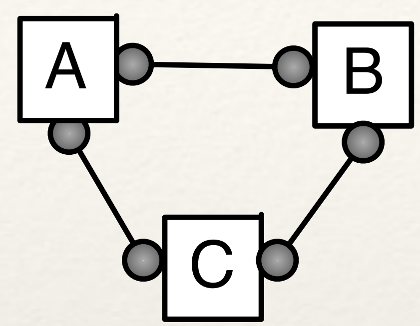
Efficient update algorithm for Kappa graph rewriting

Incremental update

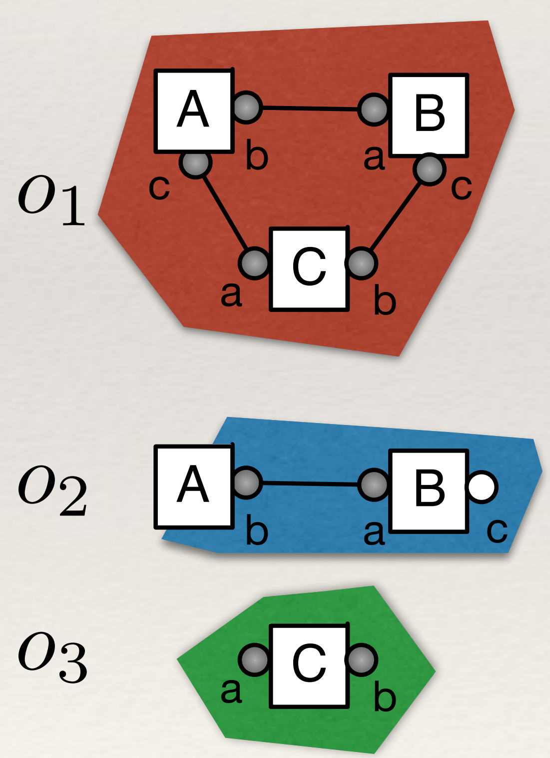
Jean Krivine
CNRS and Univ. Paris Diderot
PPS laboratory

