Polynomial χ-Boundedness of Bounded Twin-Width Classes

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Joint work with Stéphan Thomassé
École Normale Supérieure de Lyon

FPT Fest in the honour of Mike Fellows, Bergen, 2023

χ-boundedness & Twin-width

Definition (χ-bounded)

If $\mathcal{C}$ is a class of graphs, $\mathcal{C}$ is *χ-bounded* if $\forall G \in \mathcal{C}, \chi(G) \leq f(\omega(G))$ for some function f.
If f is a polynomial, $\mathcal{C}$ is *polynomially χ-bounded*.

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Theorem [Bonnet, B., Duron, Geniet, Thomassé, Trotignon '23]

For every t, there exists a triangle-free graph with twin-width t and chromatic number $t+1$.

Main result

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- Gives an efficient coloring algorithm.

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- Edge partition: $E(G) = E_1 \cup E_2$ s.t. $(V(G), E_1)$ and $(V(G), E_2)$ are simpler.
  With the product coloring: $\chi(G) \leq \chi(G_1) \cdot \chi(G_2)$.

Modular decomposition

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If $G = (V, E)$ is a graph, we say that $X \subseteq V$ is a *module* if $\forall v \in V \setminus X$, either v is complete to X or v is anti-complete to X.

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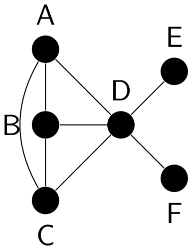
Idea: Partition G into modules such that the quotient graph is simple, iterate within each module.

Tree for modular decomposition



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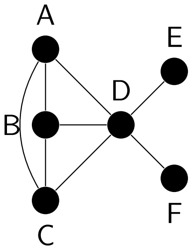

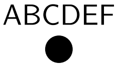

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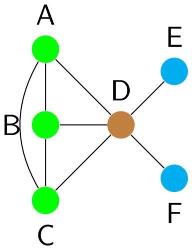

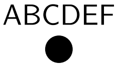

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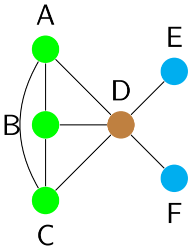

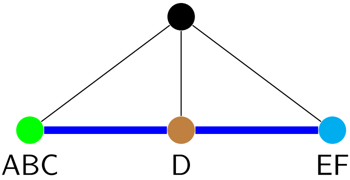

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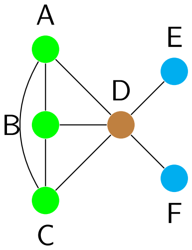

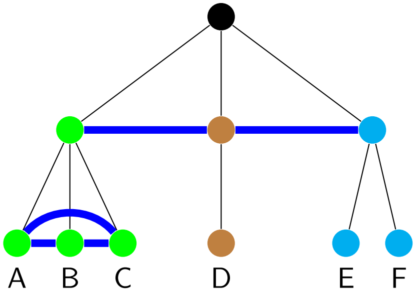

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If $\mathcal{C}$ is a class of graphs, the *substitution closure* $\mathcal{C}_s$ of $\mathcal{C}$ is the set of graphs that we can decompose such that all quotient graphs are in $\mathcal{C}$.

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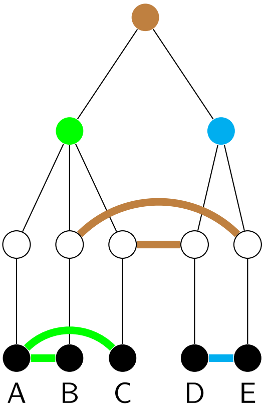
Theorem [Chudnovsky, Penev, Scott, Trotignon '13]

If $\mathcal{C}$ is polynomially χ-bounded, so is $\mathcal{C}_s$.

Delayed substitution



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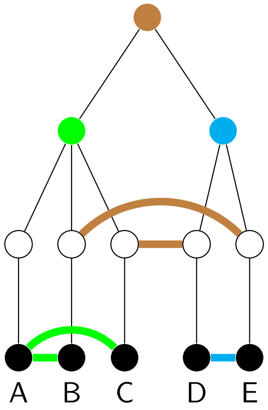

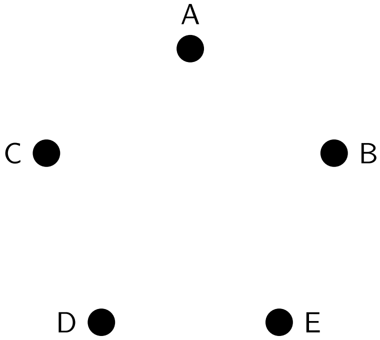

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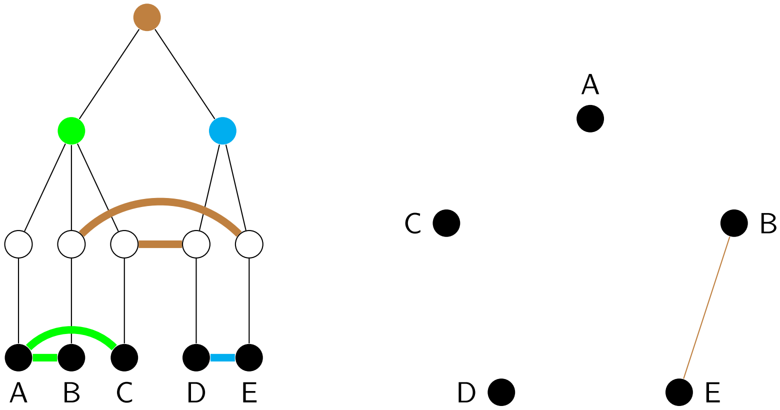

