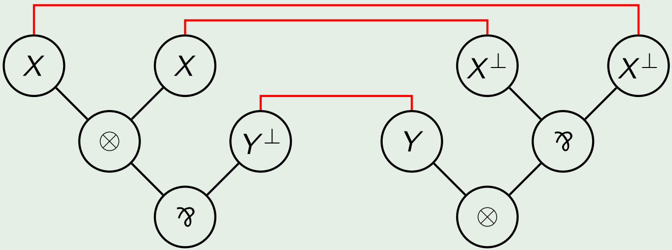
Acyclic and connected
CORRECT

Identity proof net

Identity proof structure of A

In the sequent $\vdash A^\perp, A$, link each leaf in A to the dual one in $A^\perp$.

Example: $A = Y \otimes (X^\perp \parr X^\perp)$



Lemma

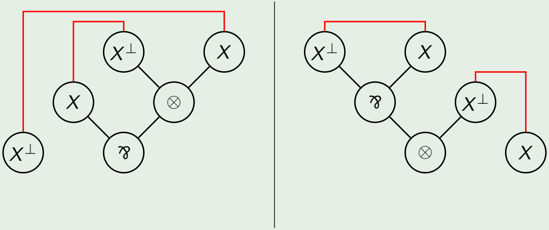
An identity proof structure is correct.

Composition by cut

Composition

Putting side by side a proof structure on $\vdash \Gamma, A$ and one on $\vdash A^\perp, \Delta$, then adding a $*$-node between the roots of A and $A^\perp$.

Example

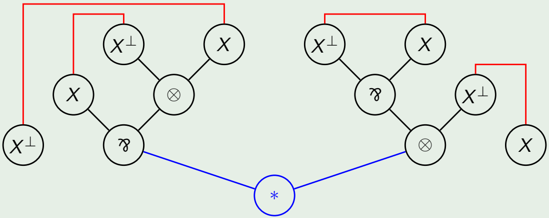


Composition by cut

Composition

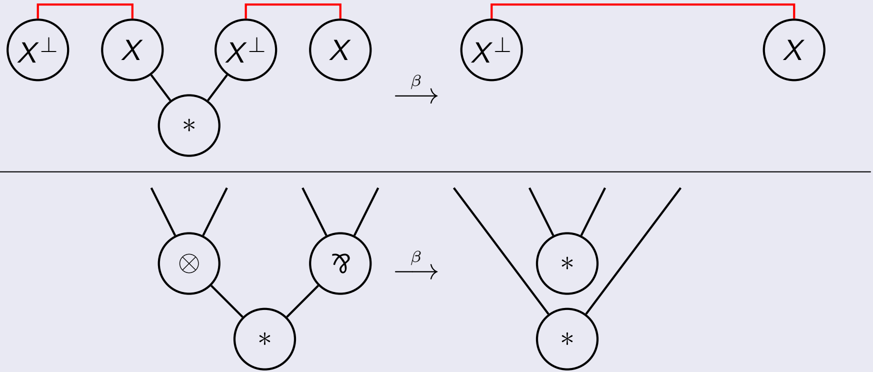
Putting side by side a proof structure on $\vdash \Gamma, A$ and one on $\vdash A^\perp, \Delta$, then adding a $*$-node between the roots of A and $A^\perp$.

Example



Cut elimination

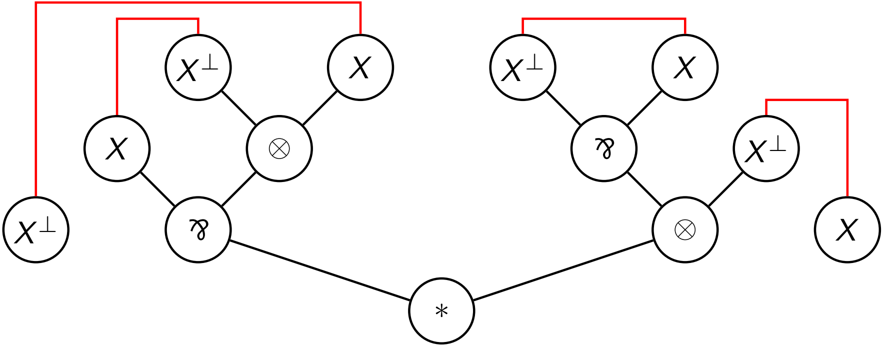
Cut elimination



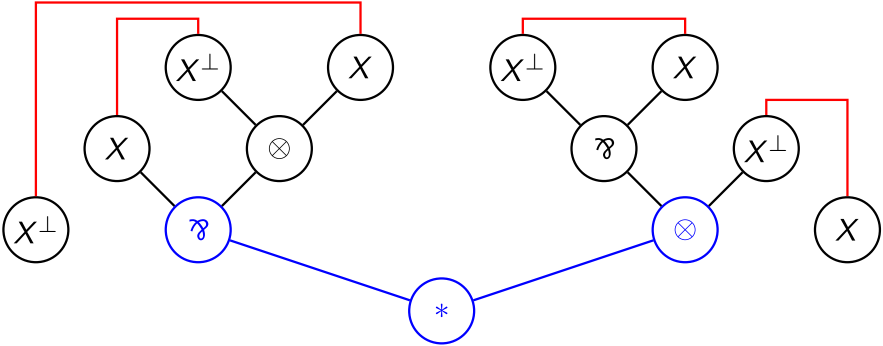
Lemma

Cut elimination preserves correction, is confluent and strongly normalizing.