Pierre Boutillier
Harvard Medical School

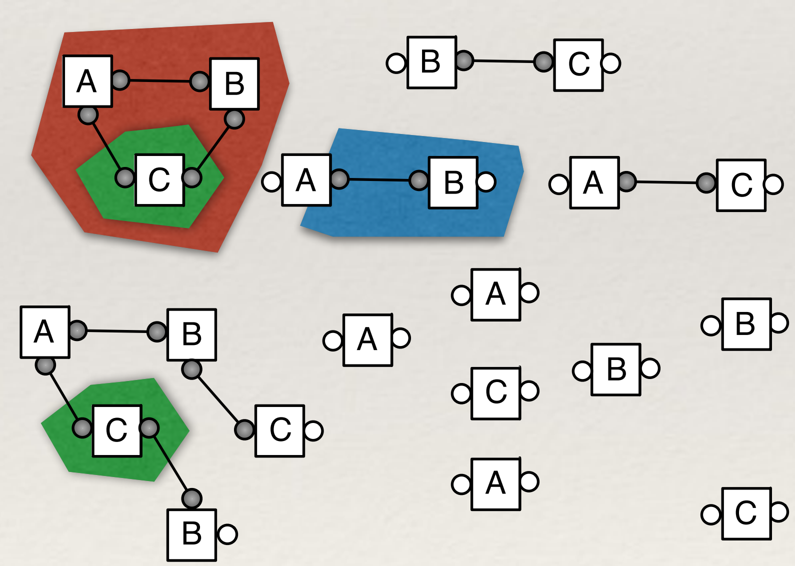
Classical algorithm



Contact graph

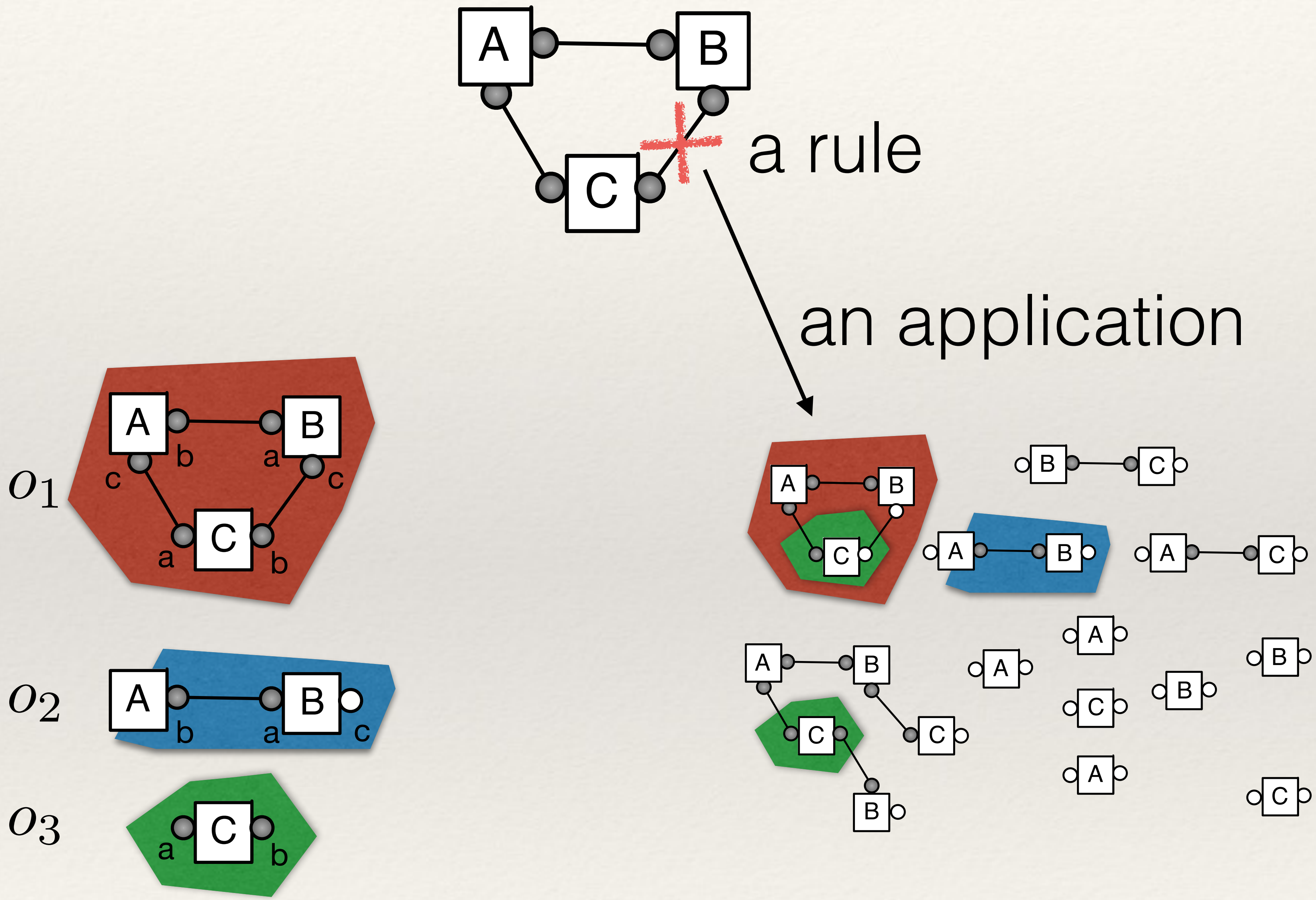


Observables

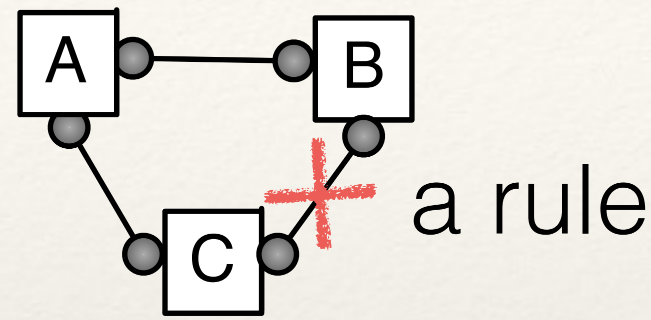


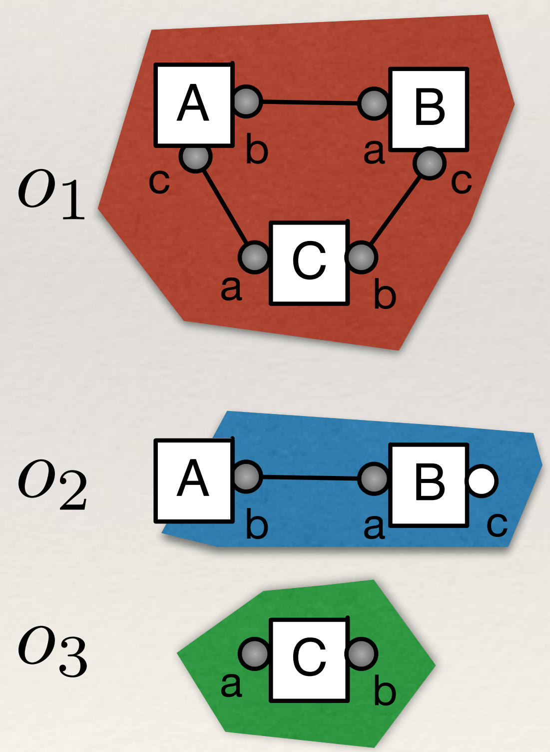
State

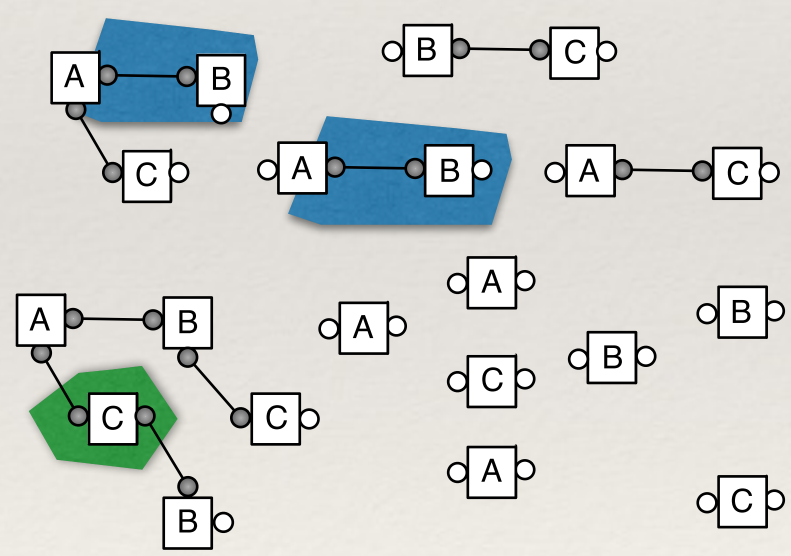
Negative update



Positive update





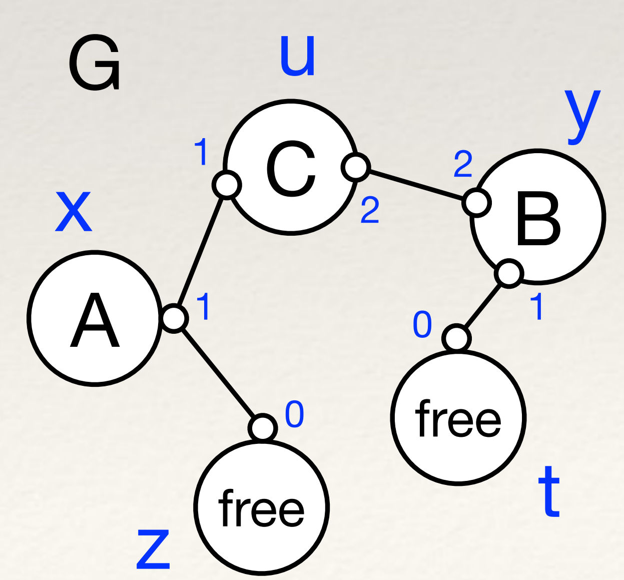


Concrete nodes

- Forget for now that observables are intentionally described
- Assume an infinite set of nodes N (meta-variables u,v,w)
- Nodes are sorted according to $\kappa : \mathsf{N} \to \{A, B, C, \dots\} = \mathsf{K}$
- Assume also a signature map $\Sigma : \mathsf{K} \to \mathbb{N}$
- We define $\mathsf{intf} : \mathsf{N} \to \mathbb{N}$ as $\Sigma \circ \kappa$ (i.e number of sites)

