A tamed family of triangle-free graphs with unbounded chromatic number

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LaBRI

Joint work with Édouard Bonnet, Julien Duron,
Colin Geniet, Stéphan Thomassé and Nicolas Trotignon[1]

September 6, 2024

[1] Source of inspiration for this presentation.

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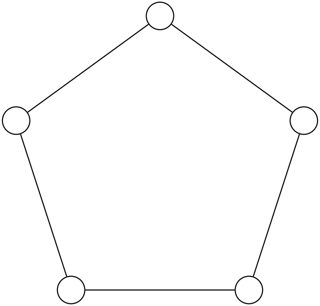

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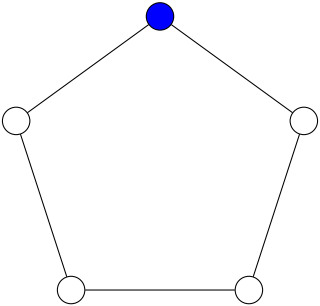

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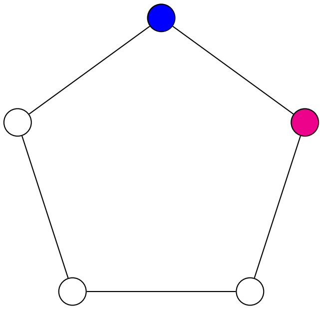

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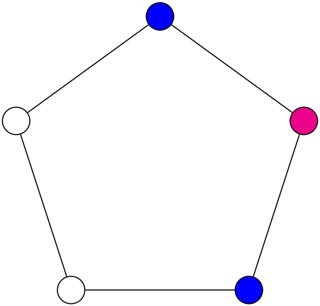

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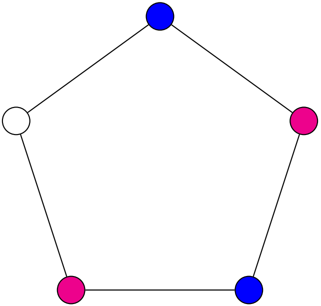

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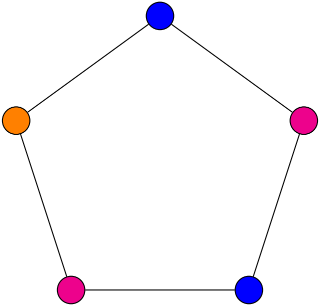

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Definition (χ-boundedness)

$\mathcal{C}$ - hereditary, is χ-bounded if there exists f such that $\chi(G) \leq f(\omega(G))$ for every $G \in \mathcal{C}$.

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- Zykov (1952)
- Blanche Descartes (1954)
- Mycielski (1955)
- Erdős (1959)
- Burling (1965)
- ...

Zykov operator

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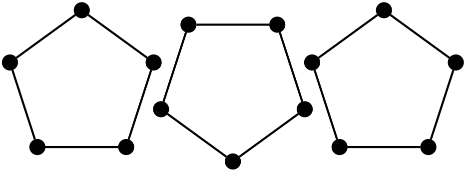

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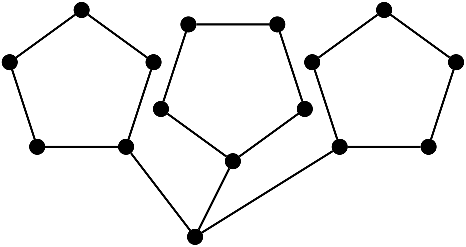

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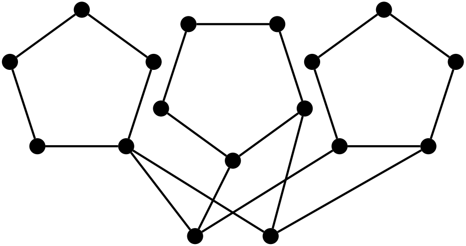

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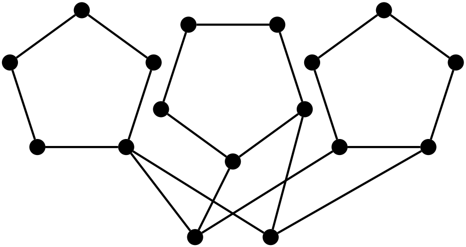

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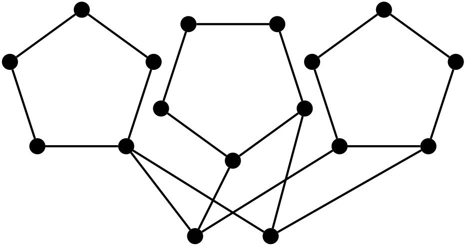