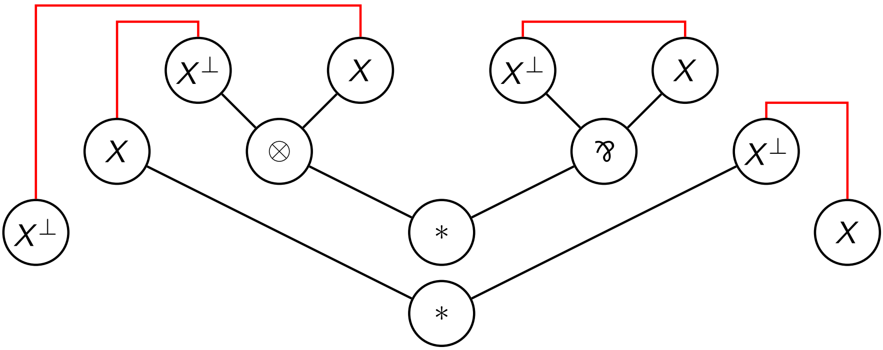
Example of cut elimination



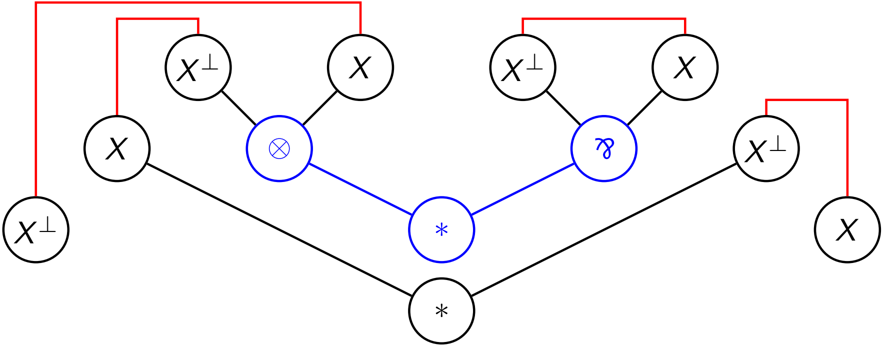
Example of cut elimination



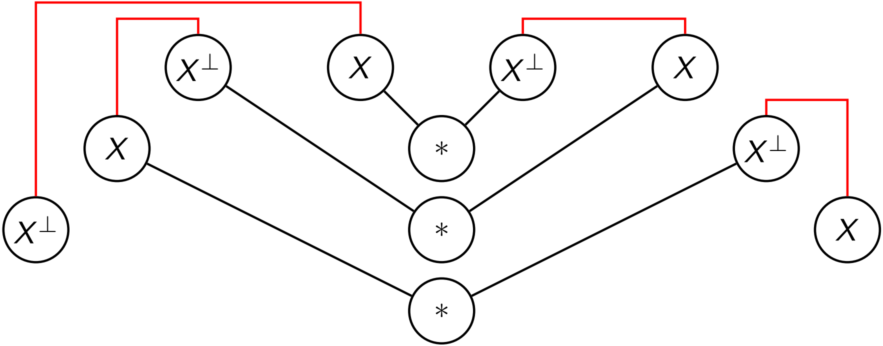
Example of cut elimination



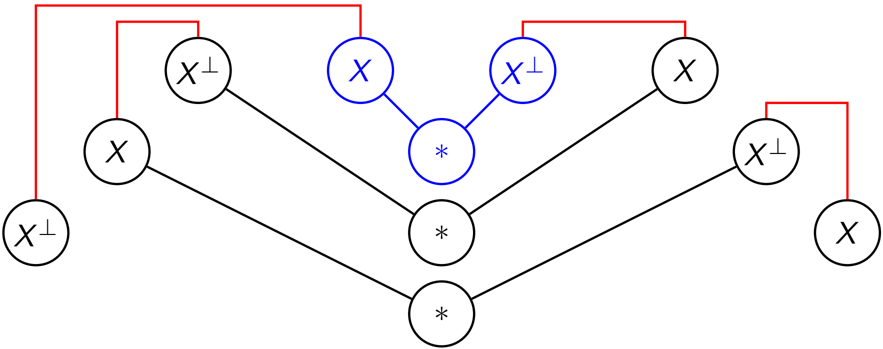
Example of cut elimination



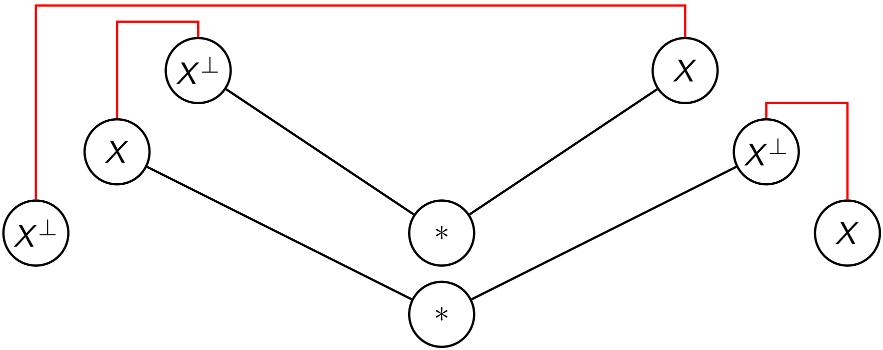
Example of cut elimination



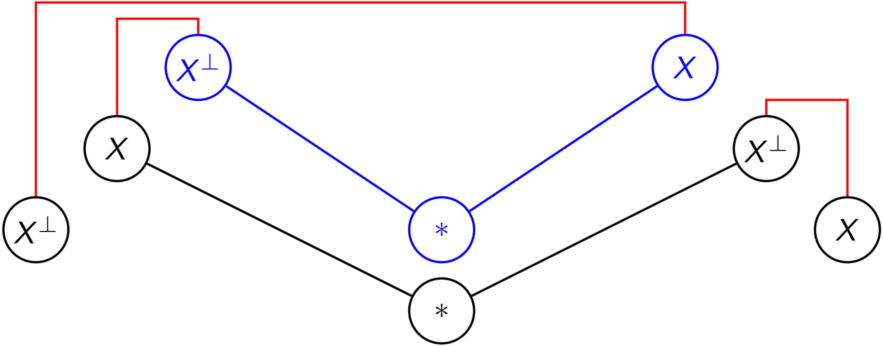
Example of cut elimination



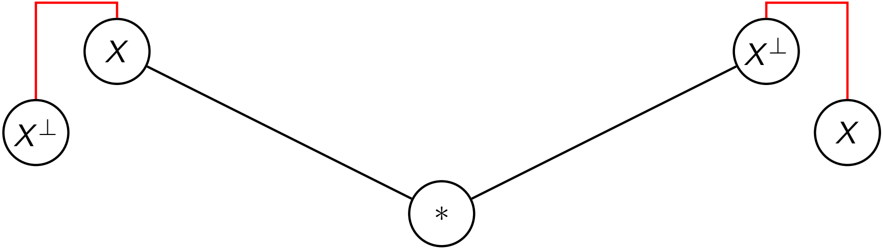
Example of cut elimination



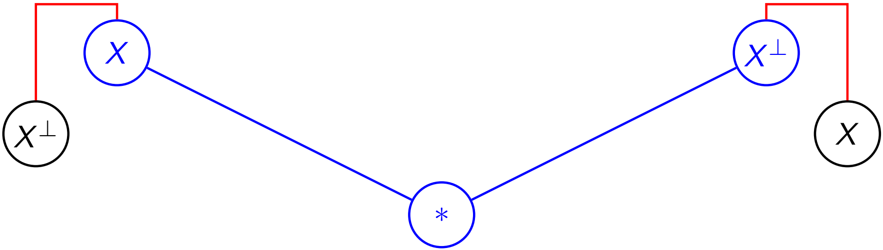
Example of cut elimination



Example of cut elimination



Example of cut elimination



Example of cut elimination

Retraction

In category theory

$$A \trianglelefteq B$$

$$\begin{tikzcd} A \arrow[loop left, "id_A"] \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd}$$

In λ-calculus

Retraction $A \trianglelefteq B$

Terms $M : A \to B$ and $N : B \to A$ such that

$$N \circ M =_{\beta\eta} \lambda x^A.x$$

Retraction

In category theory

$$A \trianglelefteq B$$

$$\begin{tikzcd} A \arrow[loop left, "id_A"] \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd}$$

In λ-calculus

Retraction $A \trianglelefteq B$

Terms $M : A \to B$ and $N : B \to A$ such that

$$N \circ M =_{\beta\eta} \lambda x^A.x$$

In multiplicative linear logic

Retraction $A \trianglelefteq B$

Proof nets $\mathcal{R}$ of $\vdash A^\perp, B$ and $\mathcal{S}$ of $\vdash B^\perp, A$ whose composition by cut over B yields, after cut elimination, the identity proof net of A.

Retraction

In category theory

$$A \trianglelefteq B$$

$$\begin{tikzcd} A \arrow[loop left, "id_A"] \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd}$$

In λ-calculus

Retraction $A \trianglelefteq B$

Terms $M : A \to B$ and $N : B \to A$ such that

$$N \circ M =_{\beta\eta} \lambda x^A.x$$

In multiplicative linear logic

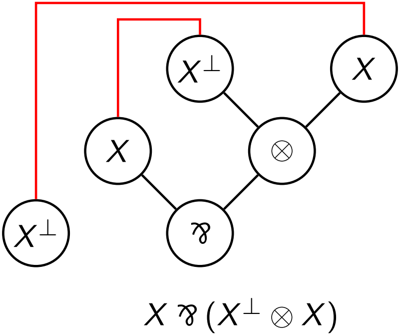
Retraction $A \trianglelefteq B$

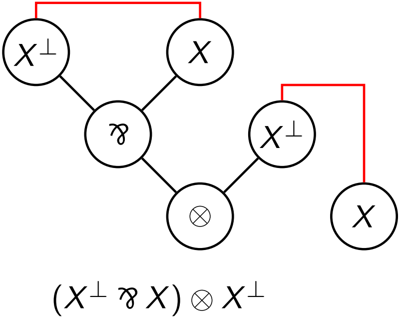
Proof nets $\mathcal{R}$ of $\vdash A^\perp, B$ and $\mathcal{S}$ of $\vdash B^\perp, A$ whose composition by cut over B yields, after cut elimination, the identity proof net of A.

$$A \trianglelefteq B \iff A^\perp \trianglelefteq B^\perp$$

Beffara's retraction

$$X \lhd X \parr (X^\perp \otimes X) \qquad \text{or dualy} \qquad X \lhd X \otimes (X^\perp \parr X)$$





Can also be seen as $X \lhd (X \multimap X) \multimap X$

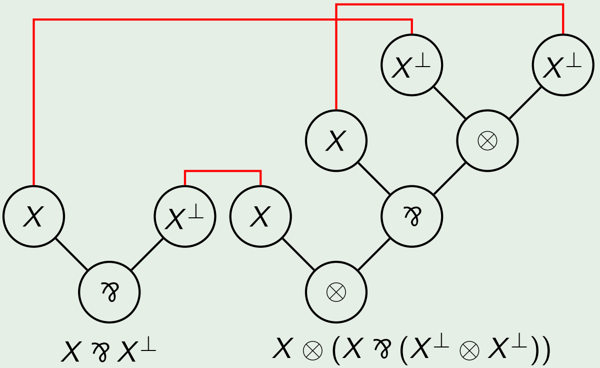
Plan

Half-Bipartiteness

Definition

A proof-net on $\vdash A, \Gamma$ is *half-bipartite* in A if there is no link between leaves of A.