Concrete edges

- An *edge* is a two element set $\{p, q\}$ where $p, q \in \mathsf{N} \times \mathbb{N}$
- where $p = (u, i) \implies i < \mathsf{intf}(u)$
- two edges are connected if they share a node
- A (concrete) site graph is a set of edges.
- We use a special kind 'free' (with arity 1)



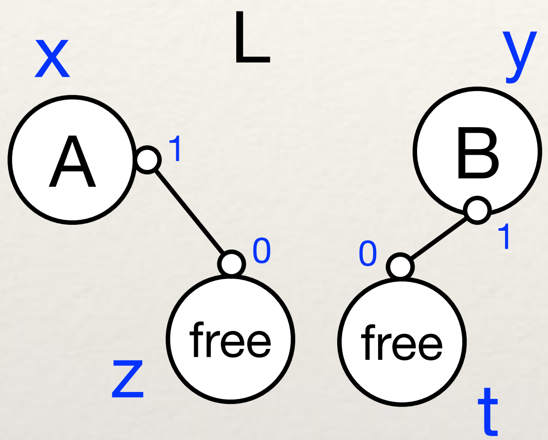
Concrete update

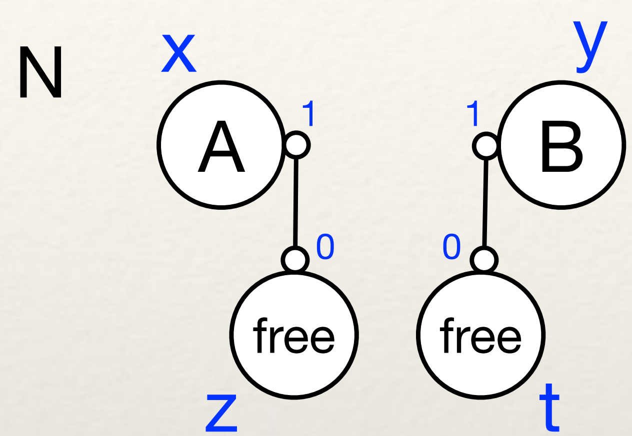
- An *update* U is a triple (L,N,P) where L,N,P are set of edges and such that $N \subseteq L$ and $P \cap L = \emptyset$
- An *instance* of U in a graph G is defined as

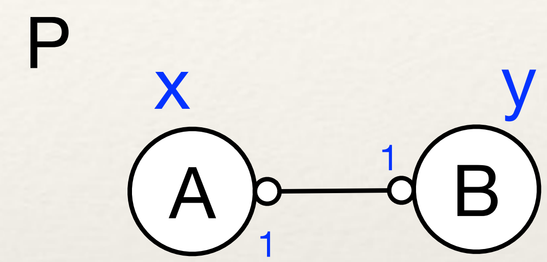
$$G \rightarrow_{\mathsf{U}} G' \text{ with } G' = (G \backslash N) \cup P.$$

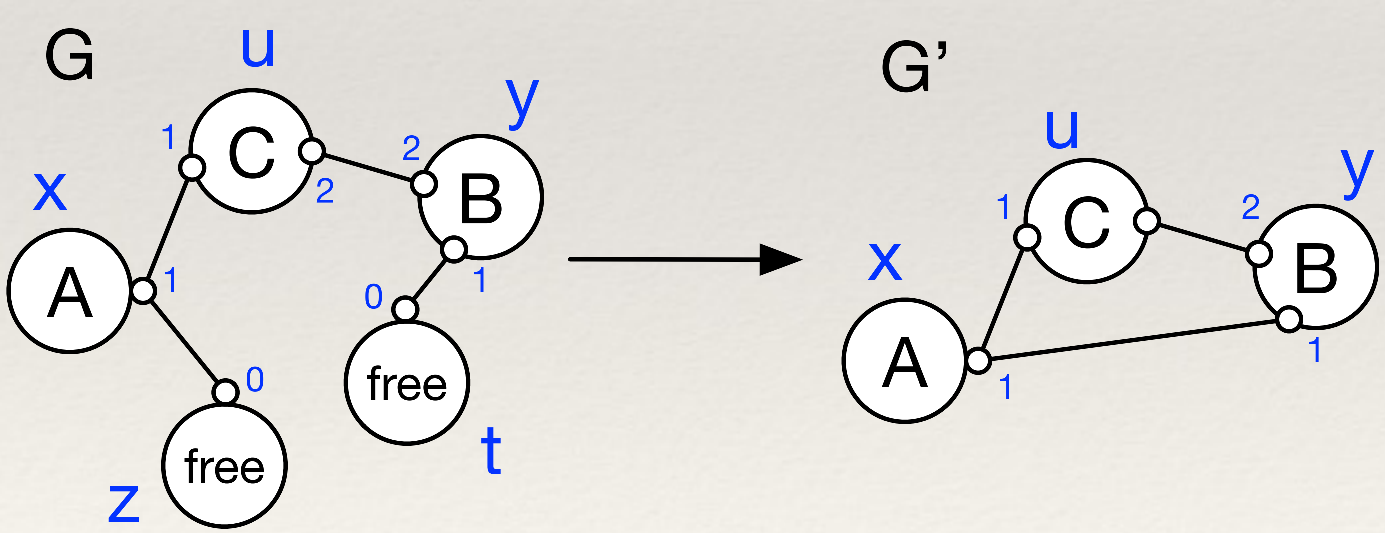
For all G such that $L \subseteq G$ and $P \cap G = \emptyset$.

