Retractions for Multiplicative Linear Logic

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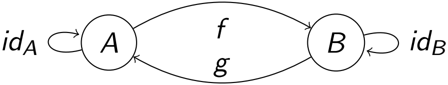
ENS Lyon (LIP)

SCALP 2023, 29 November

Isomorphisms relate types/formulas/objects A and B which are "the same"

$$A \simeq B$$



Instantiation in λ-calculus, logics,...

Equational theory for λ-calculus with products and unit type [Sol83]

$\times$	$A \times (B \times C) \simeq (A \times B) \times C$		$A \times B \simeq B \times A$
$\times$ and $\rightarrow$	$(A \times B) \rightarrow C \simeq A \rightarrow (B \rightarrow C)$		$A \rightarrow (B \times C) \simeq (A \rightarrow B) \times (A \rightarrow C)$
1	$A \times 1 \simeq A$	$1 \rightarrow A \simeq A$	$A \rightarrow 1 \simeq 1$

Introduction

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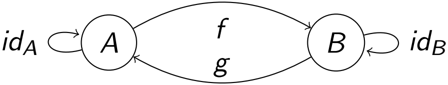


Instantiation in λ-calculus, logics,...

Equational theory for Multiplicative Linear Logic [BDC99]

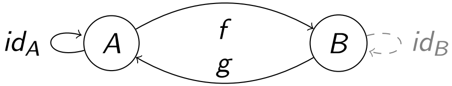
Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$	$A \parr (B \parr C) \simeq (A \parr B) \parr C$
Commutativity	$A \otimes B \simeq B \otimes A$	$A \parr B \simeq B \parr A$
Neutrality	$A \otimes 1 \simeq A$	$A \parr \bot \simeq A$

$$(A \otimes B) \multimap C = (A^\perp \parr B^\perp) \parr C \simeq A^\perp \parr (B^\perp \parr C) = A \multimap (B \multimap C)$$

Introduction

Retractions relate A and B when A is a "subtype" of B

$$A \trianglelefteq B$$



Instantiation in λ-calculus, logics,...

Equational theory for simply typed affine λ-calculus [RU02]

$\simeq$	$A \to B \to C \simeq B \to A \to C$
$\lhd \ (= \trianglelefteq \setminus \simeq)$	$A \lhd B \to A$ $A \lhd (A \to X) \to X$ if A is $Y_1 \to Y_2 \to \cdots \to X$

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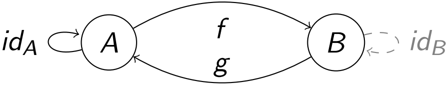


Instantiation in λ-calculus, logics,...

Equational theory for Multiplicative Linear Logic [UNKNOWN]

$\simeq$	associativity and commutativity of $\otimes$ and $\parr$, neutrality of 1 and $\bot$
$\triangleleft \ (= \trianglelefteq \setminus \simeq)$	???

Plan

1. Definitions
   - Proof Net
   - Retraction
2. Retractions of the shape $X \trianglelefteq \cdot$
   - Looking for a pattern
   - Quasi-Beffara
   - Beffara
3. Difficulties for $A \trianglelefteq B$ & Other fragments

Formula & Sequent

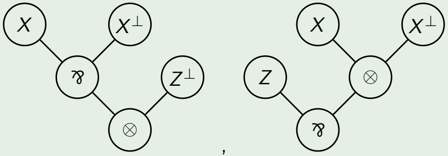
Formulas

$$A, B \ ::= \ X \mid \overset{\textit{not}}{X^\perp} \mid A \overset{\textit{and}}{\otimes} B \mid A \overset{\textit{or}}{\parr} B$$

Duality

$$(X^\perp)^\perp = X$$
$$(A \otimes B)^\perp = B^\perp \parr A^\perp$$
$$(A \parr B)^\perp = B^\perp \otimes A^\perp$$

Examples



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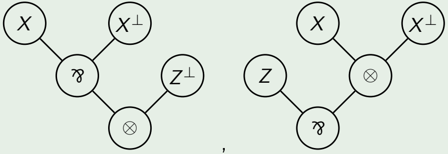


Duality

$$(X^{\perp})^{\perp} = X$$

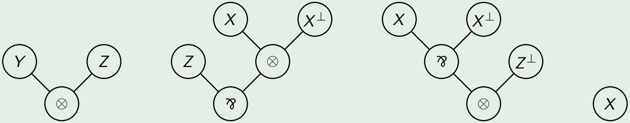
$$(A \otimes B)^{\perp} = B^{\perp} ⅋ A^{\perp}$$

$$(A ⅋ B)^{\perp} = B^{\perp} \otimes A^{\perp}$$

Sequent

$$\vdash A_1, \ldots, A_n$$

Example



Proof Structure

Proof Structure

Sequent with edges between dual leaves (some X and $X^\perp$), these edges partitioning the leaves of the sequent.

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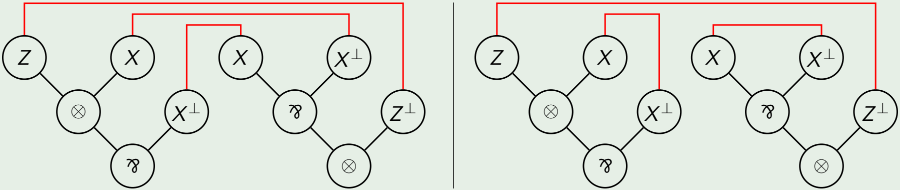


Correctness & Proof Net

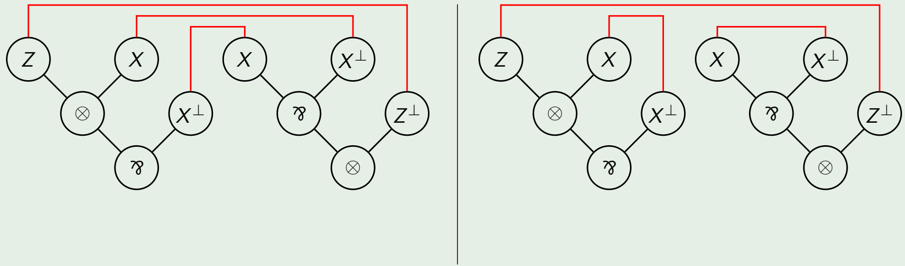
Correctness Graph

In a proof structure, keep only one premise of each ⅋-node.

Danos-Regnier Correctness Criterion

A proof structure is *correct*, and called a *proof net*, if all its correctness graphs are acyclic and connected (i.e. are trees).

Examples



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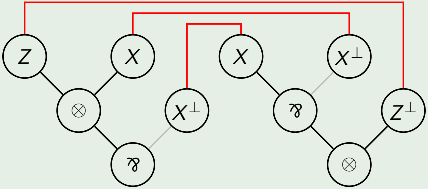
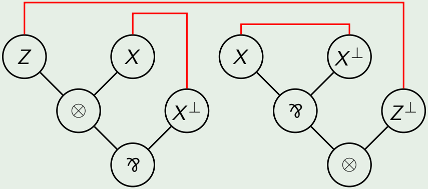


Acyclic and connected



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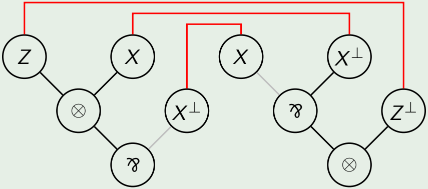
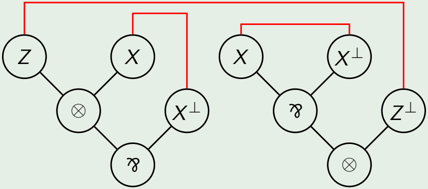


Not acyclic nor connected

INCORRECT



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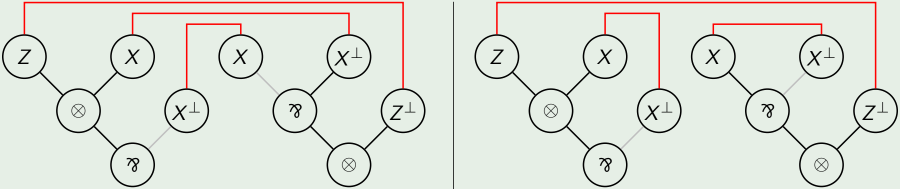