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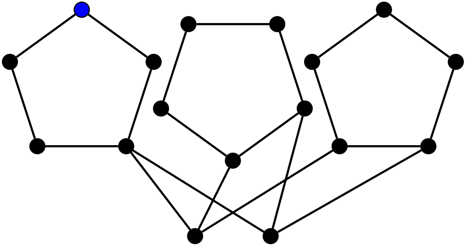

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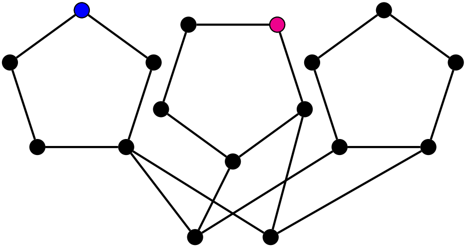

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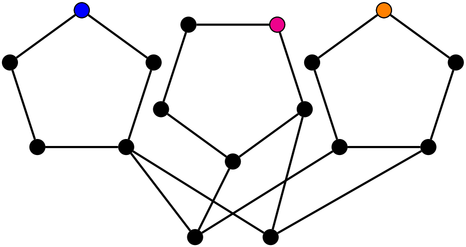

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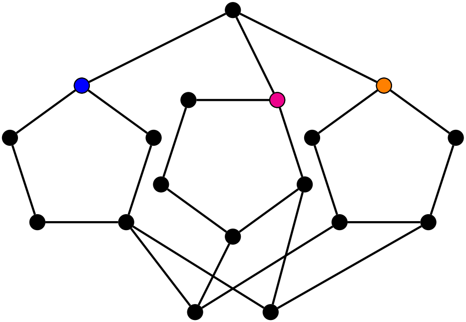

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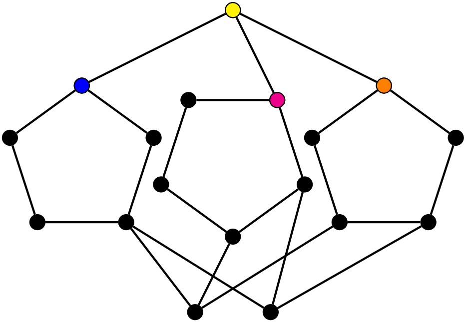

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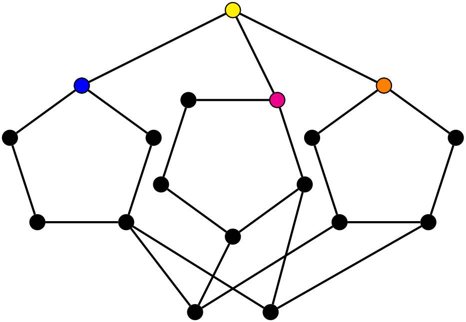

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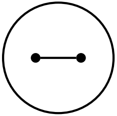

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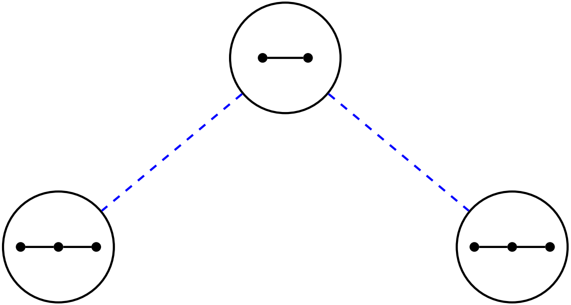

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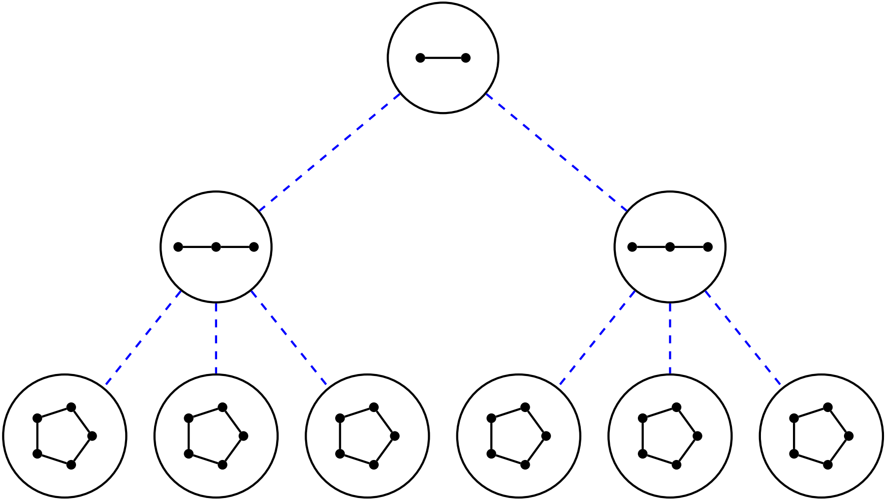