Concrete update instance (example)







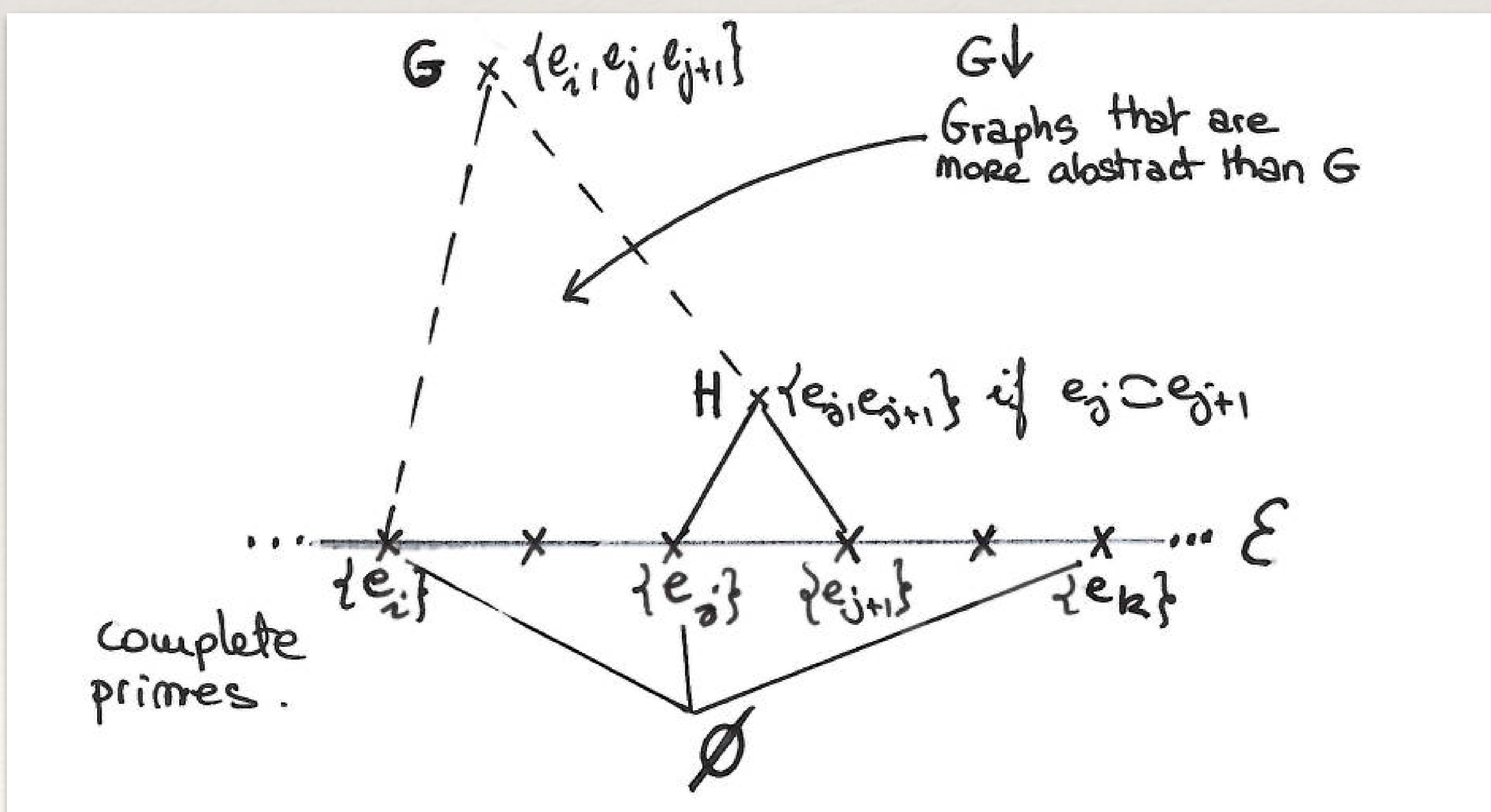


We define $\mathsf{pre}(\mathsf{U}) \triangleq L$ and $\mathsf{post}(\mathsf{U}) = (L \setminus N) \cup P$.

