A tamed family of triangle-free graphs with unbounded chromatic number

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LaBRI

Joint work with Édouard Bonnet, Julien Duron, Colin Geniet, Stéphan Thomassé and Nicolas Trotignon

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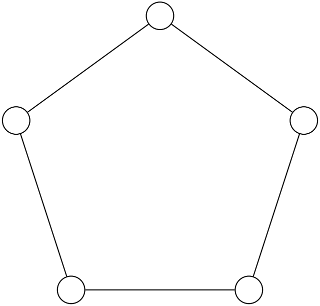

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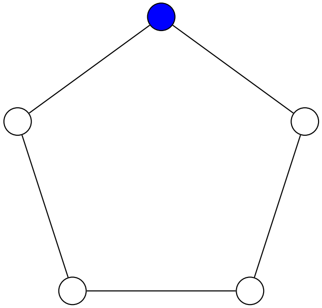

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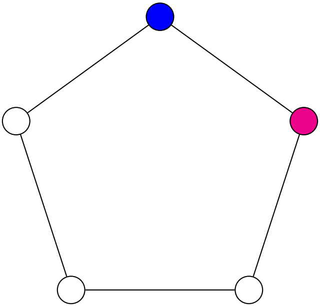

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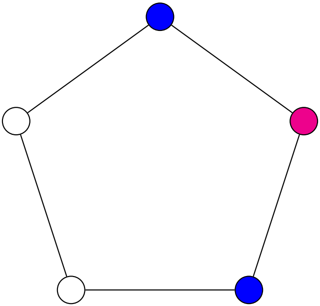

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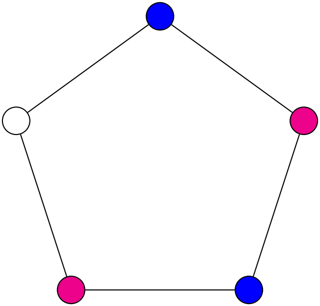

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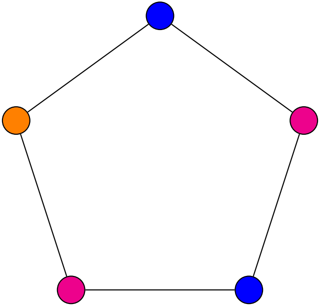

Gap between ω and χ

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There exist triangle-free graphs with arbitrarily large χ.

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- Zykov (1952)
- Blanche Descartes (1954)
- Mycielski (1955)
- Erdős (1959)
- Burling (1965)
- ...

Zykov operator

$$G_1, G_2, \ldots, G_k \longrightarrow Z(G_1, G_2, \ldots, G_k)$$

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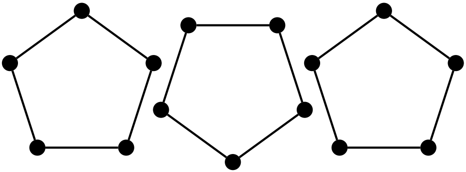

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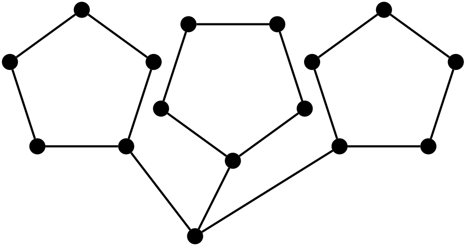

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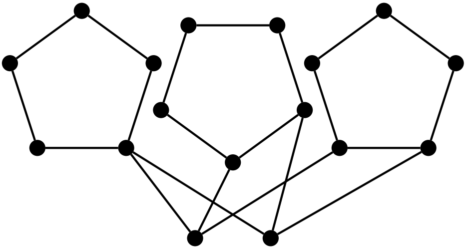

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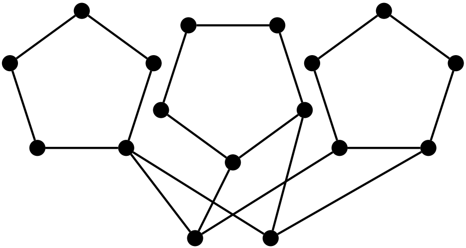