Example



Half-bipartite in $X \parr X^\perp$ but not in $X \otimes (X \parr (X^\perp \otimes X^\perp))$.

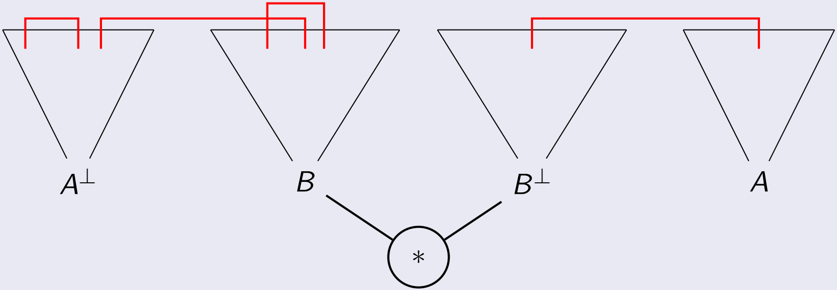
Half-Bipartiteness in retractions

Lemma

Proof nets of $A \trianglelefteq B$ are half-bipartite in $A^\perp$ and A respectively.

Proof.

A link between leaves of $A^\perp$ or A would survive cut elimination, and appears in the resulting identity proof net: contradiction.



$\square$

Consequences of Half-Bipartiteness

Corollary

Up to renaming leaves, in $A \trianglelefteq B$ one can assume leaves of A to be distinct atoms $X, Y, Z, \ldots$ without any $X^\perp, Y^\perp, Z^\perp, \ldots$ in A.

Proof.

Can rename leaves of A to respect this; no clash by half-bipartiteness.
A renaming preserves correction and steps of cut elimination. $\square$

Consequences of Half-Bipartiteness

Corollary

Up to renaming leaves, in $A \trianglelefteq B$ one can assume leaves of A to be distinct atoms $X, Y, Z, \ldots$ without any $X^\perp, Y^\perp, Z^\perp, \ldots$ in A.

Proof.

Can rename leaves of A to respect this; no clash by half-bipartiteness.
A renaming preserves correction and steps of cut elimination. $\square$

In this setting:

Retraction $A \trianglelefteq B$

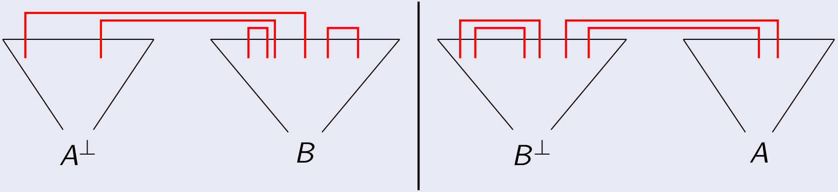
Proof nets $\mathcal{R}$ of $\vdash A^\perp, B$ and $\mathcal{S}$ of $\vdash B^\perp, A$ ~~whose composition by cut over B yields, after cut elimination, the identity proof net of A~~.

Property on sizes

Theorem

Let A and B be unit-free MLL formulas such that $A \trianglelefteq B$.
Then $s(B) = s(A) + 2 \times n$, with $n = 0$ iff $A \simeq B$.

Proof.



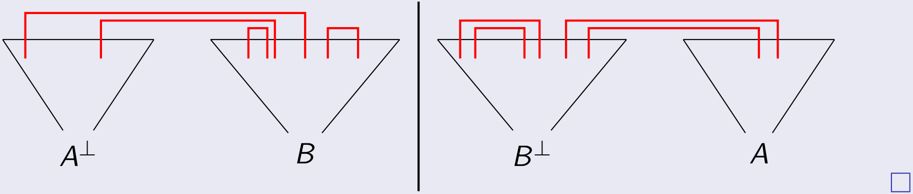
$\square$

Property on sizes

Theorem

Let A and B be unit-free MLL formulas such that $A \trianglelefteq B$.
Then $s(B) = s(A) + 2 \times n$, with $n = 0$ iff $A \simeq B$.

Proof.



Corollary

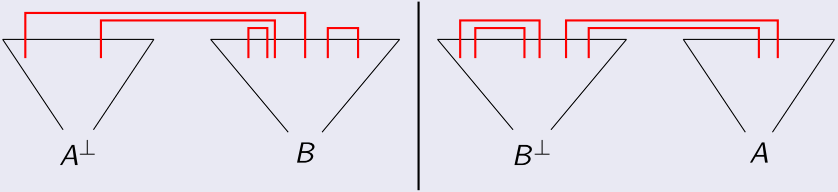
Cantor-Bernstein holds for unit-free MLL, and then for MLL.

Property on sizes

Theorem

Let A and B be unit-free MLL formulas such that $A \trianglelefteq B$.
Then $s(B) = s(A) + 2 \times n$, with $n = 0$ iff $A \simeq B$.

Proof.



□

Corollary

Cantor-Bernstein holds for unit-free MLL, and then for MLL.

$X \otimes Y \ntrianglelefteq X \parr Y$, $X \otimes (Y \parr Z) \ntrianglelefteq Y \parr (X \otimes Z)$, ...

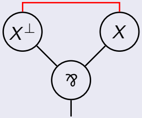
Plan

1. Retractions in (fragments of) Linear Logic
2. Definitions
   - Proof Net
   - Retraction
3. Good properties of retractions in MLL – or why it should be easy
4. **Retractions of the shape $X \trianglelefteq \cdot$ (universal super-types)**
   - Looking for a pattern
   - Quasi-Beffara
   - Beffara $X \lhd X \otimes (X^{\perp} \parr X)$
5. Difficulties for $A \trianglelefteq B$

Key Result

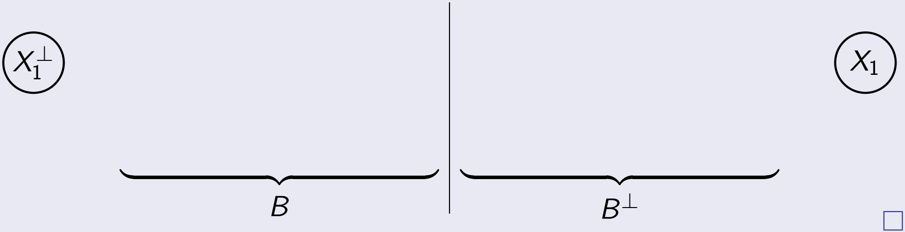
Lemma

In $X \lhd B$ one of the two proof nets contains:



Proof.

We build a sequence (GOI path) finding such a pattern.

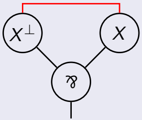


□

Key Result

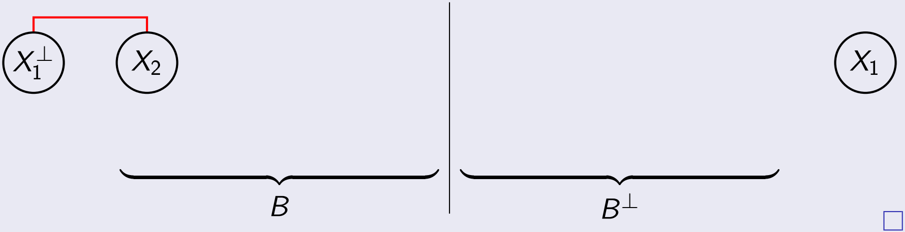
Lemma

In $X \lhd B$ one of the two proof nets contains:



Proof.

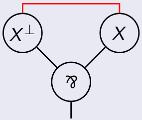
We build a sequence (GOI path) finding such a pattern.



Key Result

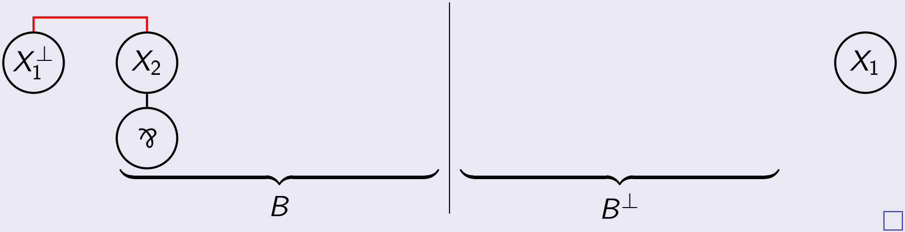
Lemma

In $X \lhd B$ one of the two proof nets contains:



Proof.