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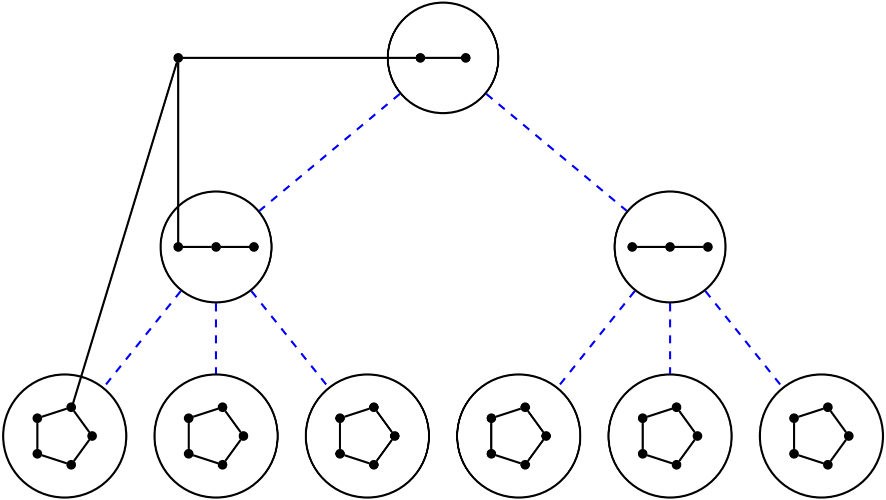

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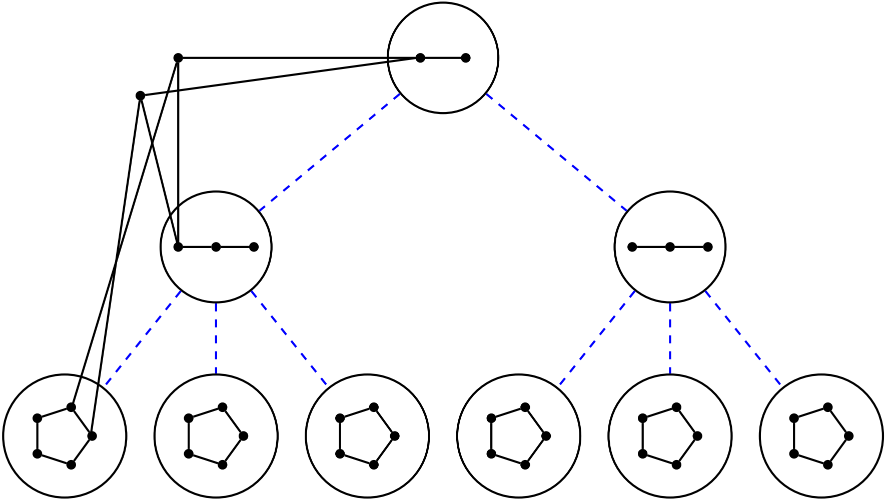

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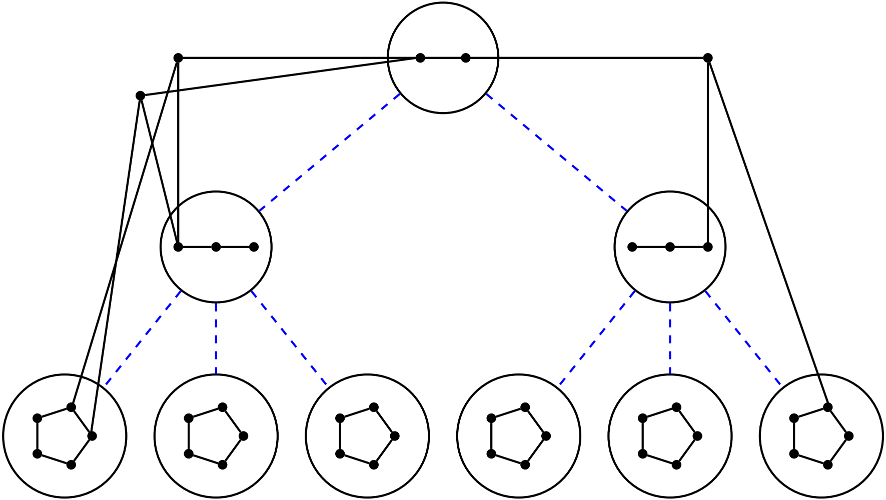