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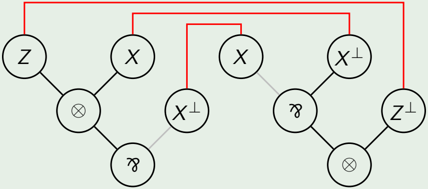

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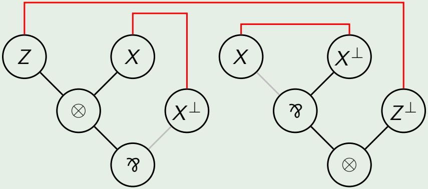

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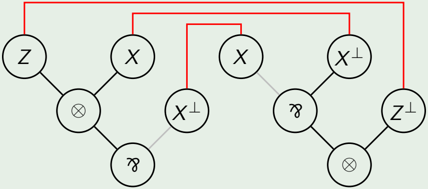

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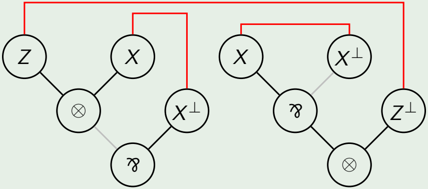

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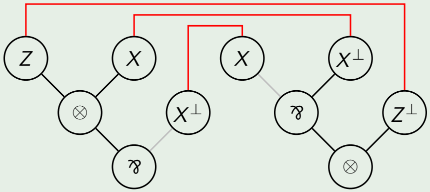

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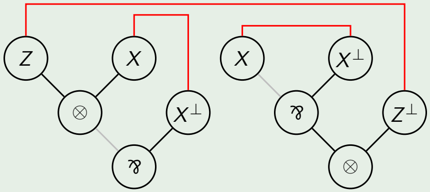
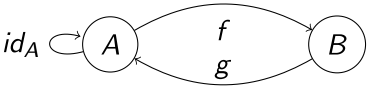


Acyclic and connected
CORRECT

Retraction

In category theory

$A \trianglelefteq B$



In λ-calculus

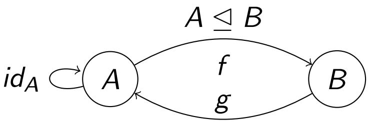
Retraction $A \trianglelefteq B$

Terms $M : A \to B$ and $N : B \to A$ such that

$$N \circ M =_{\beta\eta} \lambda x^A.x$$

Retraction

In category theory



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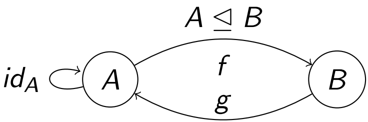
In multiplicative linear logic

Retraction $A \trianglelefteq B$

Proof nets $\mathcal{R} \vdash A^\perp, B$ and $\mathcal{S} \vdash B^\perp, A$ whose composition by cut over B yields, after cut elimination, the identity proof net of A.

Retraction

In category theory



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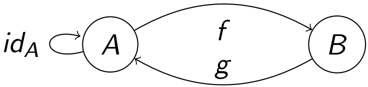

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$$A \trianglelefteq B \iff A^\perp \trianglelefteq B^\perp$$

Beffara's retraction

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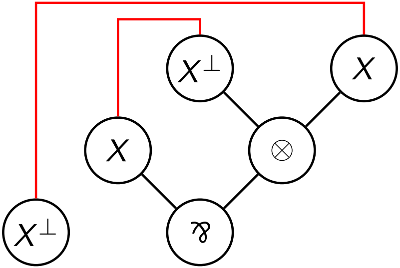
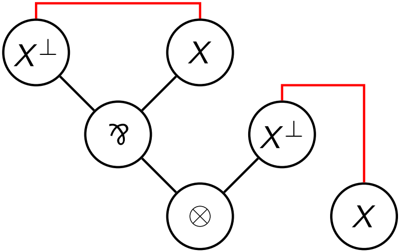
$$X \lhd X \parr (X^\perp \otimes X) \qquad \text{or dualy} \qquad X \lhd X \otimes (X^\perp \parr X)$$



$X \parr (X^\perp \otimes X)$



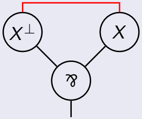
$(X^\perp \parr X) \otimes X^\perp$

Plan

Key Result

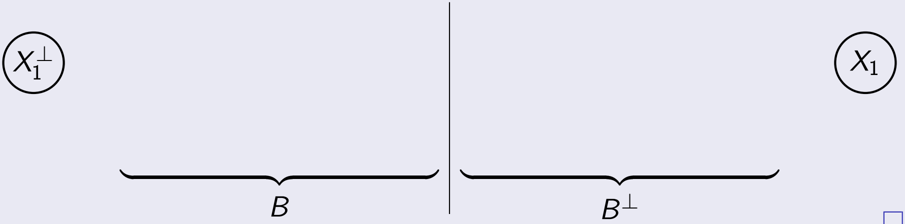
Lemma

In $X \lhd B$ one of the two proof nets contains:



Proof.

We build a sequence finding such a pattern.



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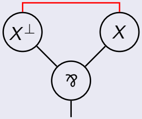

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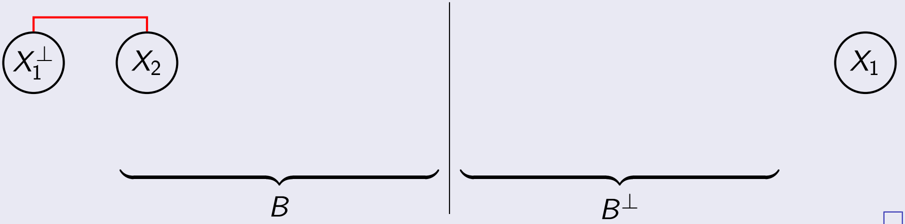

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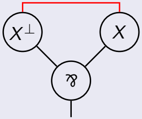

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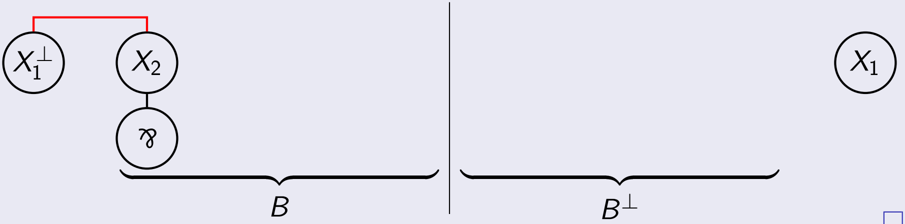

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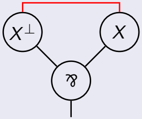

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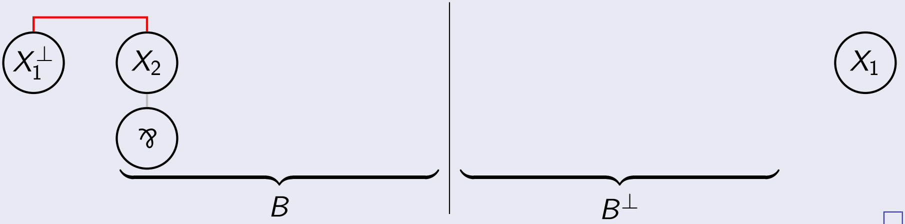

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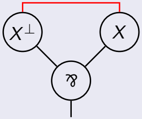

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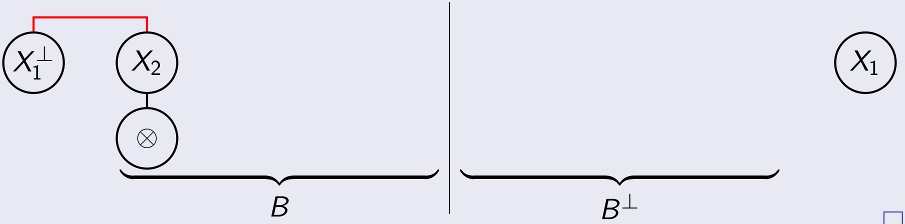

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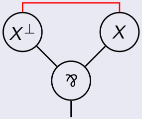