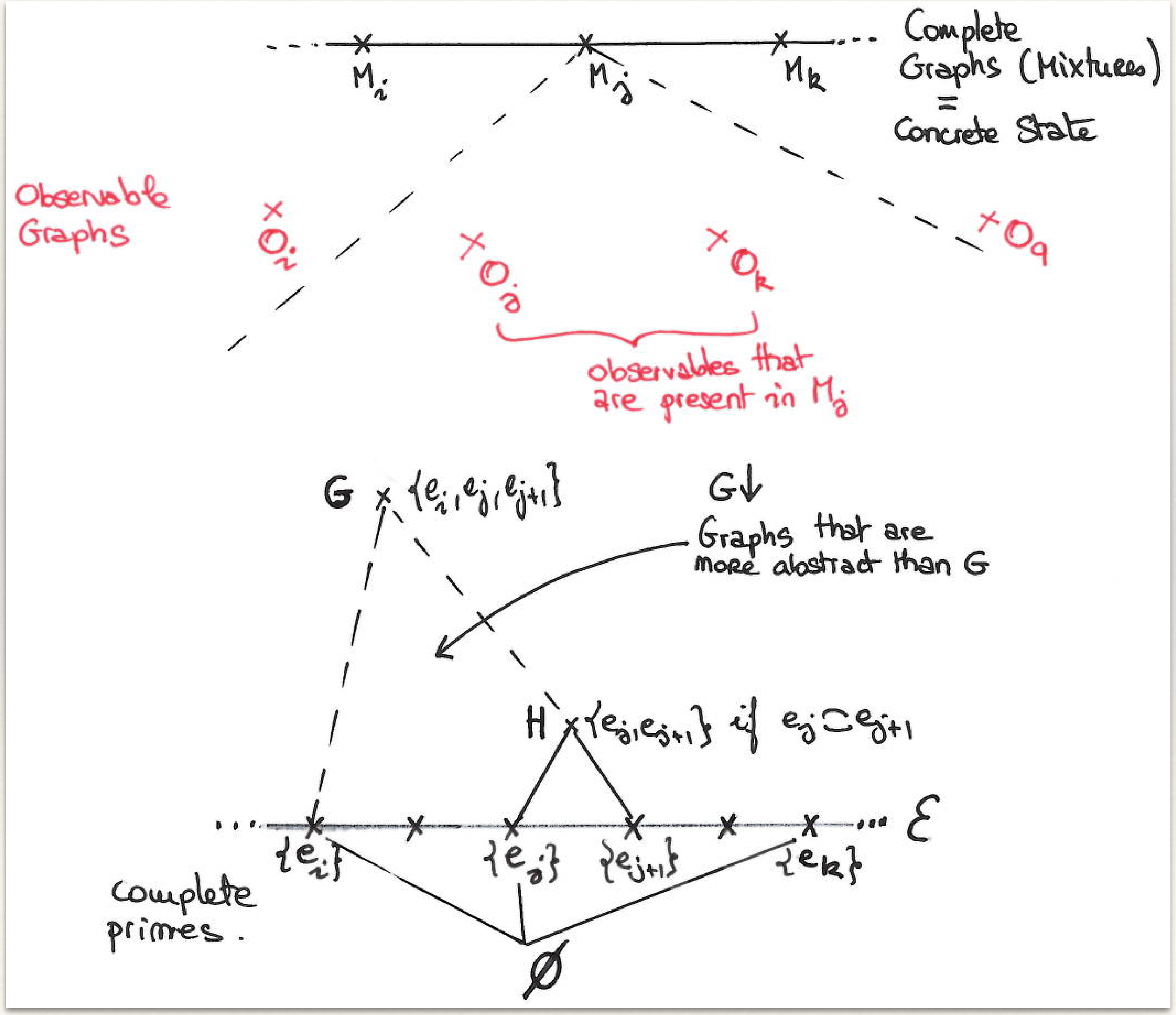
Domain of consistent graphs

The domain of consistent graphs can be seen as the *coherent space* $\mathcal{G}$ generated by $(\mathcal{E}, \frown_{\mathcal{E}})$. The graphs of $\mathcal{G}$ are sets of edges (they are the *cliques* of the graph induced by the coherence relation) that satisfy the coherence relation $\frown_{\mathcal{E}} \subseteq \mathcal{E} \times \mathcal{E}$ defined as:

$$\forall e, e' \in \mathcal{E}. e \underset{\mathcal{E}}{\frown} e' \iff e \cap e' = \emptyset \lor e = e'$$



Observables



Complete graph M:

$$\forall u \in M, \forall i < \mathsf{intf}(u) : \exists e \in M.(u, i) \in e$$

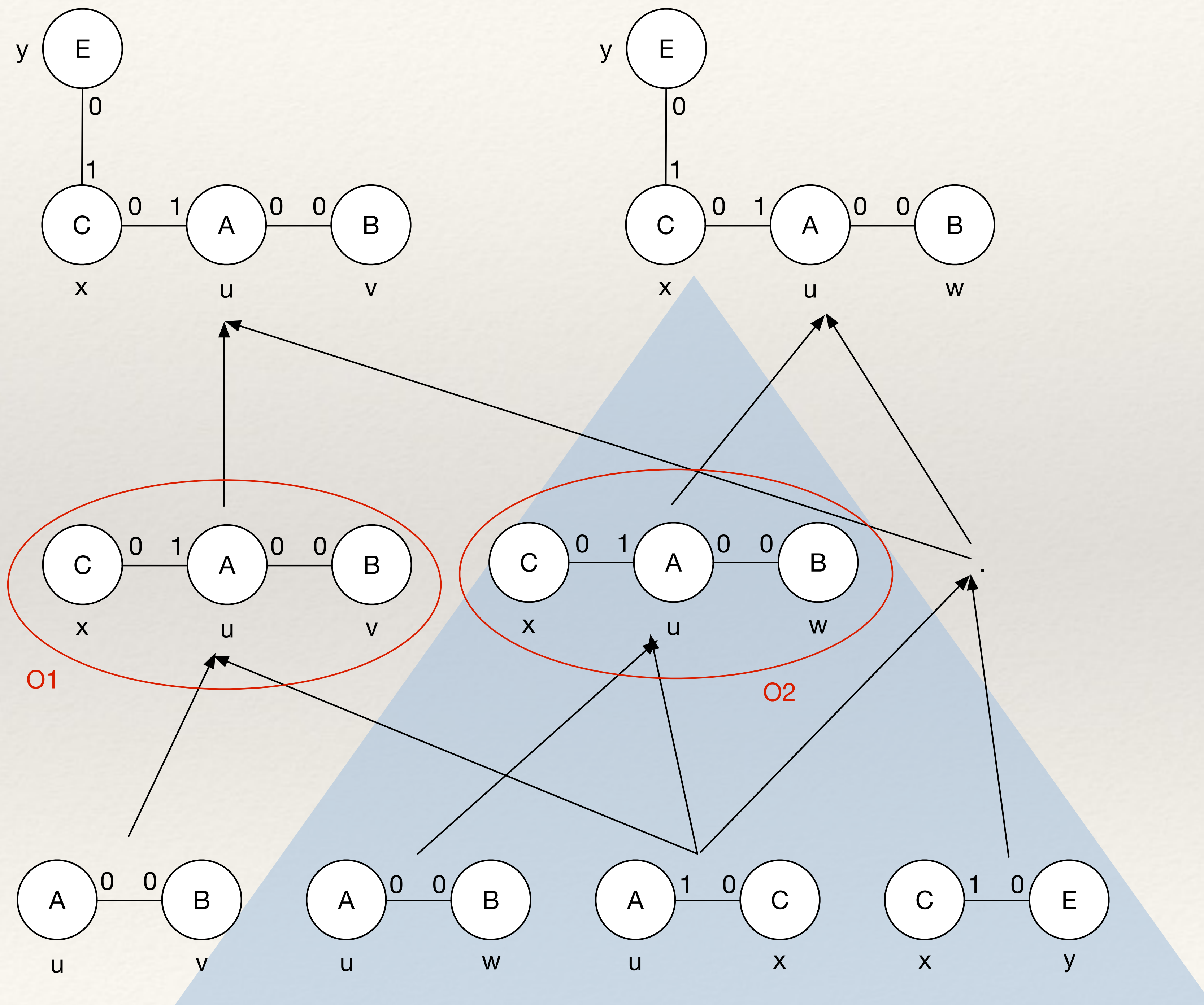
Observable:

Assume $\mathsf{Obs} : \mathcal{G} \to 2$

Define

$$\mathcal{O}_M = \{G \subseteq M \mid \mathsf{Obs}(G)\}$$

Example



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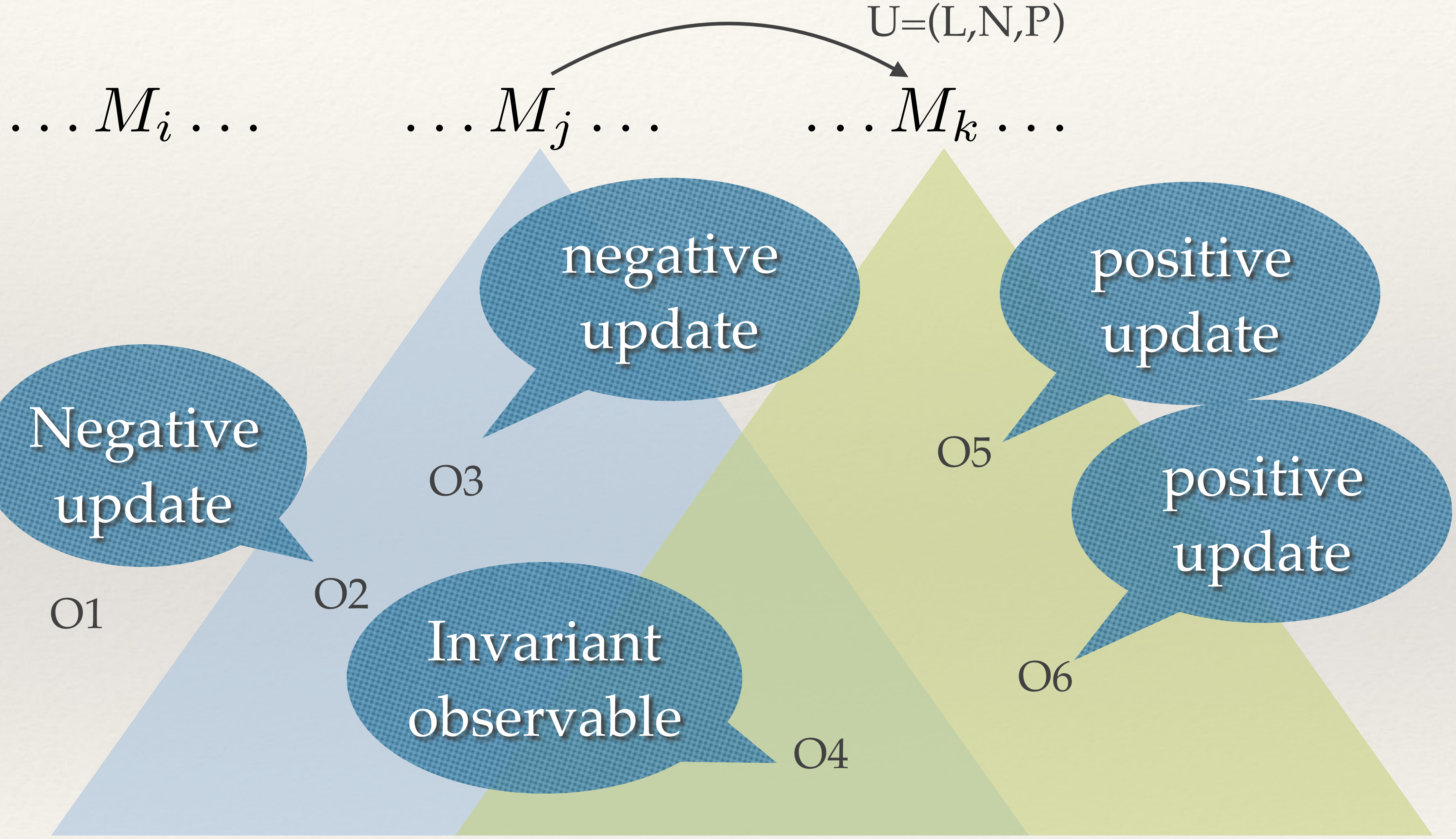
The update problem



Update without exploring M_k !