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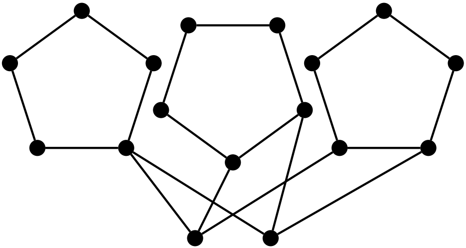

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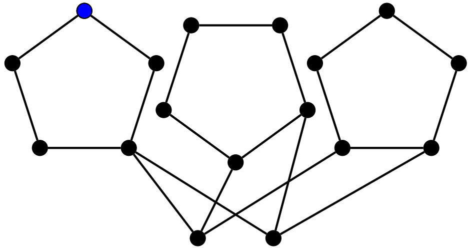

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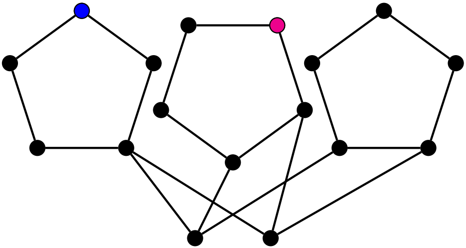

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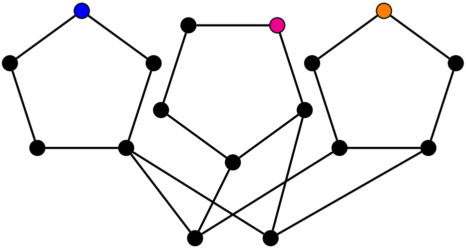

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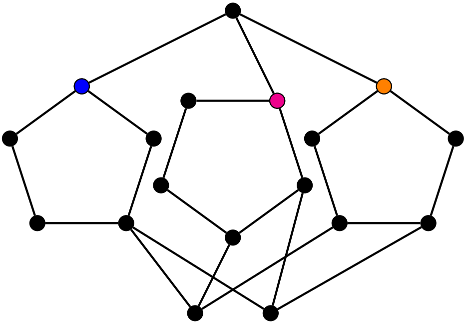

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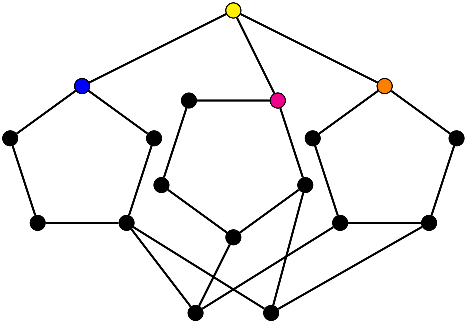

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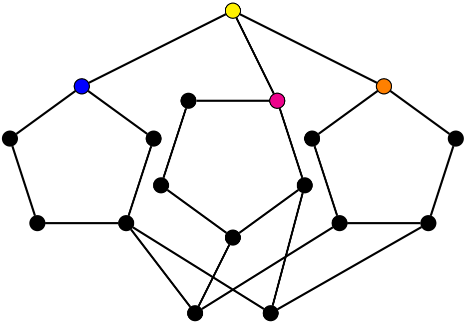

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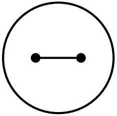

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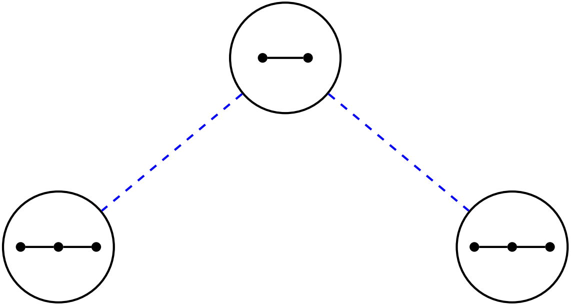