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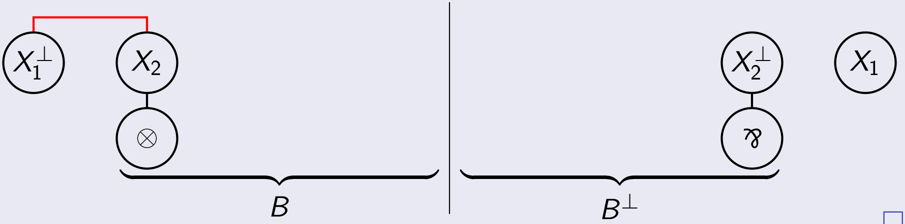

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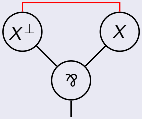

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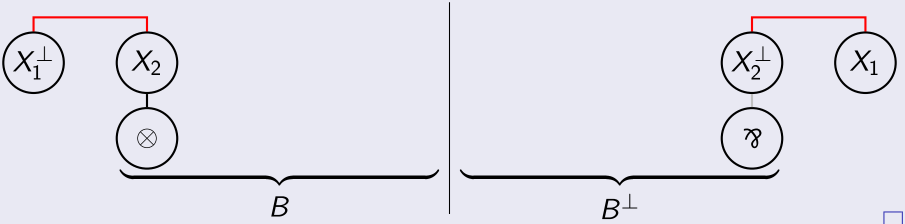

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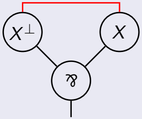

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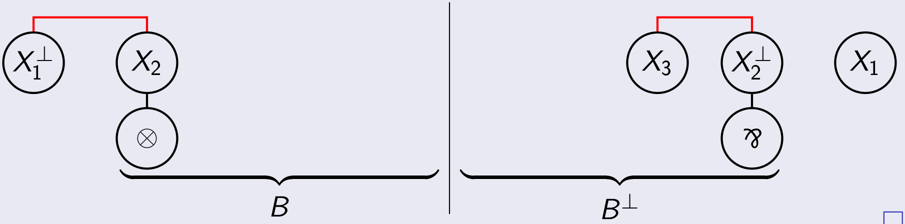

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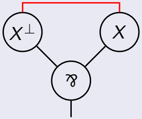

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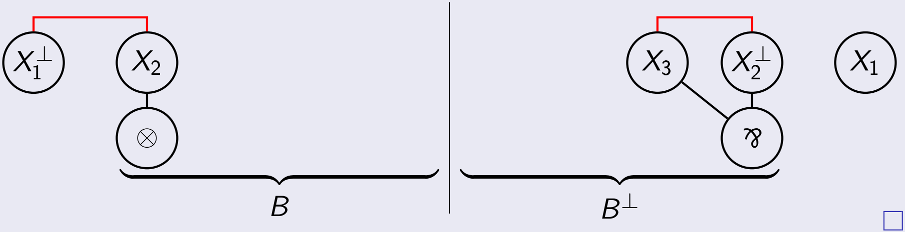

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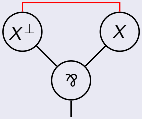

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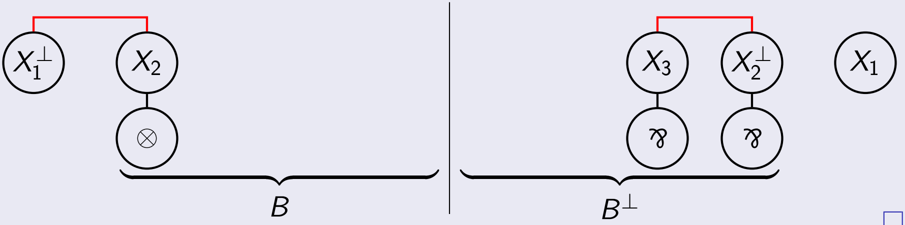

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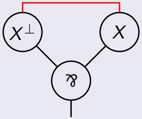

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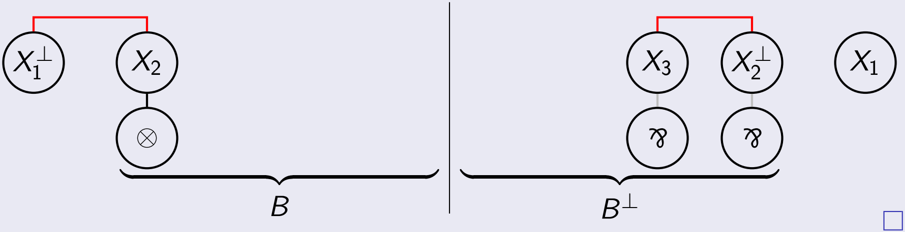

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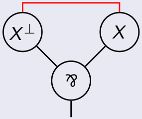

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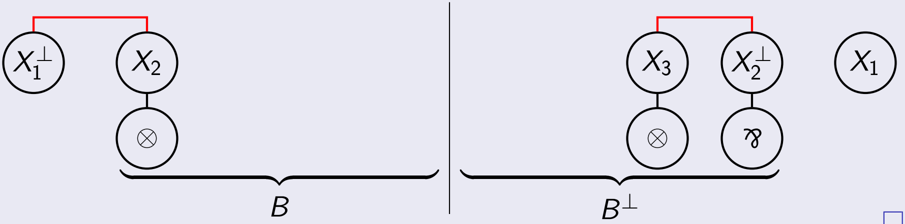
Invariant: every X of B is above a $\otimes$, and every $X^\perp$ above a ⅋.

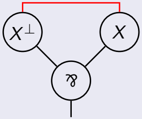


□

Key Result

Lemma

In $X \lhd B$ one of the two proof nets contains:



Proof.

We build a sequence finding such a pattern.

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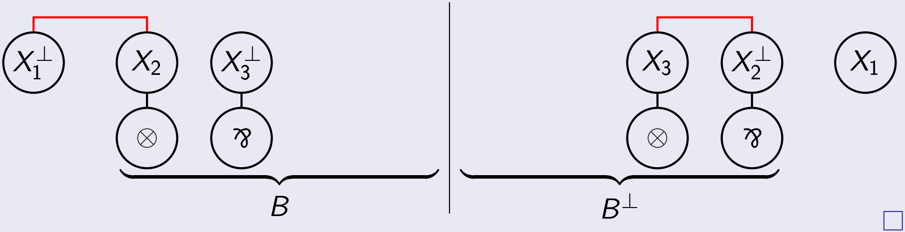

□

Key Result

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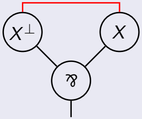

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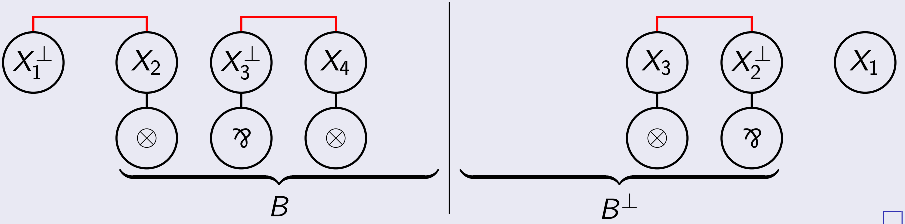
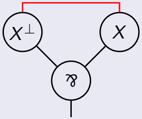


Key Result

Lemma

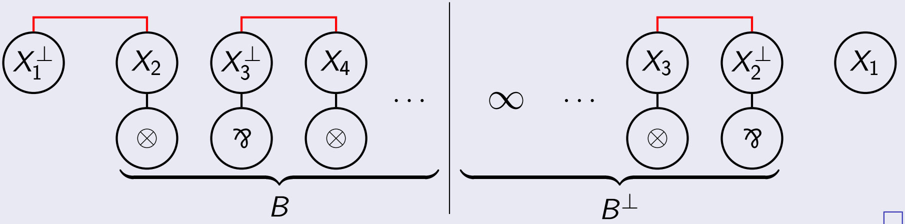
In $X \lhd B$ one of the two proof nets contains:



Proof.

We build a sequence finding such a pattern.

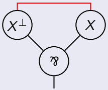
Invariant: every X of B is above a $\otimes$, and every $X^\perp$ above a $\parr$.

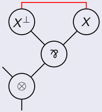


□

Extended pattern

Lemma

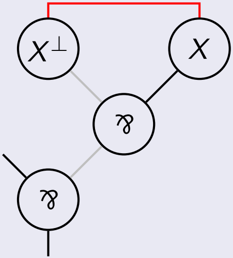
If 

has a node below it, then this is a 

.