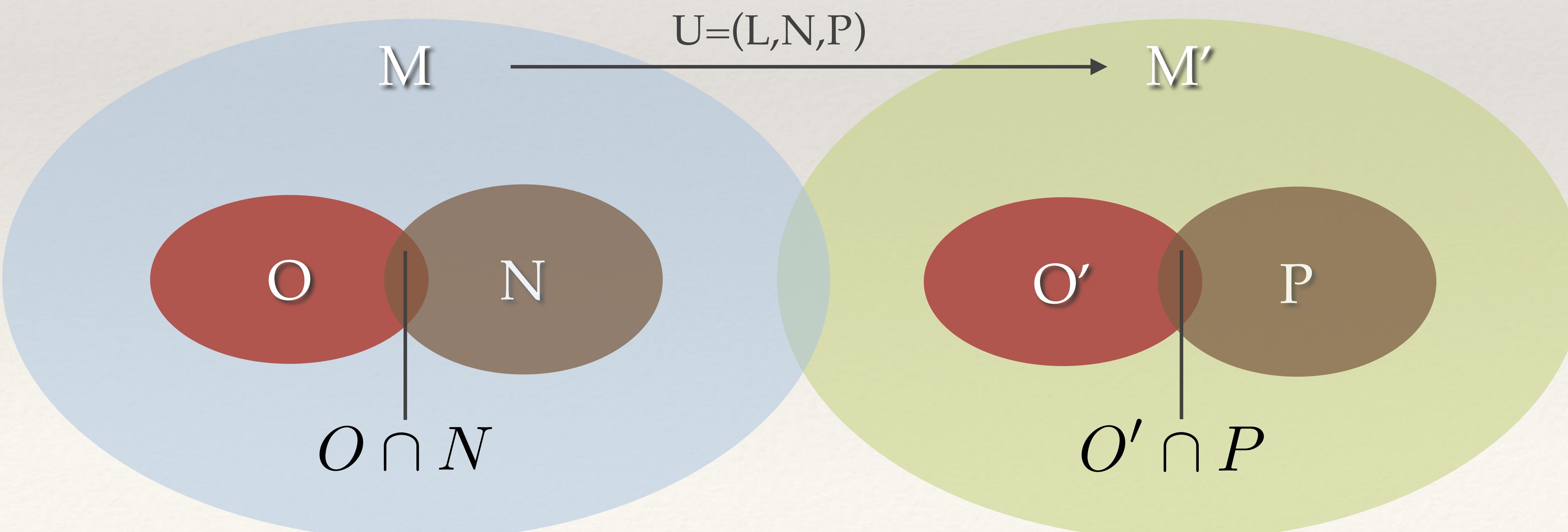
Simple update

Property 1. *Let* $\mathsf{U} = (L, N, P)$, *for all* $M \rightarrow_{\mathsf{U}} M'$ *we have:*

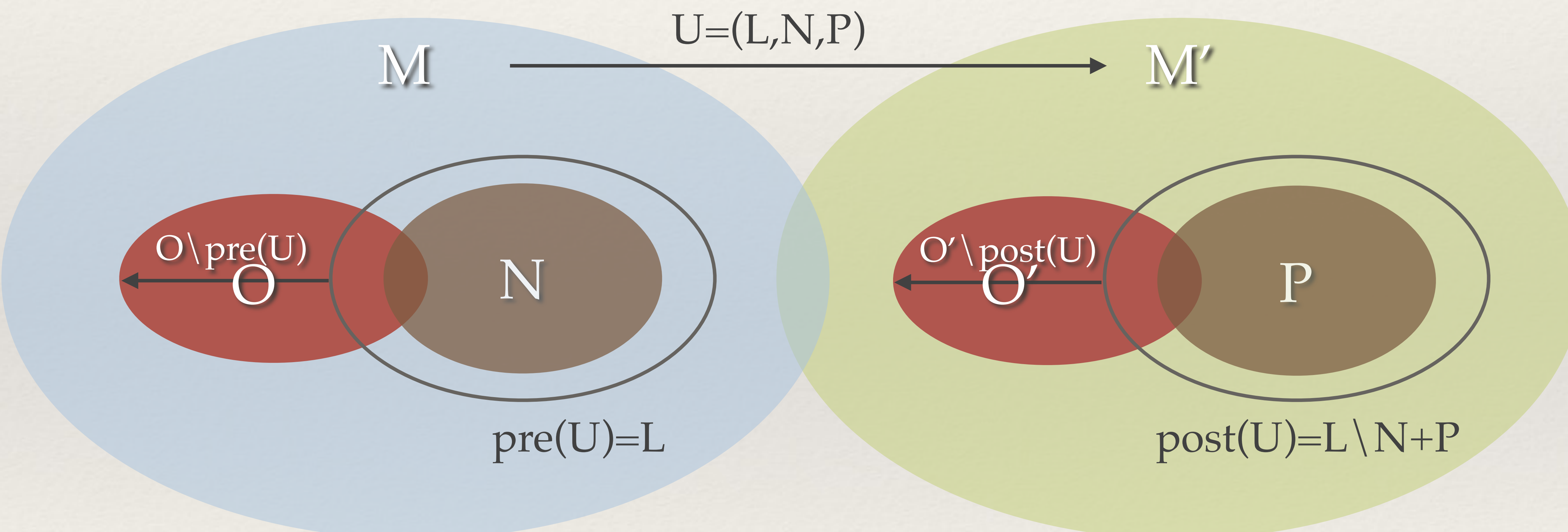
$$\mathcal{O}_{M'} = (\mathcal{O}_M \backslash \Delta_M(N)) \uplus \Delta_{M'}(P)$$

with, for all $H \in \{N, P\}$*:*

$$\Delta_M(H) \triangleq \{O \in \mathcal{G} \mid \mathsf{Obs}(O) \wedge (H \cap O) \neq \emptyset \wedge (H \cup O) \subseteq M\}$$