We build a sequence (GOI path) finding such a pattern.

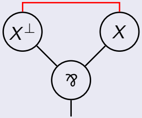


□

Key Result

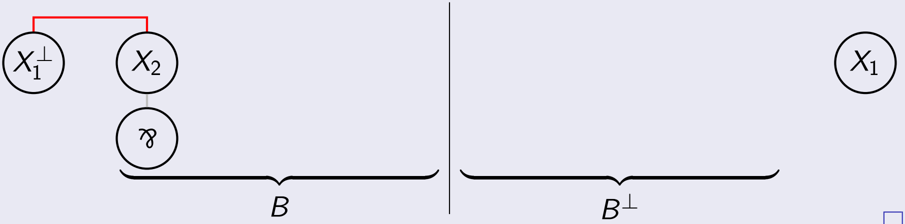
Lemma

In $X \lhd B$ one of the two proof nets contains:



Proof.

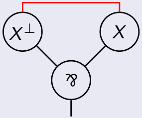
We build a sequence (GOI path) finding such a pattern.



Key Result

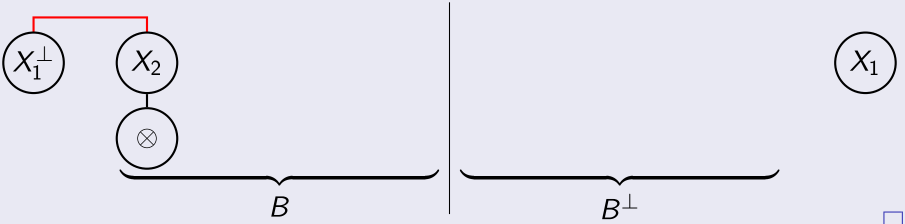
Lemma

In $X \triangleleft B$ one of the two proof nets contains:



Proof.

We build a sequence (GOI path) finding such a pattern.

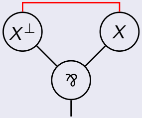


□

Key Result

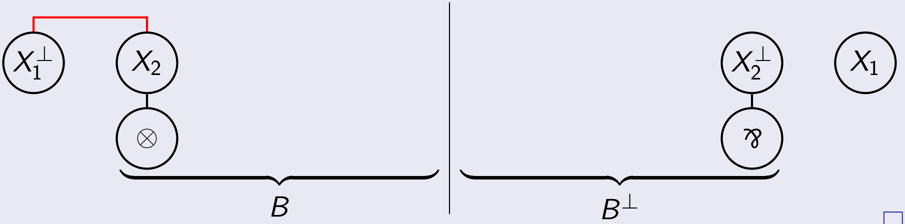
Lemma

In $X \lhd B$ one of the two proof nets contains:



Proof.

We build a sequence (GOI path) finding such a pattern.

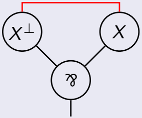


□

Key Result

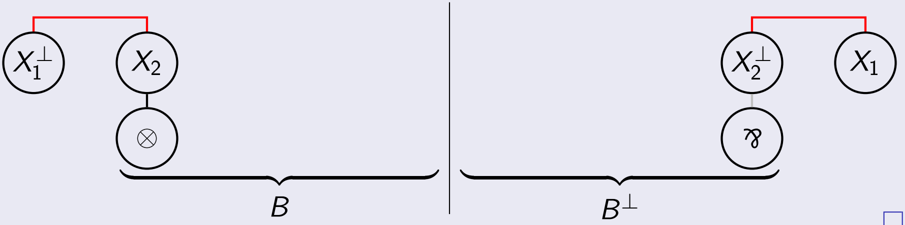
Lemma

In $X \lhd B$ one of the two proof nets contains:



Proof.

We build a sequence (GOI path) finding such a pattern.

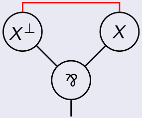


□

Key Result

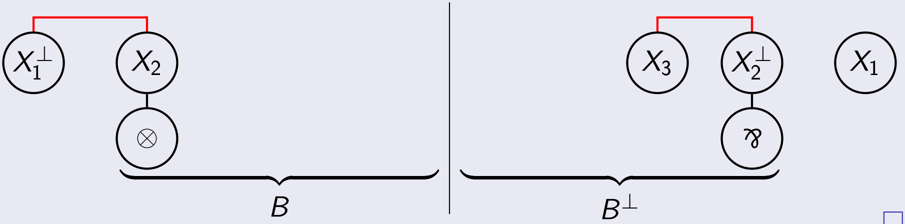
Lemma

In $X \lhd B$ one of the two proof nets contains:



Proof.

We build a sequence (GOI path) finding such a pattern.

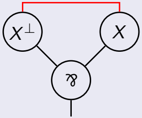


□

Key Result

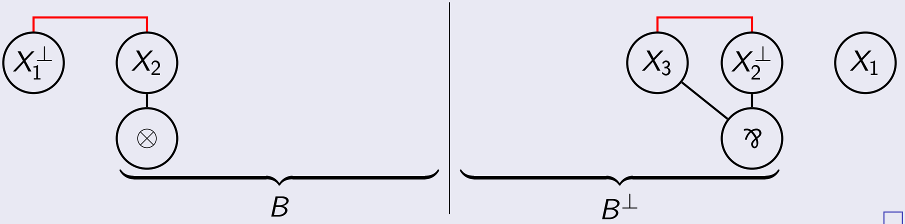
Lemma

In $X \lhd B$ one of the two proof nets contains:



Proof.

We build a sequence (GOI path) finding such a pattern.

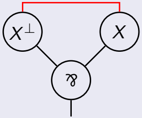


□

Key Result

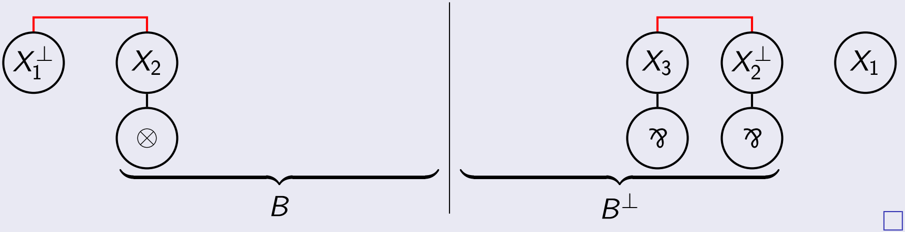
Lemma

In $X \lhd B$ one of the two proof nets contains:



Proof.

We build a sequence (GOI path) finding such a pattern.

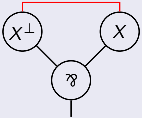


□

Key Result

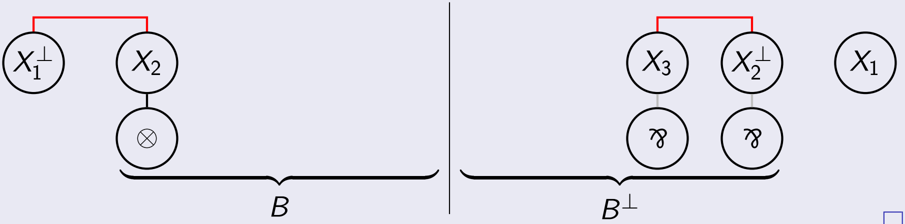
Lemma

In $X \lhd B$ one of the two proof nets contains:



Proof.

We build a sequence (GOI path) finding such a pattern.

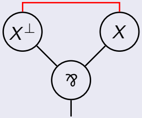


$\square$

Key Result

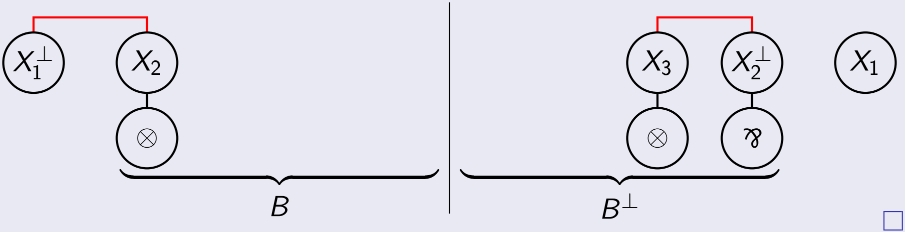
Lemma

In $X \lhd B$ one of the two proof nets contains:



Proof.