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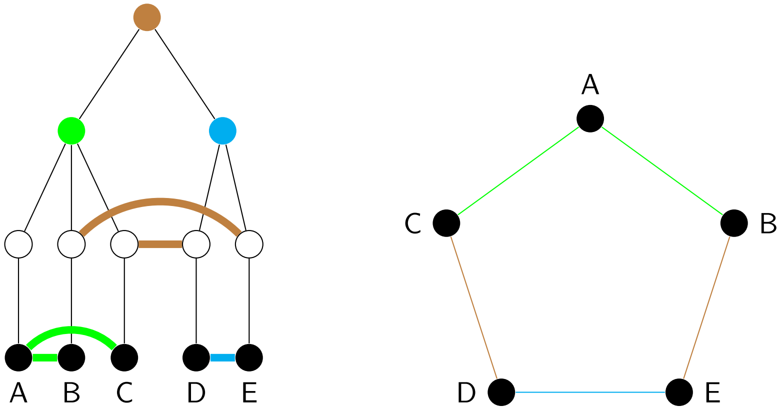

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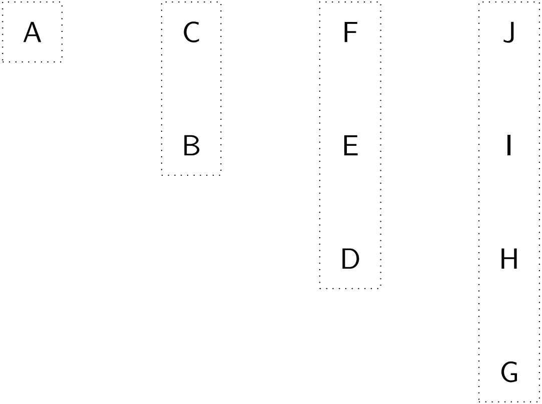

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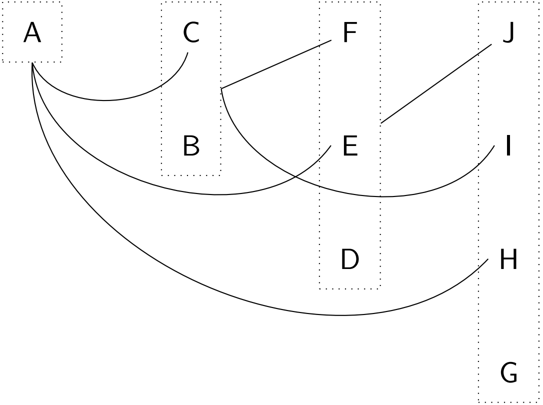

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If $\mathcal{C}$ is a class of graphs, the *right extension* $RM(\mathcal{C})$ of $\mathcal{C}$ is the set of graphs that have an RMP where all transversals are in $\mathcal{C}$.

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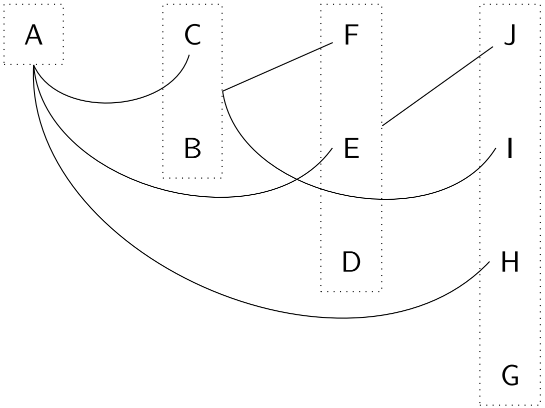
- Issue: with this definition, the right extension of the class of all forests is not χ-bounded.
- Shift graphs: triangle-free, unbounded χ.