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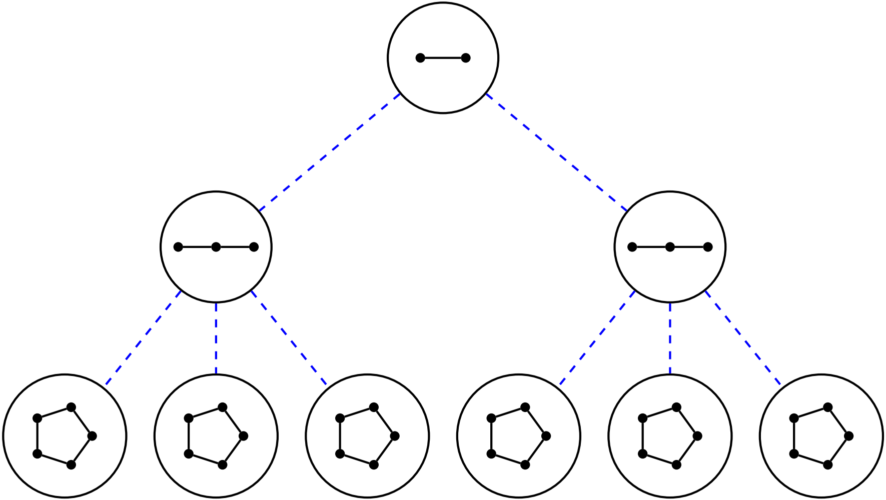

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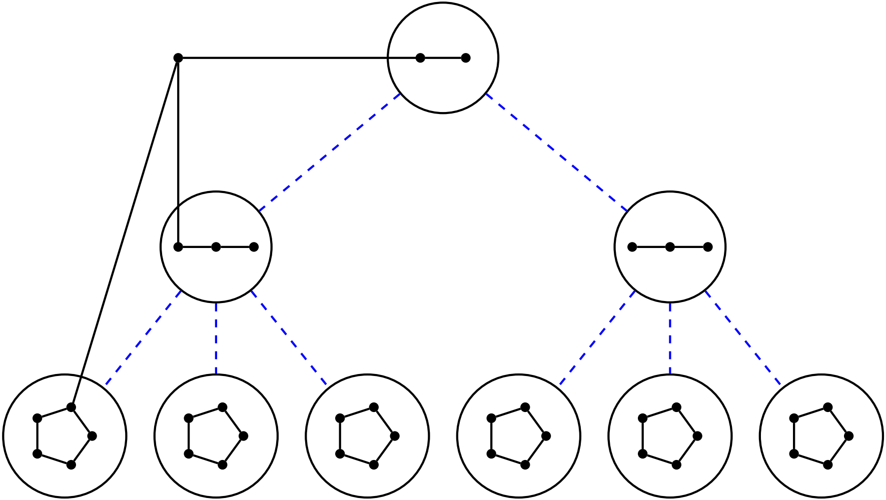

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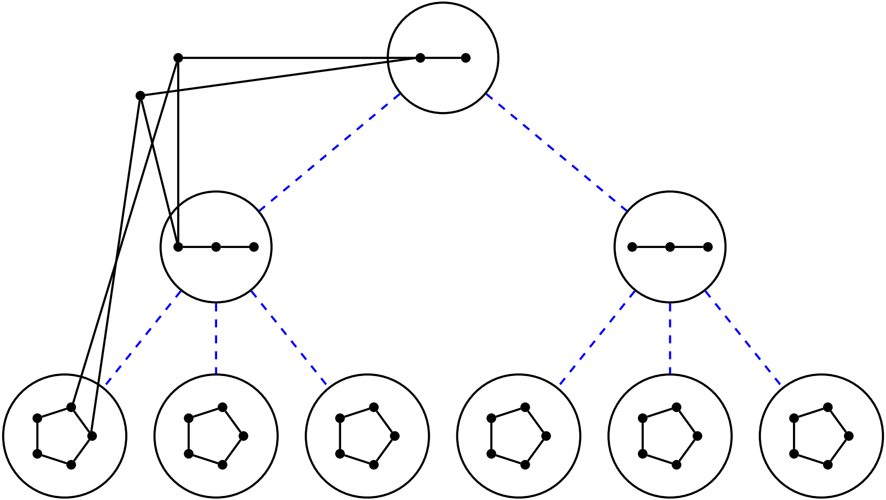