We build a sequence (GOI path) finding such a pattern.
Invariant: every X of B is above a $\otimes$, and every $X^\perp$ above a $\parr$.

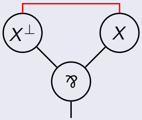


$\square$

Key Result

Lemma

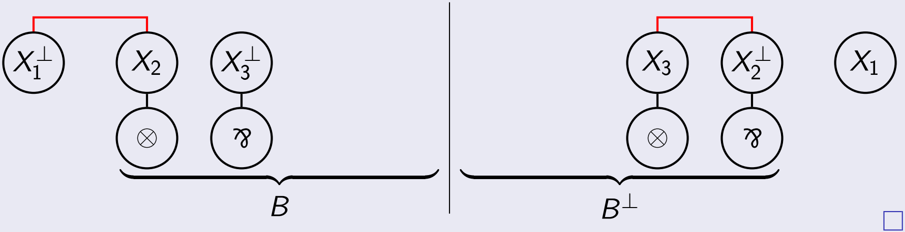
In $X \lhd B$ one of the two proof nets contains:



Proof.

We build a sequence (GOI path) finding such a pattern.

Invariant: every X of B is above a $\otimes$, and every $X^\perp$ above a $\parr$.

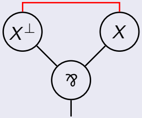


$\square$

Key Result

Lemma

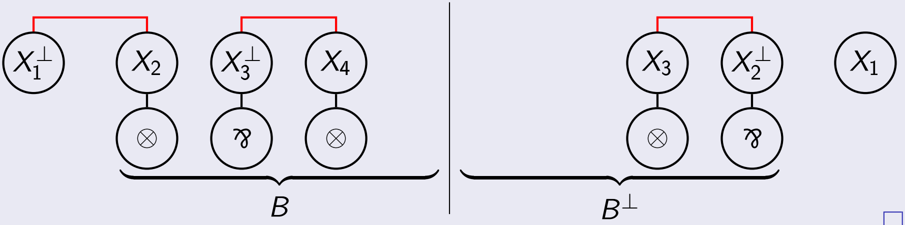
In $X \lhd B$ one of the two proof nets contains:



Proof.

We build a sequence (GOI path) finding such a pattern.

<u>Invariant:</u> every X of B is above a $\otimes$, and every $X^\perp$ above a $\parr$.

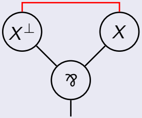


□

Key Result

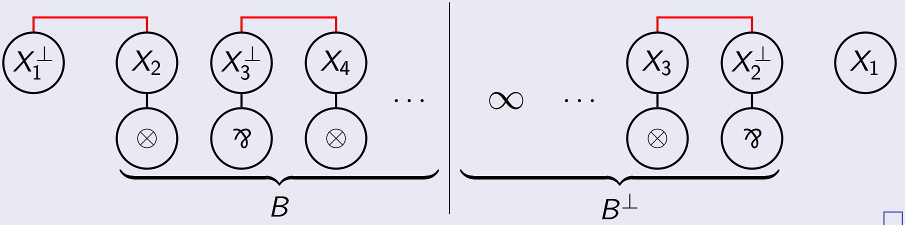
Lemma

In $X \lhd B$ one of the two proof nets contains:



Proof.

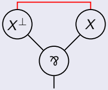
We build a sequence (GOI path) finding such a pattern.
<u>Invariant:</u> every X of B is above a $\otimes$, and every $X^\perp$ above a $⅋$.

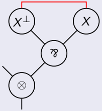


□

Extended pattern

Lemma

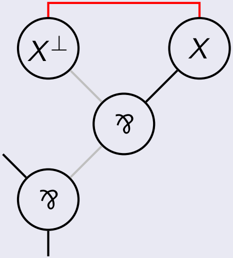
If 

has a node below it, then this is a 

.

Proof.

The connector below the pattern cannot be a ⅋ by connectivity:

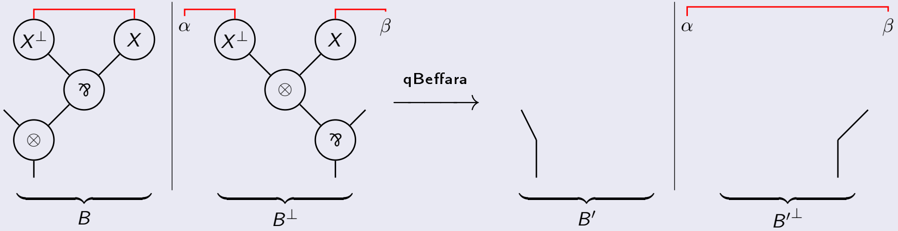


□

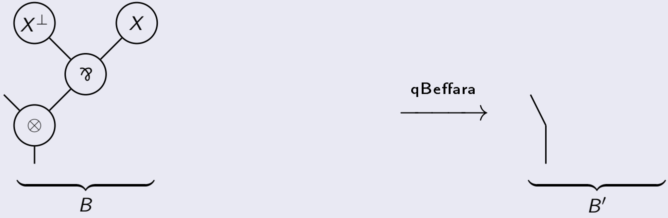
Quasi-Beffara

Definition

Quasi-Beffara is this local transformation on proofs of a retraction $A \trianglelefteq B$:



By extension, this defines two transformations on a formula B (by duality):



Coherence of Quasi-Beffara

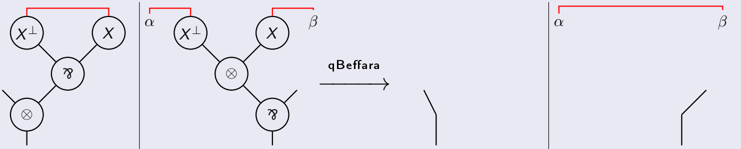
Lemma

If $(\mathcal{R}, \mathcal{S})$ *are proofs of* $A \trianglelefteq B$ *and* $(\mathcal{R}, \mathcal{S}) \xrightarrow{\text{qBeffara}} (\mathcal{R}', \mathcal{S}')$*, then* $(\mathcal{R}', \mathcal{S}')$ *are proofs of* $A \trianglelefteq B'$ *with* $B \xrightarrow{\text{qBeffara}} B'$.

Proof.

Quasi-Beffara preserves:

- being a proof structure



$\square$

Coherence of Quasi-Beffara

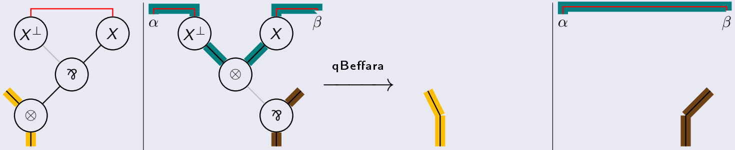
Lemma

If $(\mathcal{R}, \mathcal{S})$ *are proofs of* $A \trianglelefteq B$ *and* $(\mathcal{R}, \mathcal{S}) \xrightarrow{\text{qBeffara}} (\mathcal{R}', \mathcal{S}')$, *then* $(\mathcal{R}', \mathcal{S}')$ *are proofs of* $A \trianglelefteq B'$ *with* $B \xrightarrow{\text{qBeffara}} B'$.

Proof.

Quasi-Beffara preserves:

- being a proof structure
- acyclicity of correctness graphs



Coherence of Quasi-Beffara

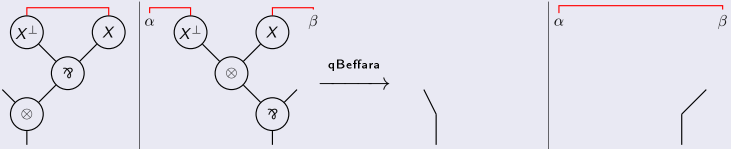
Lemma

If $(\mathcal{R}, \mathcal{S})$ are proofs of $A \trianglelefteq B$ and $(\mathcal{R}, \mathcal{S}) \xrightarrow{\text{qBeffara}} (\mathcal{R}', \mathcal{S}')$, then $(\mathcal{R}', \mathcal{S}')$ are proofs of $A \trianglelefteq B'$ with $B \xrightarrow{\text{qBeffara}} B'$.

Proof.