Proof.

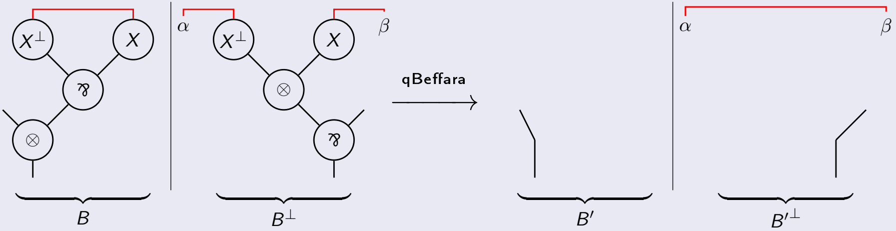
The connector below the pattern cannot be a ⅋ by connectivity:



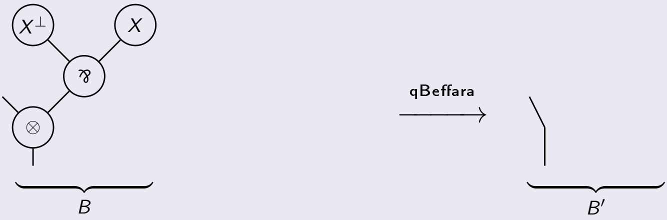
Quasi-Beffara

Definition

Quasi-Beffara is this local transformation on proofs of a retraction $A \trianglelefteq B$:



By extension, this defines two transformations on a formula B (by duality):



Coherence of Quasi-Beffara

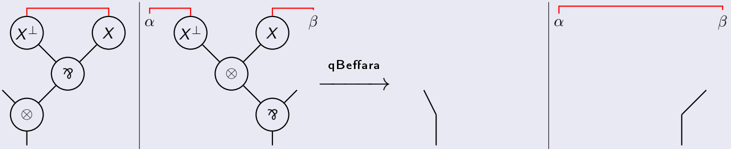
Lemma

If $(\mathcal{R}, \mathcal{S})$ *are proofs of* $A \trianglelefteq B$ *and* $(\mathcal{R}, \mathcal{S}) \xrightarrow{\text{qBeffara}} (\mathcal{R}', \mathcal{S}')$*, then* $(\mathcal{R}', \mathcal{S}')$ *are proofs of* $A \trianglelefteq B'$ *with* $B \xrightarrow{\text{qBeffara}} B'$.

Proof.

Quasi-Beffara preserves:

- being a proof structure



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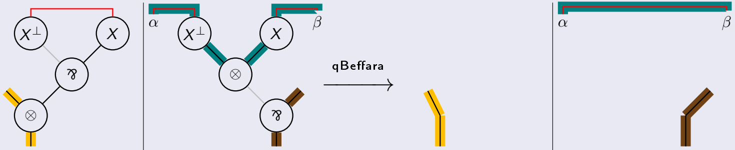
Proof.

Quasi-Beffara preserves:

- being a proof structure
- acyclicity of correctness graphs



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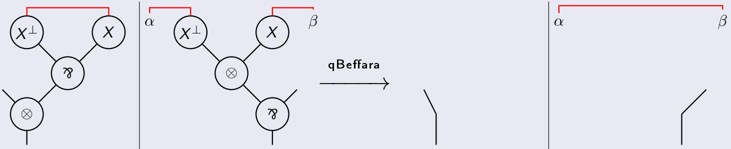
If $(\mathcal{R}, \mathcal{S})$ *are proofs of* $A \trianglelefteq B$ *and* $(\mathcal{R}, \mathcal{S}) \xrightarrow{\text{qBeffara}} (\mathcal{R}', \mathcal{S}')$*, then* $(\mathcal{R}', \mathcal{S}')$ *are proofs of* $A \trianglelefteq B'$ *with* $B \xrightarrow{\text{qBeffara}} B'$.

Proof.

Quasi-Beffara preserves:

- being a proof structure
- acyclicity of correctness graphs
- the number $|V| + |\parr| - |E|$ of cc. of any correctness graph: it removes 4 vertices, including 1 $\parr$, and 5 edges



□

Coherence of Quasi-Beffara

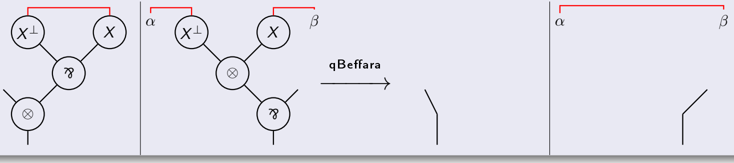
Lemma

If $(\mathcal{R}, \mathcal{S})$ are proofs of $A \trianglelefteq B$ and $(\mathcal{R}, \mathcal{S}) \xrightarrow{\text{qBeffara}} (\mathcal{R}', \mathcal{S}')$, then $(\mathcal{R}', \mathcal{S}')$ are proofs of $A \trianglelefteq B'$ with $B \xrightarrow{\text{qBeffara}} B'$.

Proof.

Quasi-Beffara preserves:

- being a proof structure
- acyclicity of correctness graphs
- the number $|V| + |\parr| - |E|$ of cc. of any correctness graph
- (normal form for cut elimination)



□

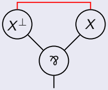
Completeness of Quasi-Beffara

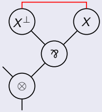
Proposition

If $X \trianglelefteq B$ *then* $B \xrightarrow{\text{qBeffara}}{}^{*} X$.

Proof.

By induction on the size of B. Trivial if $B = X$.
Else, by previous results:

1. we find some 

2. which is a 

3. $B \xrightarrow{\text{qBeffara}} B'$, $X \trianglelefteq B'$ and B' of strictly smaller size

□

Quasi-Beffara & Beffara (statement)

- Remember Beffara's retraction:

$$X \lhd X \otimes (X^\perp \parr X) \qquad X \lhd X \parr (X^\perp \otimes X)$$

- Corresponding transformations inside a formula:

$$X \otimes (X^\perp \parr X) \xrightarrow{\text{Beffara}} X \qquad X \parr (X^\perp \otimes X) \xrightarrow{\text{Beffara}} X$$

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- Corresponding transformations inside a formula:

$$X \otimes (X^\perp \parr X) \xrightarrow{\text{Beffara}} X \qquad X \parr (X^\perp \otimes X) \xrightarrow{\text{Beffara}} X$$

Proposition

If $B \xrightarrow{\text{qBeffara}}{}^{*} X$, *then* $B \xrightarrow{\text{Beffara}}{}^{*} X$ ***up to isomorphism***
(associativity and commutativity of $\parr$ *and* $\otimes$*)*

Quasi-Beffara & Beffara (proof)

By induction on the size of B.

Base cases: $B \in \{X; X \parr (X^\perp \otimes X); X \otimes (X^\perp \parr X)\}$

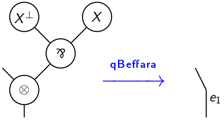
Inductive case: $B \xrightarrow{\text{qBeffara}} B_1 \xrightarrow{\text{Beffara}} B_2 \xrightarrow{\text{Beffara}}{}^* X$ by induction hypothesis.

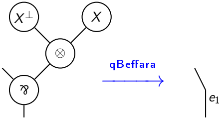
Quasi-Beffara & Beffara (proof)

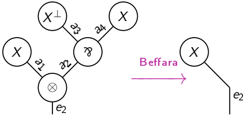
By induction on the size of B.

Base cases: $B \in \{X; X \parr (X^\perp \otimes X); X \otimes (X^\perp \parr X)\}$

Inductive case: $B \xrightarrow{\text{qBeffara}} B_1 \xrightarrow{\text{Beffara}} B_2 \xrightarrow{\text{Beffara}}{}^* X$ by induction hypothesis.

$B \xrightarrow{\text{qBeffara}} B_1$ is 

or 

$B_1 \xrightarrow{\text{Beffara}} B_2$ is 

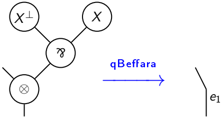
(up to duality)

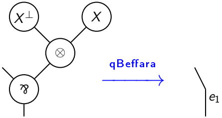
Quasi-Beffara & Beffara (proof)

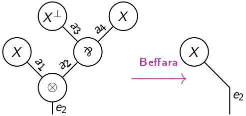
By induction on the size of B.