Transversal minor



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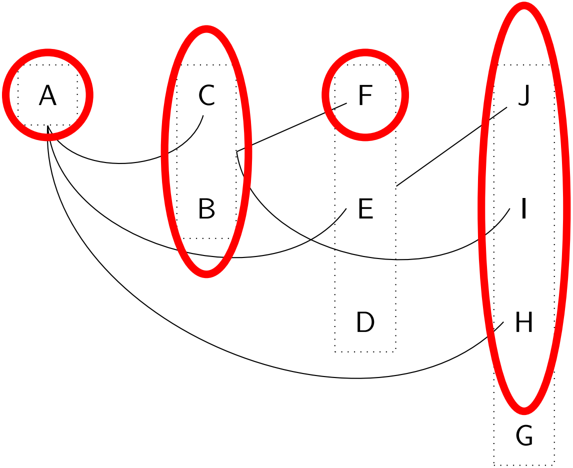

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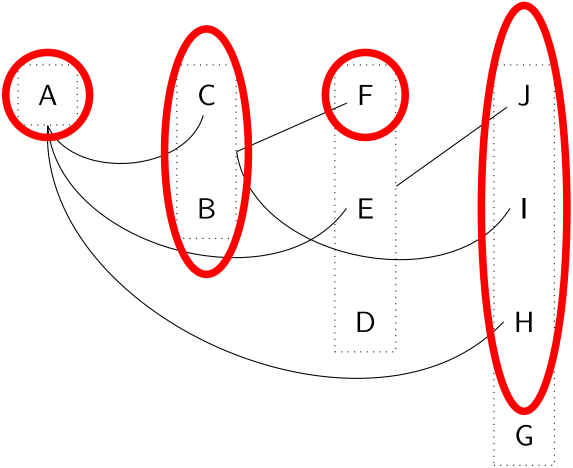

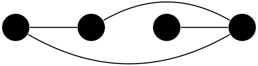

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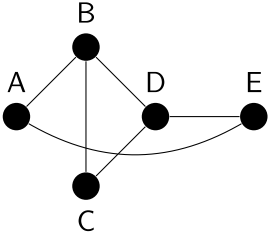
Theorem

If $\mathcal{C}$ is polynomially χ-bounded, then the class of graphs that have a "nice" $\mathcal{C}$-RMP is also polynomially χ-bounded.

How to obtain a delayed decomposition?

Start from an ordered graph G.



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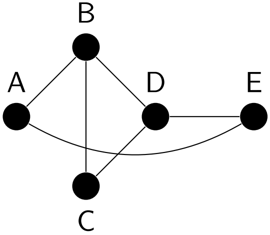

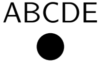

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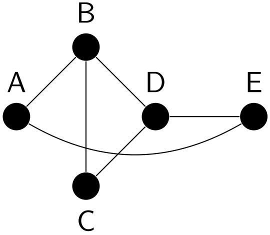

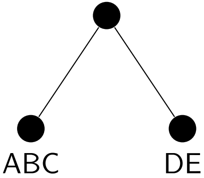

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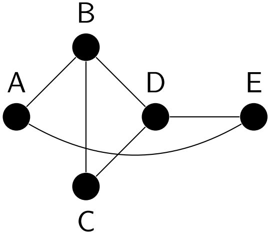

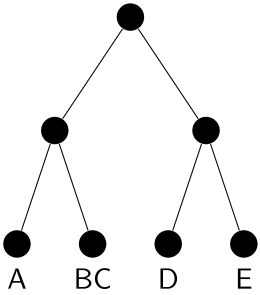

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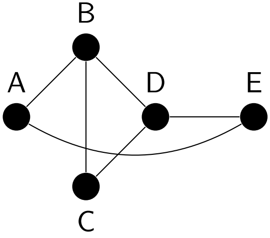

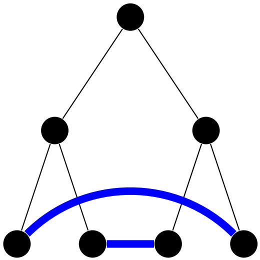

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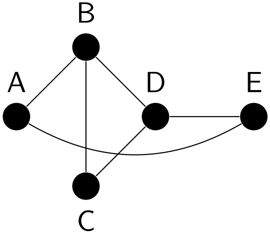

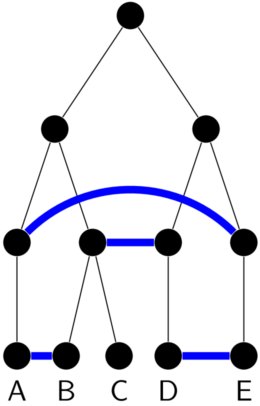

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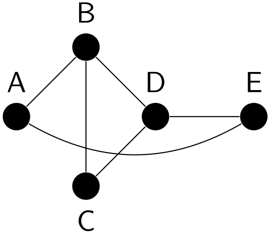

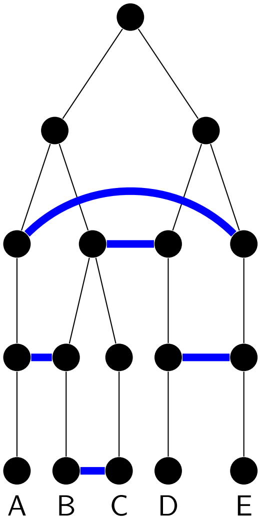

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- Prove that all transversal minors have twin-width $\leq t - 1$.

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- For triangle-free graphs of twin-width t, do we have $\chi \leq t + 1$?