Quasi-Beffara preserves:

- being a proof structure
- acyclicity of correctness graphs
- the number $|V| + |\parr| - |E|$ of cc. of any correctness graph: it removes 4 vertices, including 1 $\parr$, and 5 edges



$\square$

Coherence of Quasi-Beffara

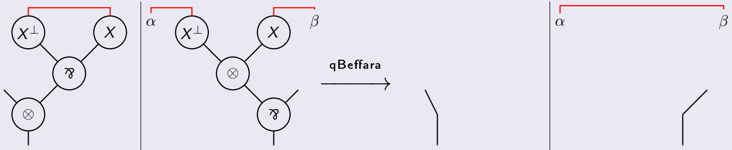
Lemma

If $(\mathcal{R}, \mathcal{S})$ are proofs of $A \trianglelefteq B$ and $(\mathcal{R}, \mathcal{S}) \xrightarrow{\text{qBeffara}} (\mathcal{R}', \mathcal{S}')$, then $(\mathcal{R}', \mathcal{S}')$ are proofs of $A \trianglelefteq B'$ with $B \xrightarrow{\text{qBeffara}} B'$.

Proof.

Quasi-Beffara preserves:

- being a proof structure
- acyclicity of correctness graphs
- the number $|V| + |\parr| - |E|$ of cc. of any correctness graph
- (normal form for cut elimination)



$\square$

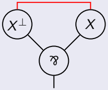
Completeness of Quasi-Beffara

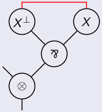
Proposition

If $X \trianglelefteq B$ *then* $B \xrightarrow{\text{qBeffara}}{}^{*} X$.

Proof.

By induction on the size of B. Trivial if $B = X$.
Else, by previous results:

1. we find some 

2. which is a 

3. $B \xrightarrow{\text{qBeffara}} B'$, $X \trianglelefteq B'$ and B' of strictly smaller size

$\square$

Quasi-Beffara & Beffara (statement)

- Remember Beffara's retraction:

$$X \lhd X \otimes (X^\perp \parr X) \qquad X \lhd X \parr (X^\perp \otimes X)$$

- Corresponding transformations inside a formula:

$$X \otimes (X^\perp \parr X) \xrightarrow{\text{Beffara}} X \qquad X \parr (X^\perp \otimes X) \xrightarrow{\text{Beffara}} X$$

Quasi-Beffara & Beffara (statement)

- Remember Beffara's retraction:

$$X \lhd X \otimes (X^\perp \parr X) \qquad X \lhd X \parr (X^\perp \otimes X)$$

- Corresponding transformations inside a formula:

$$X \otimes (X^\perp \parr X) \xrightarrow{\text{Beffara}} X \qquad X \parr (X^\perp \otimes X) \xrightarrow{\text{Beffara}} X$$

Proposition

If $B \xrightarrow{\text{qBeffara}}{}^* X$, *then* $B \xrightarrow{\text{Beffara}}{}^* X$ ***up to isomorphism***
(associativity and commutativity of $\parr$ *and* $\otimes$*)*

Quasi-Beffara & Beffara (proof)

By induction on the size of B.

Base cases: $B \in \{X; X \parr (X^\perp \otimes X); X \otimes (X^\perp \parr X)\}$

Inductive case: $B \xrightarrow{\text{qBeffara}} B_1 \xrightarrow{\text{Beffara}} B_2 \xrightarrow{\text{Beffara}}{}^{*} X$ by induction hypothesis.

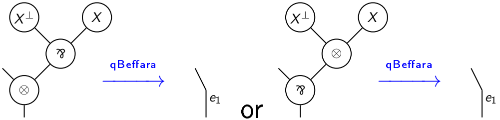
Quasi-Beffara & Beffara (proof)

By induction on the size of B.

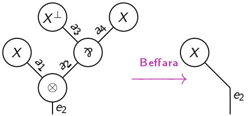
<u>Base cases:</u> $B \in \{X; X \parr (X^\perp \otimes X); X \otimes (X^\perp \parr X)\}$

<u>Inductive case:</u> $B \xrightarrow{\text{qBeffara}} B_1 \xrightarrow{\text{Beffara}} B_2 \xrightarrow{\text{Beffara}}{}^{*} X$ by induction hypothesis.

$B \xrightarrow{\text{qBeffara}} B_1$ is



$B_1 \xrightarrow{\text{Beffara}} B_2$ is



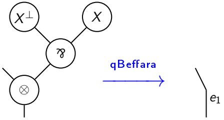
(up to duality)

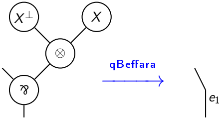
Quasi-Beffara & Beffara (proof)

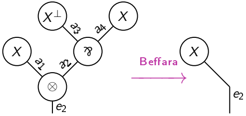
By induction on the size of B.

<u>Base cases:</u> $B \in \{X; X \parr (X^\perp \otimes X); X \otimes (X^\perp \parr X)\}$

<u>Inductive case:</u> $B \xrightarrow{\text{qBeffara}} B_1 \xrightarrow{\text{Beffara}} B_2 \xrightarrow{\text{Beffara}}{}^{*} X$ by induction hypothesis.

$B \xrightarrow{\text{qBeffara}} B_1$ is 

or 

$B_1 \xrightarrow{\text{Beffara}} B_2$ is 

(up to duality)

- $e_1 \notin \{a_1; a_2; a_3; a_4\}$ (including $e_1 = e_2$)

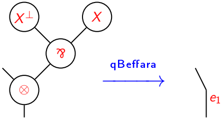
  The rewritings commute: $B \xrightarrow{\text{Beffara}} B'_1 \xrightarrow{\text{qBeffara}} B_2 \xrightarrow{\text{Beffara}}{}^{*} X$, so by induction $B \xrightarrow{\text{Beffara}} B'_1 \xrightarrow{\text{Beffara}}{}^{*} X$

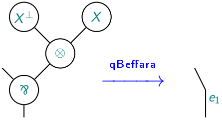
Quasi-Beffara & Beffara (proof)

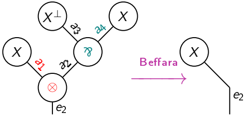
By induction on the size of B.

Base cases: $B \in \{X; X \parr (X^\perp \otimes X); X \otimes (X^\perp \parr X)\}$

Inductive case: $B \xrightarrow{\text{qBeffara}} B_1 \xrightarrow{\text{Beffara}} B_2 \xrightarrow{\text{Beffara}}{}^{*} X$ by induction hypothesis.

$B \xrightarrow{\text{qBeffara}} B_1$ is 

or 

$B_1 \xrightarrow{\text{Beffara}} B_2$ is 

(up to duality)

- $e_1 \notin \{a_1; a_2; a_3; a_4\}$ (including $e_1 = e_2$) $\checkmark$
- $e_1 = a_2$
  Up to isomorphism $e_1 = a_1$ or $e_1 = a_4$

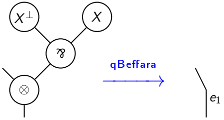
Quasi-Beffara & Beffara (proof)

By induction on the size of B.

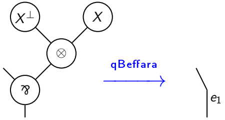
<u>Base cases:</u> $B \in \{X; X \parr (X^\perp \otimes X); X \otimes (X^\perp \parr X)\}$

<u>Inductive case:</u> $B \xrightarrow{\text{qBeffara}} B_1 \xrightarrow{\text{Beffara}} B_2 \xrightarrow{\text{Beffara}}{}^{*} X$ by induction hypothesis.