<u>Base cases:</u> $B \in \{X; X \parr (X^\perp \otimes X); X \otimes (X^\perp \parr X)\}$

<u>Inductive case:</u> $B \xrightarrow{\text{qBeffara}} B_1 \xrightarrow{\text{Beffara}} B_2 \xrightarrow{\text{Beffara}}{}^* X$ by induction hypothesis.

$B \xrightarrow{\text{qBeffara}} B_1$ is 

or 

$B_1 \xrightarrow{\text{Beffara}} B_2$ is 

(up to duality)

- $e_1 \notin \{a_1; a_2; a_3; a_4\}$ (including $e_1 = e_2$)

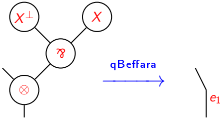
  The rewritings commute: $B \xrightarrow{\text{Beffara}} B_1' \xrightarrow{\text{qBeffara}} B_2 \xrightarrow{\text{Beffara}}{}^* X$, so by induction $B \xrightarrow{\text{Beffara}} B_1' \xrightarrow{\text{Beffara}}{}^* X$

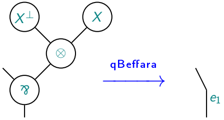
Quasi-Beffara & Beffara (proof)

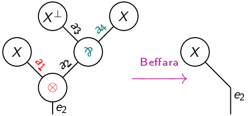
By induction on the size of B.

Base cases: $B \in \{X; X ⅋ (X^\perp \otimes X); X \otimes (X^\perp ⅋ X)\}$

Inductive case: $B \xrightarrow{\text{qBeffara}} B_1 \xrightarrow{\text{Beffara}} B_2 \xrightarrow{\text{Beffara}}{}^* X$ by induction hypothesis.

$B \xrightarrow{\text{qBeffara}} B_1$ is 

or 

$B_1 \xrightarrow{\text{Beffara}} B_2$ is 

(up to duality)

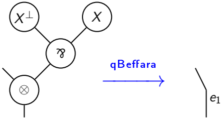
- $e_1 \notin \{a_1; a_2; a_3; a_4\}$ (including $e_1 = e_2$) $\surd$
- $e_1 = a_2$
  Up to isomorphism $e_1 = a_1$ or $e_1 = a_4$

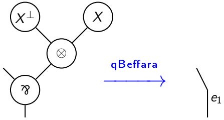
Quasi-Beffara & Beffara (proof)

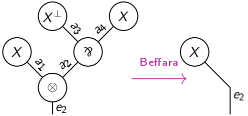
By induction on the size of B.

Base cases: $B \in \{X; X \parr (X^\perp \otimes X); X \otimes (X^\perp \parr X)\}$

Inductive case: $B \xrightarrow{\text{qBeffara}} B_1 \xrightarrow{\text{Beffara}} B_2 \xrightarrow{\text{Beffara}}{}^{*} X$ by induction hypothesis.

$B \xrightarrow{\text{qBeffara}} B_1$ is 

or 

$B_1 \xrightarrow{\text{Beffara}} B_2$ is 

(up to duality)

- $e_1 \notin \{a_1; a_2; a_3; a_4\}$ (including $e_1 = e_2$) $\surd$
- $e_1 = a_2$ $\surd$
- $e_1 \in \{a_1; a_3; a_4\}$

  $B \xrightarrow{\text{qBeffara}} B_1$ is also a $B \xrightarrow{\text{Beffara}} B_1$

Characterization of $X \trianglelefteq B$

Theorem

The followings are equivalent:

1. $X \trianglelefteq B$
2. $B \xrightarrow{\text{qBeffara}}{}^{*} X$
3. $B \xrightarrow{\text{Beffara}}{}^{*} X$ *(up to iso)*

Characterization of $X \trianglelefteq B$

Theorem

The followings are equivalent:

1. $X \trianglelefteq B$
2. $B \xrightarrow{\text{qBeffara}}{}^{*} X$
3. $B \xrightarrow{\text{Beffara}}{}^{*} X$ *(up to iso)*
4. $B \in P$ *(up to iso)*

$$P ::= X \mid P \otimes (N \parr P) \mid P \parr (N \otimes P)$$
$$N ::= X^{\perp} \mid N \otimes (P \parr N) \mid N \parr (P \otimes N)$$

Characterization of $X \trianglelefteq B$

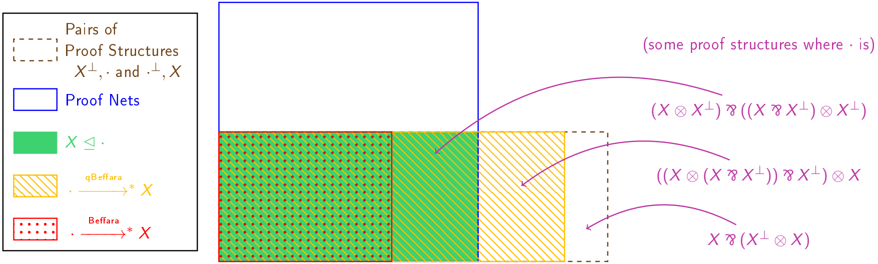
Theorem

The followings are equivalent:

1. $X \trianglelefteq B$
2. $B \xrightarrow{\text{qBeffara}}{}^{*} X$
3. $B \xrightarrow{\text{Beffara}}{}^{*} X$ *(up to iso)*
4. $B \in P$ *(up to iso)*

$$P ::= X \mid P \otimes (N \parr P) \mid P \parr (N \otimes P)$$
$$N ::= X^{\perp} \mid N \otimes (P \parr N) \mid N \parr (P \otimes N)$$

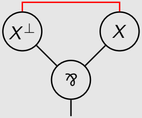
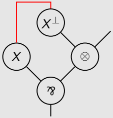
…but this is when looking at *formulas*! Looking at *proofs*, this is messier:



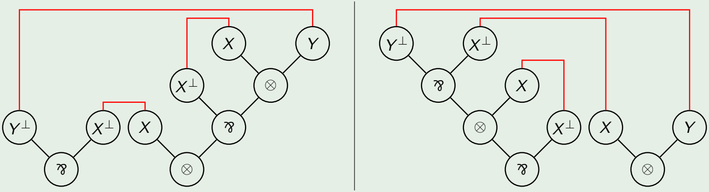
Plan

1. Definitions
   - Proof Net
   - Retraction
2. Retractions of the shape $X \trianglelefteq \cdot$
   - Looking for a pattern
   - Quasi-Beffara
   - Beffara
3. Difficulties for $A \trianglelefteq B$ & Other fragments

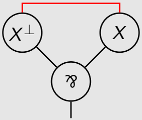
Difficulties for $A \trianglelefteq B$

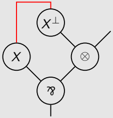
Not only  as a proof pattern, also  (and others?)

Example: $X \otimes Y \trianglelefteq X \otimes (X^{\perp} \parr (X \otimes Y))$



Difficulties for $A \trianglelefteq B$

Not only 

as a proof pattern, also 

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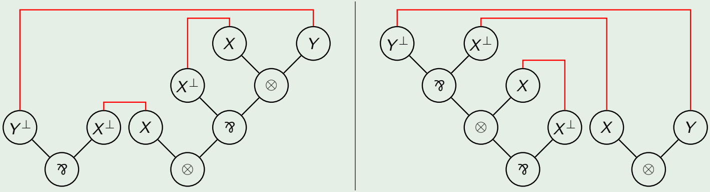
Example: $X \otimes Y \trianglelefteq X \otimes (X^\perp \parr (X \otimes Y))$



May not be finitely axiomatisable (on formulas)?

$\{\otimes X_i\} \triangleleft \{\otimes X_i\} \parr (X_1 \otimes (X_1^\perp \parr (\ldots (X_{n-1} \otimes (X_{n-1}^\perp \parr (X_n \otimes X_n^\perp)) \ldots)))$

And $(A \otimes X) \parr B \ntrianglelefteq (A \otimes X) \parr (X \otimes (X^\perp \parr B))$

What about other fragments?