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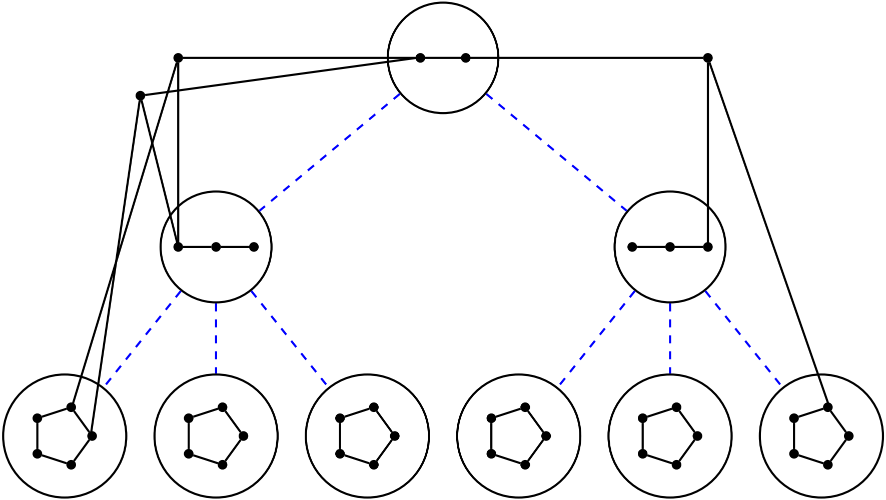

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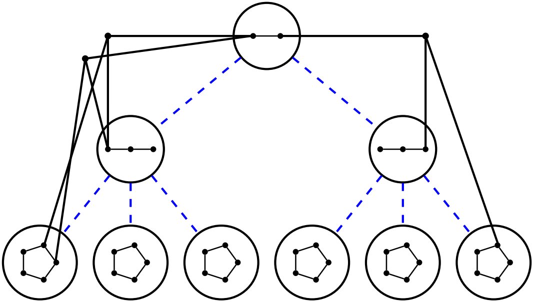

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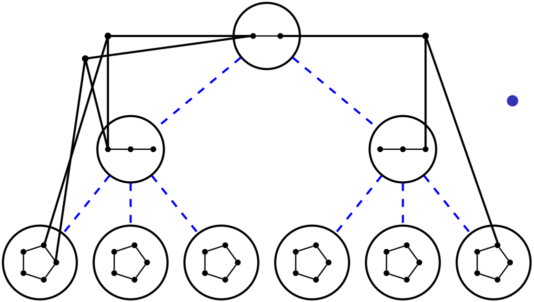

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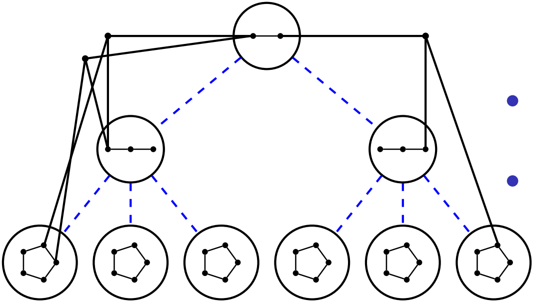

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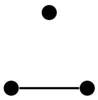

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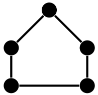

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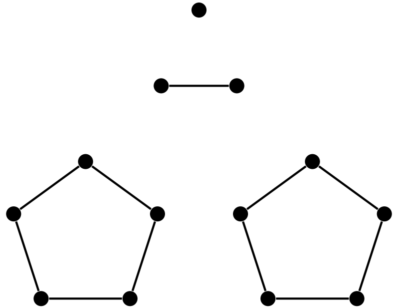

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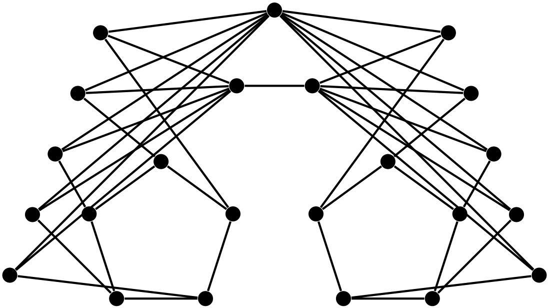

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Equivalently, every nontrivial induced subgraph of a T_k either has false twins or a separation of order at most 2.

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Theorem [Bonnet, Geniet, Kim, Thomassé, Watrigant '21]

Every triangle-free graph of chromatic number k has twin-width at least $k - 2$.

Open questions

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