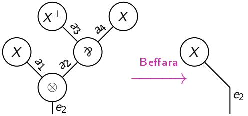
$B \xrightarrow{\text{qBeffara}} B_1$ is



or



$B_1 \xrightarrow{\text{Beffara}} B_2$ is



(up to duality)

- $e_1 \notin \{a_1; a_2; a_3; a_4\}$ (including $e_1 = e_2$) $\surd$
- $e_1 = a_2$ $\surd$
- $e_1 \in \{a_1; a_3; a_4\}$
  $B \xrightarrow{\text{qBeffara}} B_1$ is also a $B \xrightarrow{\text{Beffara}} B_1$

Characterization of $X \trianglelefteq B$

Theorem

The followings are equivalent:

1. $X \trianglelefteq B$
2. $B \xrightarrow{\text{qBeffara}}{}^{*} X$
3. $B \xrightarrow{\text{Beffara}}{}^{*} X$ *(up to iso)*

Characterization of $X \trianglelefteq B$

Theorem

The followings are equivalent:

1. $X \trianglelefteq B$
2. $B \xrightarrow{\text{qBeffara}}{}^{*} X$
3. $B \xrightarrow{\text{Beffara}}{}^{*} X$ *(up to iso)*
4. $B \in P$ *(up to iso)*

$$P ::= X \mid P \otimes (N \parr P) \mid P \parr (N \otimes P)$$
$$N ::= X^{\perp} \mid N \otimes (P \parr N) \mid N \parr (P \otimes N)$$

Characterization of $X \trianglelefteq B$

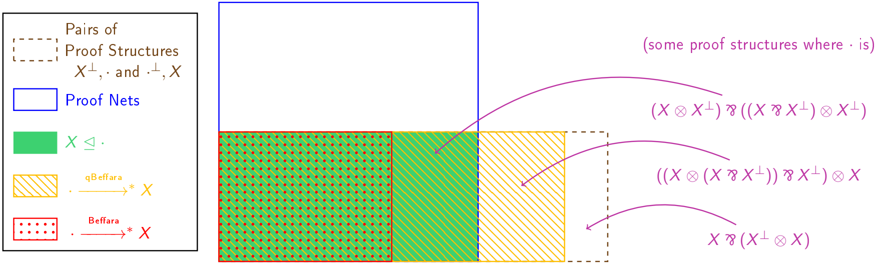
Theorem

The followings are equivalent:

1. $X \trianglelefteq B$
2. $B \xrightarrow{\text{qBeffara}}{}^{*} X$
3. $B \xrightarrow{\text{Beffara}}{}^{*} X$ *(up to iso)*
4. $B \in P$ *(up to iso)*

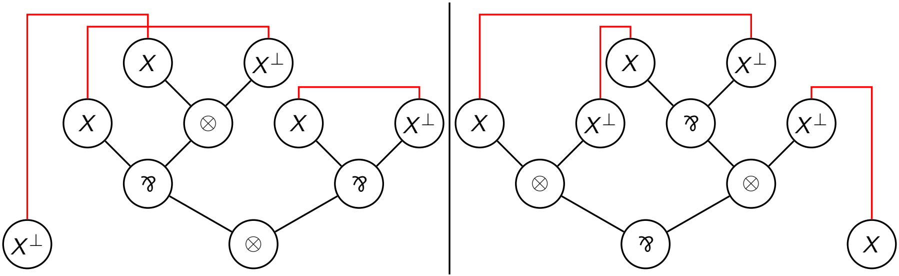
$$P \ ::= \ X \mid P \otimes (N \parr P) \mid P \parr (N \otimes P)$$
$$N \ ::= \ X^{\perp} \mid N \otimes (P \parr N) \mid N \parr (P \otimes N)$$

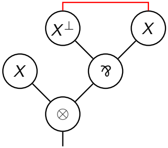
...but this is when looking at *formulas*! Looking at *proofs*, this is messier:



Retraction not generated by Beffara

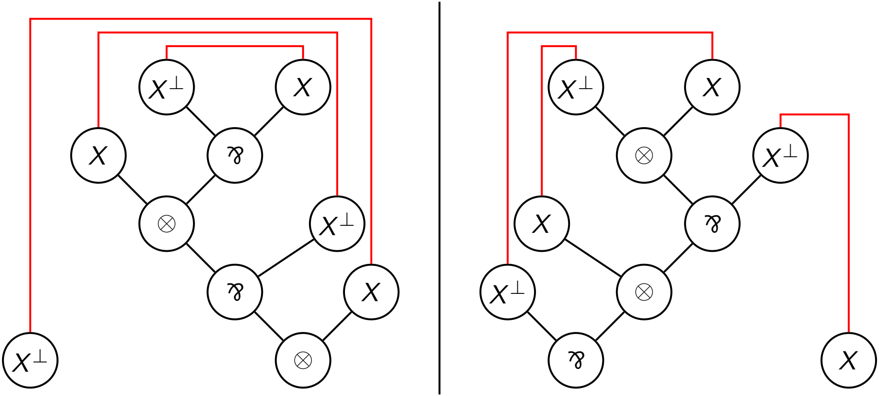
Proof of $X \lhd (X \otimes X^\perp) \parr ((X \parr X^\perp) \otimes X^\perp)$



Not generated by Beffara as no  in either proof nets

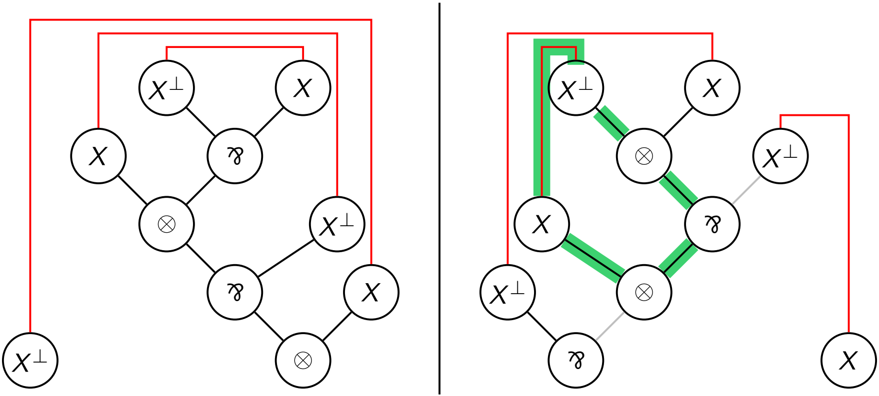
Incorrect retraction generated by Quasi-Beffara

Not-Proof of $X \lhd ((X \otimes (X \parr X^\perp)) \parr X^\perp) \otimes X$



Incorrect retraction generated by Quasi-Beffara

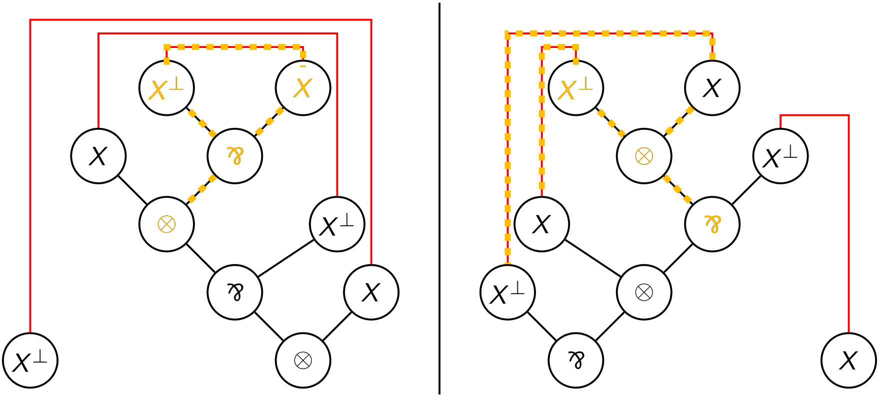
Not-Proof of $X \lhd ((X \otimes (X \parr X^\perp)) \parr X^\perp) \otimes X$



Incorrect

Incorrect retraction generated by Quasi-Beffara

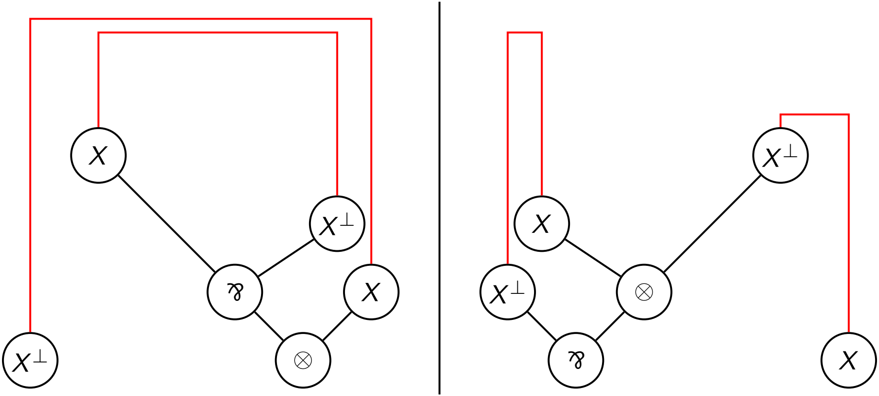
Not-Proof of $X \lhd ((X \otimes (X \parr X^\perp)) \parr X^\perp) \otimes X$