- Adding **exponentials** gives new retractions

$$?A \trianglelefteq ??A \qquad ?!A \trianglelefteq ?!?!A$$

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- Adding **exponentials** gives new retractions

$$?A \trianglelefteq ??A \qquad ?!A \trianglelefteq ?!?!A$$

- In **ALL** retractions seems easier

$$A \trianglelefteq A \,\&\, B \iff \vdash A^\perp, B$$

Generates all (unit-free?) retraction proofs? *(proof to be checked)*

What about other fragments?

- Adding **exponentials** gives new retractions

$$?A \trianglelefteq ??A \qquad\qquad ?!A \trianglelefteq ?!?!A$$

- In **ALL** retractions seems easier

$$A \trianglelefteq A \,\&\, B \iff \vdash A^{\perp}, B$$

  Generates all (unit-free?) retraction proofs? *(proof to be checked)*

- In **ALL**, **MELL**, **MALL** and **LL** finding if a pair of formulas is a retraction is at least harder than provability:

$$!X \trianglelefteq\ !X \otimes !(X \otimes A) \iff A \text{ is provable}$$
$$X \trianglelefteq X \,\&\, (X \otimes A) \iff A \text{ is provable}$$

  Thus PSPACE-hard in MALL and undecidable in LL!

Thank you
for your attention!

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In Jörg Flum and Mario Rodríguez-Artalejo, editors, *Computer Science Logic*, volume 1683 of *Lecture Notes in Computer Science*, pages 250–265. Springer, 1999.

Laurent Regnier and Pawel Urzyczyn.
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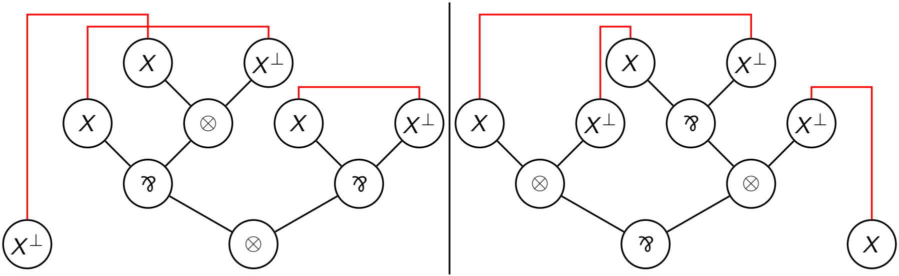
http://arxiv.org/abs/cs/0212005.

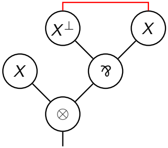
Sergei Soloviev.
The category of finite sets and cartesian closed categories.
Journal of Soviet Mathematics, 22(3):1387–1400, 1983.

Retraction not generated by Beffara

Proof of $X \lhd (X \otimes X^{\perp}) \parr ((X \parr X^{\perp}) \otimes X^{\perp})$



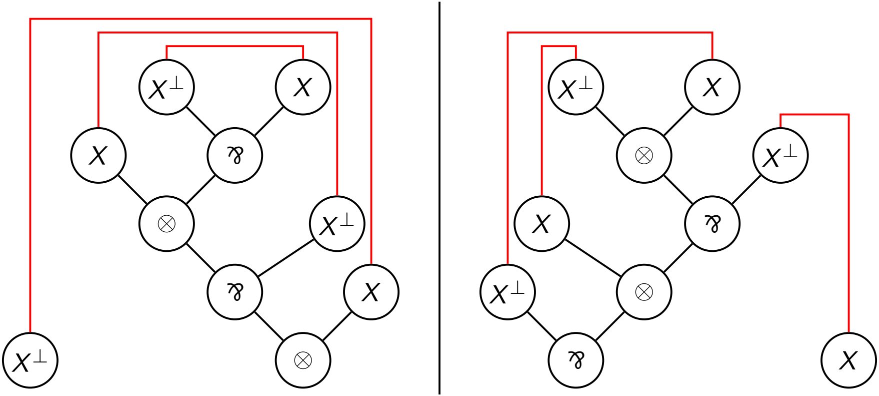
Not generated by Beffara as no



in either proof nets

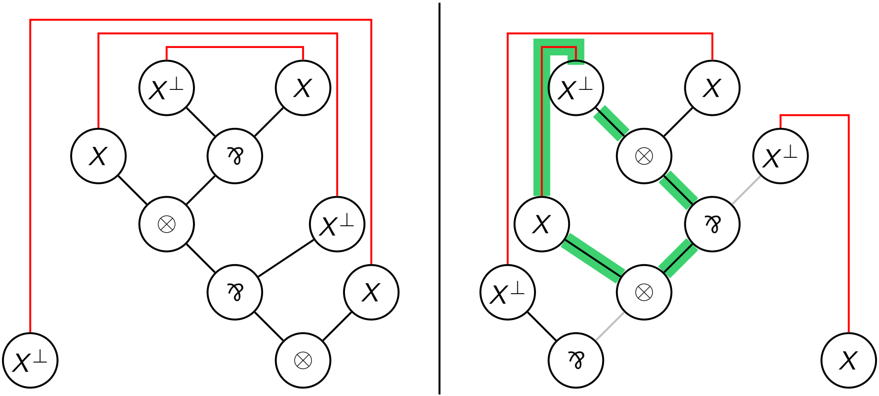
Incorrect retraction generated by Quasi-Beffara

Not-Proof of $X \lhd ((X \otimes (X ⅋ X^\perp)) ⅋ X^\perp) \otimes X$



Incorrect retraction generated by Quasi-Beffara

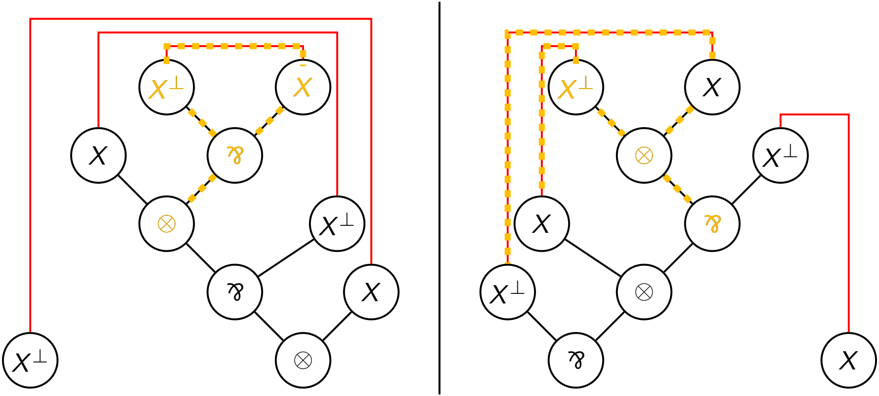
Not-Proof of $X \lhd ((X \otimes (X \parr X^{\perp})) \parr X^{\perp}) \otimes X$



Incorrect

Incorrect retraction generated by Quasi-Beffara

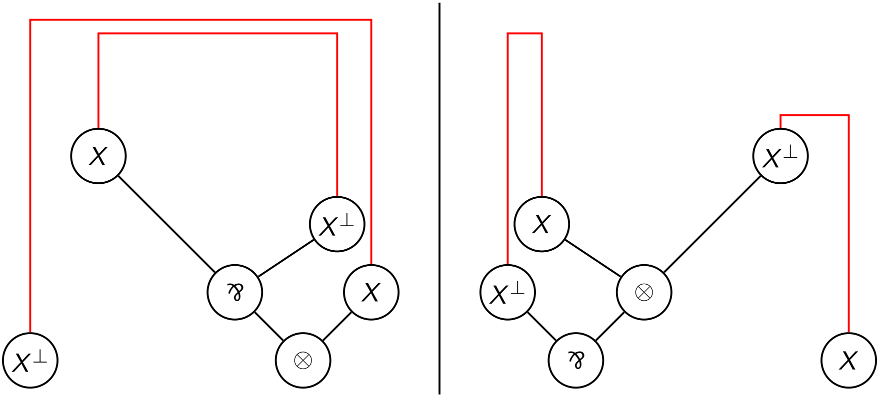
Not-Proof of $X \lhd ((X \otimes (X \parr X^\perp)) \parr X^\perp) \otimes X$