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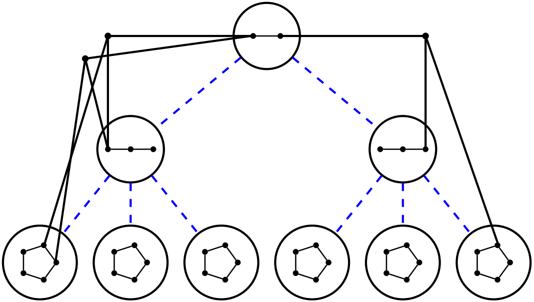

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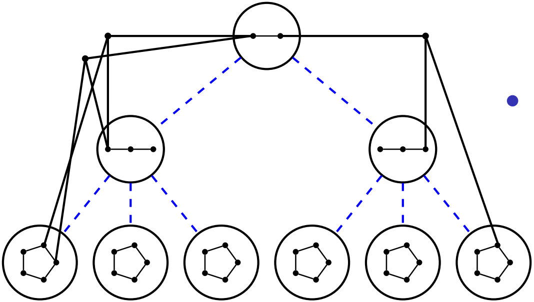

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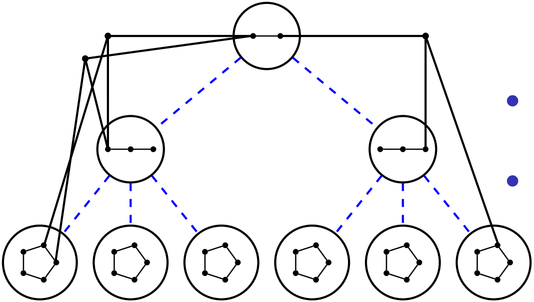

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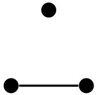

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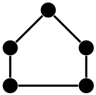

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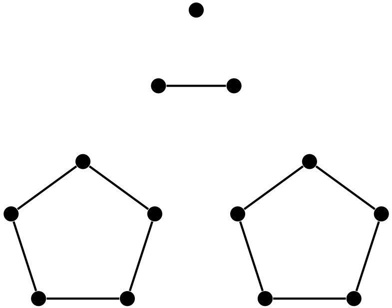

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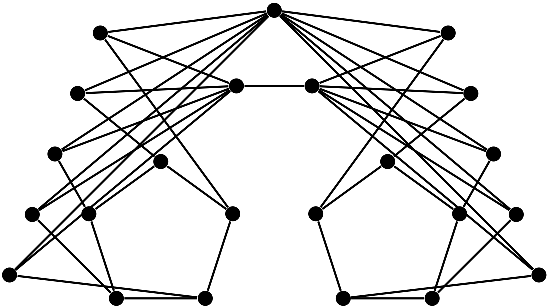

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- K_5-minor-free graphs
- Pattern-avoiding permutations
- Totally unimodular matrices

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Remark

Also true for (polynomially) χ-bounded.

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Theorem [Chudnovsky, Penev, Scott, Trotignon '13]

If $\mathcal{C}$ is hereditary and (polynomially) χ-bounded then so is $\mathcal{C}^*$.

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Proof: Induction on the number of substitutions.
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Use the product of the colorings.

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Theorem [Chudnovsky, Penev, Scott, Trotignon '13]

- If $\mathcal{C}$ is hereditary and χ-bounded then the closure of $\mathcal{C}$ under gluing on cliques and gluing on sets of size $\leq k$ is χ-bounded.

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NO: Twincut sequence!

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Every T_k can be built from $\{K_1, K_2, \overline{K_2}\}$ by gluing on sets of size at most 2 and substituting by $\overline{K_2}$.

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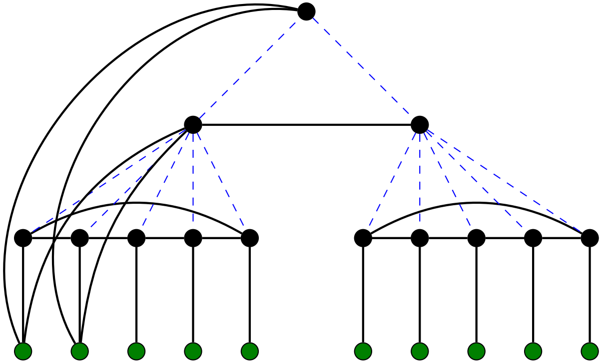

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Proposition

- Deciding whether a twincut graph is 3-colorable is NP-complete.
- The maximum stable set problem is NP-hard on twincut graphs.

Other properties

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Every T_k is edge-critical: for every edge e of T_k, $\chi(T_k - e) = k - 1$.

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Theorem [Bonnet, Geniet, Kim, Thomassé, Watrigant '21]

Every triangle-free graph of chromatic number k has twin-width at least $k - 2$.

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Thank you! Questions?