Can apply one step of Quasi-Beffara

Incorrect retraction generated by Quasi-Beffara

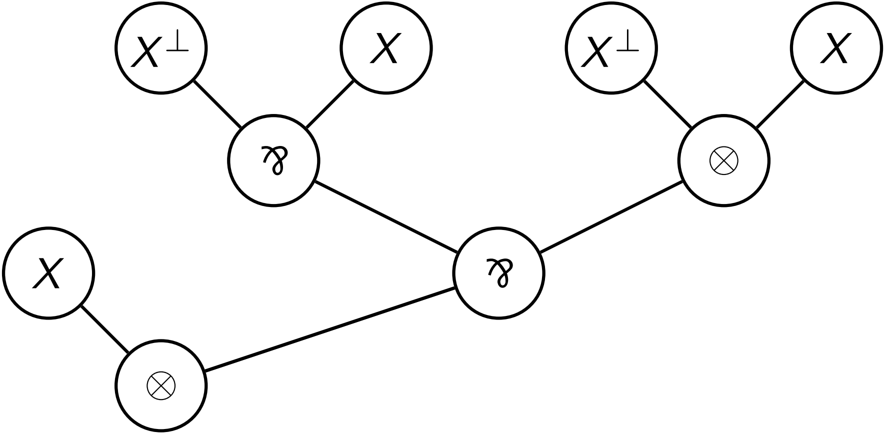
Not-Proof of $X \lhd ((X \otimes (X \parr X^\perp)) \parr X^\perp) \otimes X$



This is Beffara, attainable from X by one step of Quasi-Beffara

Formula not generated by Beffara without iso

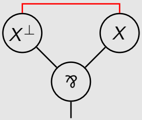
$$X \lhd X \otimes ((X^\perp \parr X) \parr (X^\perp \otimes X))$$

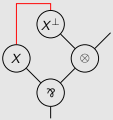


Generated by Beffara only up to isomorphism!

Plan

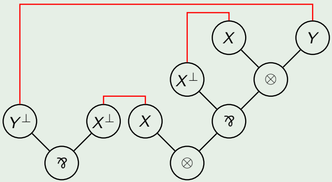
Difficulties for $A \trianglelefteq B$

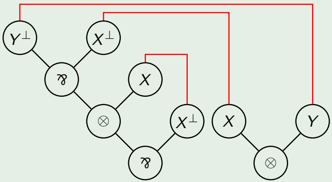
Not only 

as a pattern, also 

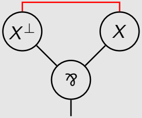
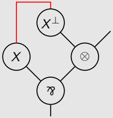
(and others?)

Example: $X \otimes Y \trianglelefteq X \otimes (X^\perp ⅋ (X \otimes Y))$

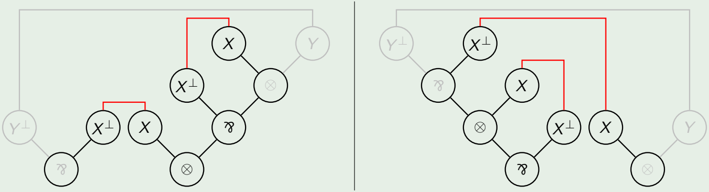




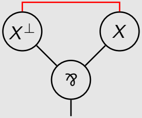
Difficulties for $A \trianglelefteq B$

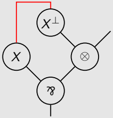
Not only  as a pattern, also  (and others?)

Example: $X \otimes Y \trianglelefteq X \otimes (X^\perp ⅋ (X \otimes Y))$



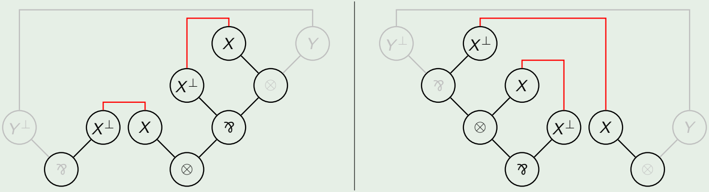
Difficulties for $A \trianglelefteq B$

Not only 

as a pattern, also 

(and others?)

Example: $X \otimes Y \trianglelefteq X \otimes (X^\perp \parr (X \otimes Y))$



May not be finitely axiomatisable (on formulas)?

$\{\otimes X_i\} \triangleleft \{\otimes X_i\} \parr (X_1 \otimes (X_1^\perp \parr (\ldots (X_{n-1} \otimes (X_{n-1}^\perp \parr (X_n \otimes X_n^\perp)) \ldots)))$

And $(A \otimes X) \parr B \ntrianglelefteq (A \otimes X) \parr (X \otimes (X^\perp \parr B))$

What about the other "simple" fragments?

- For **exponential** formulas, there are new retractions:

$$?A \trianglelefteq ??A \qquad\qquad ?!A \trianglelefteq ?!?!A$$

Look like the only "basic" ones?

What about the other “simple” fragments?

- For **exponential** formulas, there are new retractions:

$$?A \trianglelefteq ??A \qquad ?!A \trianglelefteq ?!?!A$$

  Look like the only “basic” ones?

- For **additive** formulas, only one “basic” retraction (with units too):

$$A \trianglelefteq A \,\&\, B \iff \vdash A^\perp, B \qquad \text{or} \qquad A \trianglelefteq A \oplus B \iff \vdash A, B^\perp$$

  Retraction of an atom manageable.
  But generally composition is bad due to the side condition:

$$X \oplus Y \triangleleft ((X \oplus Z) \,\&\, (X \oplus Y)) \oplus Y$$

  comes from $X \oplus Y \triangleleft (X \oplus Y) \oplus Y$ without $\vdash X \oplus Z, (X \oplus Y)^\perp$

What about the other "simple" fragments?

- For **exponential** formulas, there are new retractions:

$$?A \trianglelefteq ??A \qquad ?!A \trianglelefteq ?!?!A$$

  Look like the only "basic" ones?

- For **additive** formulas, only one "basic" retraction (with units too):

$$A \trianglelefteq A \& B \iff \vdash A^{\perp}, B \qquad \text{or} \qquad A \trianglelefteq A \oplus B \iff \vdash A, B^{\perp}$$

  Retraction of an atom manageable.
  But generally composition is bad due to the side condition:

$$X \oplus Y \lhd ((X \oplus Z) \& (X \oplus Y)) \oplus Y$$

  comes from $X \oplus Y \lhd (X \oplus Y) \oplus Y$ without $\vdash X \oplus Z, (X \oplus Y)^{\perp}$

- Cantor-Bernstein holds in ALL. More complicated in MALL. . .

Conclusion

- $X \trianglelefteq B \iff B \xrightarrow{\text{Beffara}}{}^{*} X$ up to isomorphism
  with some subtilities on the proof morphisms
- good properties: Cantor-Bernstein, result on sizes, only provability of a particular shape no consider, ...
- still the problem may be difficult?!

Conclusion

- $X \trianglelefteq B \iff B \xrightarrow{\text{Beffara}}{}^{*} X$ up to isomorphism
  with some subtilities on the proof morphisms
- good properties: Cantor-Bernstein, result on sizes, only provability of a particular shape no consider, ...
- still the problem may be difficult?!

Thank you
for your attention!

References I

Vincent Balat and Roberto Di Cosmo.
A linear logical view of linear type isomorphisms.
In Jörg Flum and Mario Rodríguez-Artalejo, editors, *Computer Science Logic*, volume 1683 of *Lecture Notes in Computer Science*, pages 250–265. Springer, 1999.

Rémi Di Guardia and Olivier Laurent.
Type isomorphisms for multiplicative-additive linear logic.
In Marco Gaboardi and Femke van Raamsdonk, editors, *International Conference on Formal Structures for Computation and Deduction (FSCD)*, volume 260 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 26:1–26:21. Schloss Dagstuhl – Leibniz-Zentrum fuer Informatik, July 2023.

References II

Patrick Lincoln.
Deciding provability of linear logic formulas.
In Jean-Yves Girard, Yves Lafont, and Laurent Regnier, editors, *Advances in Linear Logic*, volume 222, pages 109–122, 1995.

Laurent Regnier and Pawel Urzyczyn.
Retractions of types with many atoms, 2002.
http://arxiv.org/abs/cs/0212005.

Sergei Soloviev.
The category of finite sets and cartesian closed categories.
Journal of Soviet Mathematics, 22(3):1387–1400, 1983.