Can apply one step of Quasi-Beffara

Incorrect retraction generated by Quasi-Beffara

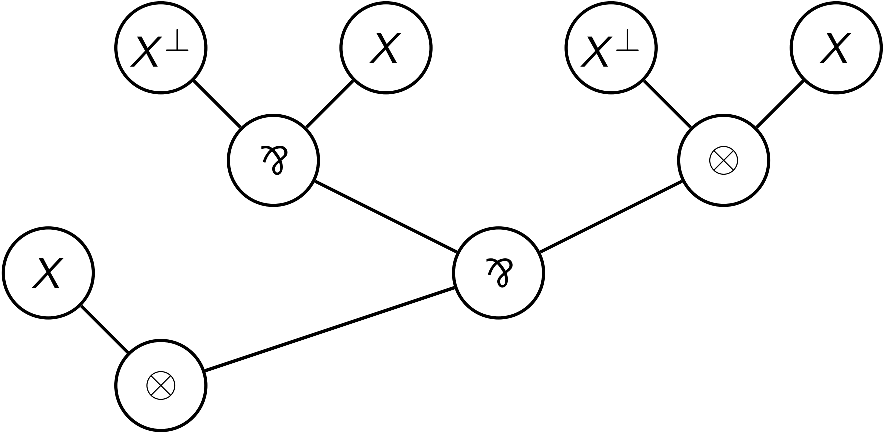
Not-Proof of $X \lhd ((X \otimes (X \parr X^\perp)) \parr X^\perp) \otimes X$



This is Beffara, attainable from X by one step of Quasi-Beffara

Formula not generated by Beffara without iso

$$X \lhd X \otimes ((X^\perp \parr X) \parr (X^\perp \otimes X))$$



Generated by Beffara only up to isomorphism!

Multiplicative Linear Logic - Cut & Sequent

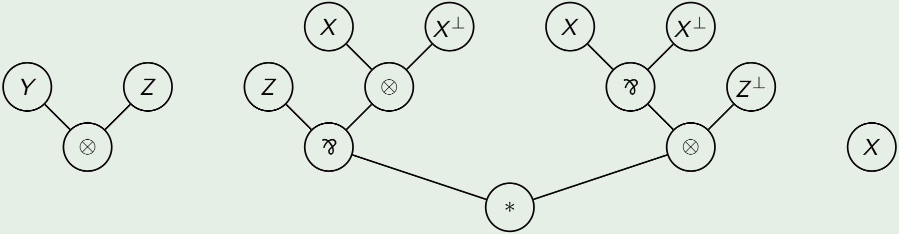
Cut

$$C \ ::= \ A * A^{\perp}$$

Sequent

$$\vdash A_1, \ldots, A_n, C_1, \ldots, C_k$$

Example

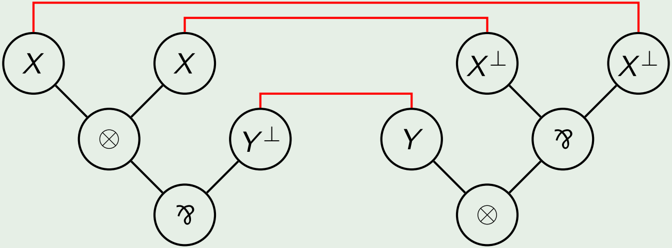


Identity proof net

Identity proof structure of A

In the sequent $\vdash A^\perp, A$, link each leaf in A to the dual one in $A^\perp$.

Example: $A = Y \otimes (X^\perp \parr X^\perp)$



Lemma

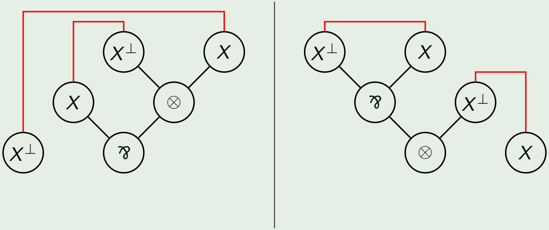
An identity proof structure is correct.

Composition by cut

Composition

Putting side by side a proof structure on $\vdash \Gamma, A$ and one on $\vdash A^\perp, \Delta$, then adding a $*$-node between the roots of A and $A^\perp$, yields a proof structure on $\vdash \Gamma, A * A^\perp, \Delta$.

Example

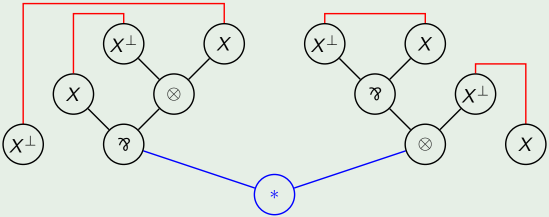


Composition by cut

Composition

Putting side by side a proof structure on $\vdash \Gamma, A$ and one on $\vdash A^\perp, \Delta$, then adding a $*$-node between the roots of A and $A^\perp$, yields a proof structure on $\vdash \Gamma, A * A^\perp, \Delta$.

Example

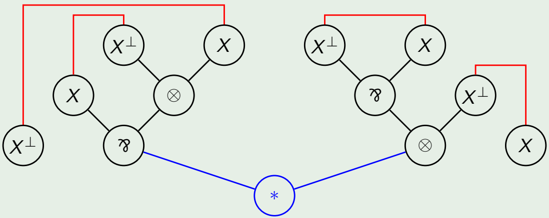


Composition by cut

Composition

Putting side by side a proof structure on $\vdash \Gamma, A$ and one on $\vdash A^\perp, \Delta$, then adding a $*$-node between the roots of A and $A^\perp$, yields a proof structure on $\vdash \Gamma, A * A^\perp, \Delta$.

Example

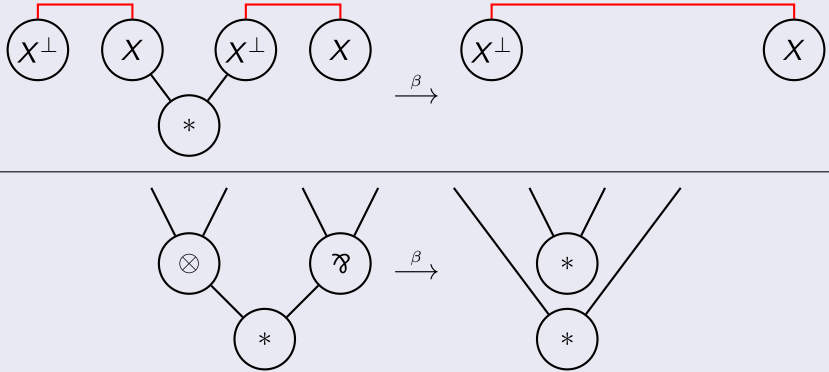


Lemma

The composition of two correct proof structures is correct.

Cut elimination

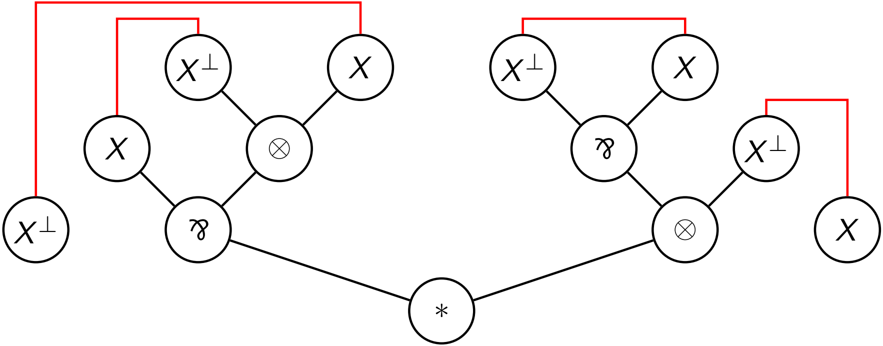
Cut elimination



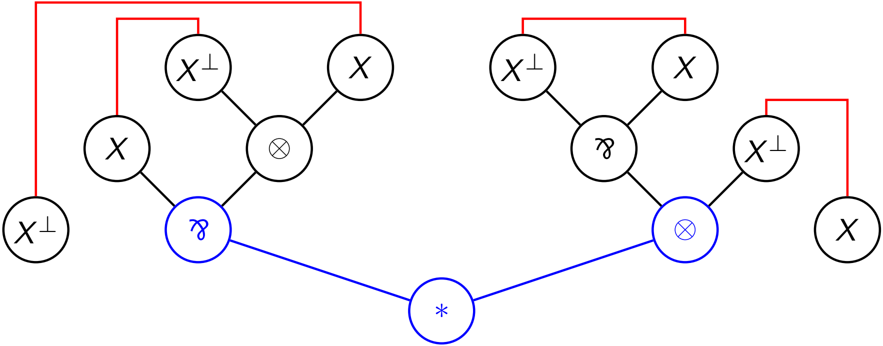
Lemma

Cut elimination preserves correction, is confluent and strongly normalizing.

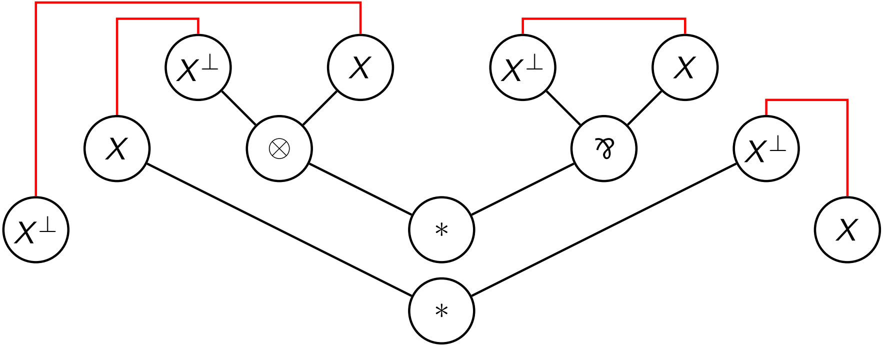
Example of cut elimination



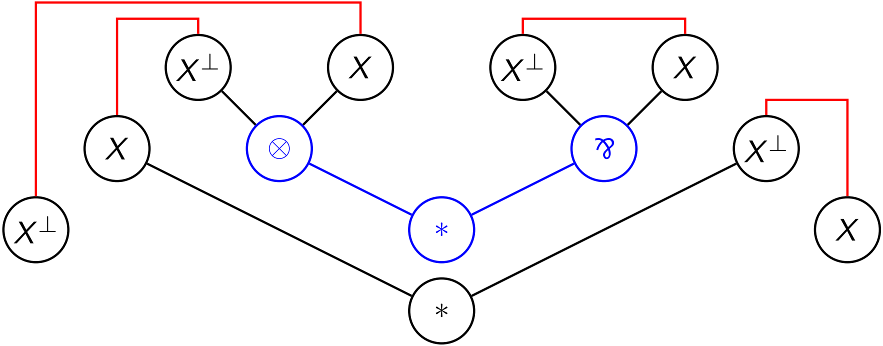
Example of cut elimination



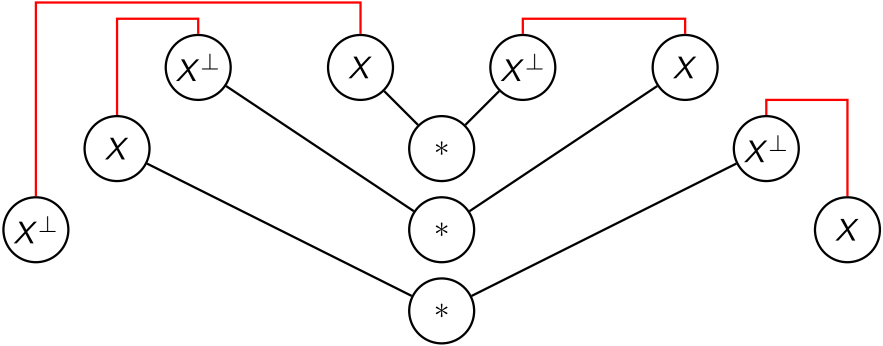
Example of cut elimination



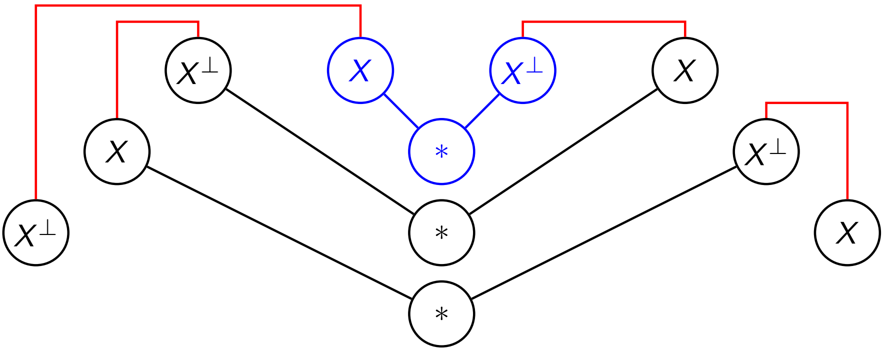
Example of cut elimination



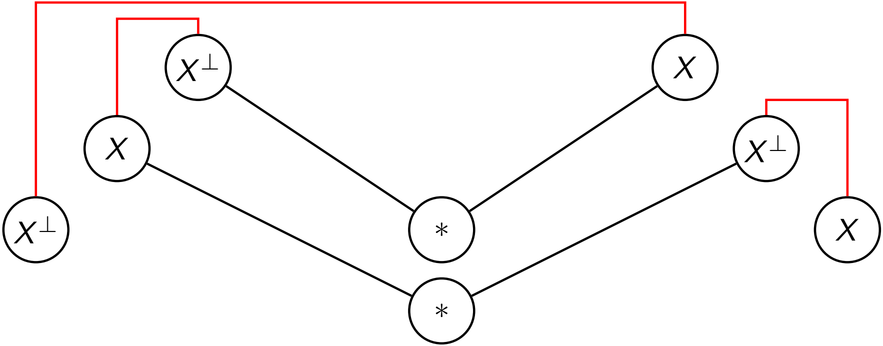
Example of cut elimination



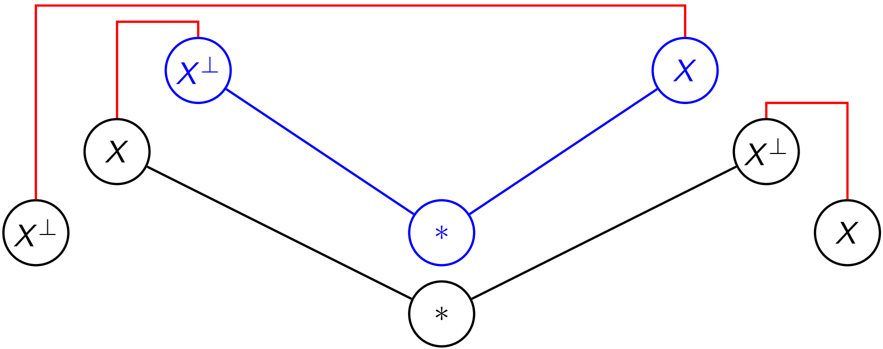
Example of cut elimination



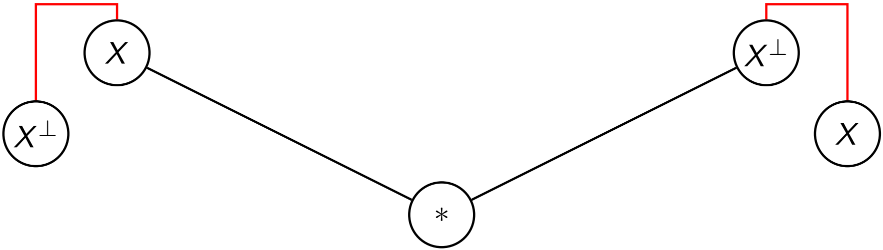
Example of cut elimination



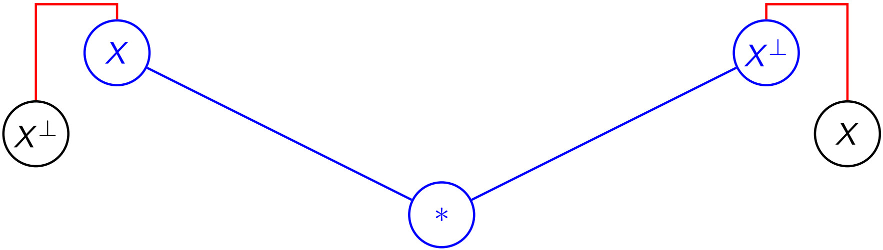
Example of cut elimination



Example of cut elimination



Example of cut elimination



Example of cut elimination