Incremental update



$O \cap N \neq \emptyset \implies$

O is removed by U iff O\pre(U) was in M

$O' \cap P \neq \emptyset \implies$

O′ is added by U iff O\post(U) is in M′

Concrete Update Structure

Let

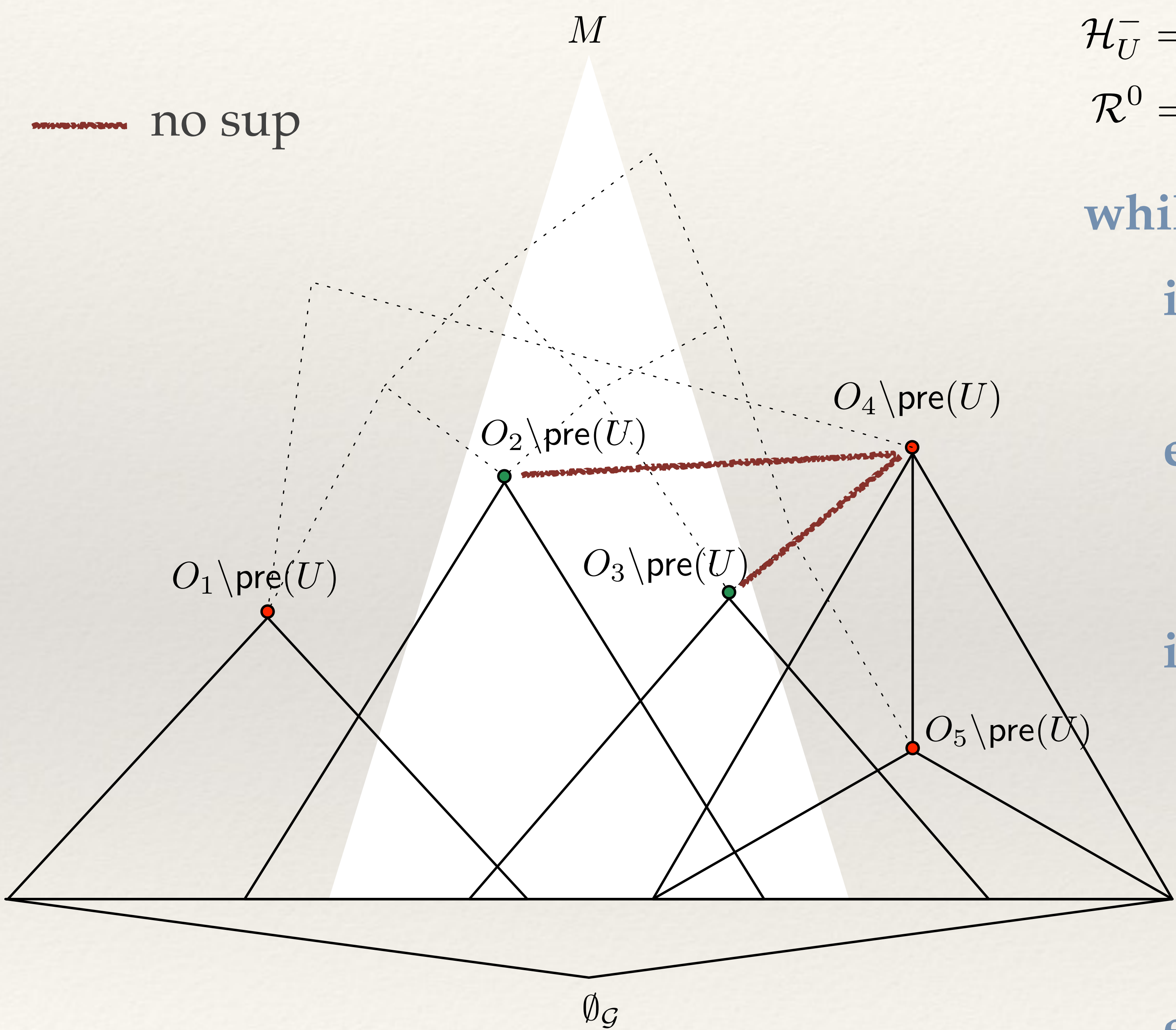
$$\mathcal{H}_U^- = \{H \mid H = O \backslash \mathsf{pre}(U) \wedge O \cap N_U \neq \emptyset\}$$

$$\mathcal{H}_U^+ = \{H \mid H = O \backslash \mathsf{post}(U) \wedge O \cap P_U \neq \emptyset\}$$

The concrete update structure is the meet semi lattice:

$$\mathcal{D}^\pm = (\downarrow \mathcal{H}_U^\pm, \subset)$$

Concrete algorithm (neg. case)



Update domain for U=(L,N,P)

$$\mathcal{H}_U^- = \{H \mid H = O\backslash\mathsf{pre}(U) \wedge O \cap N_U \neq \emptyset\}$$

$$\mathcal{R}^0 = \mathcal{H}_U^- \qquad X^0 = \emptyset_{\mathcal{G}} \qquad \mathcal{F}^0 = \emptyset \qquad p = X^0$$

while true

if $p \in \mathcal{H}_U^-$

$\mathcal{F}^{i+1} = \mathcal{F}^i \cup \{p\}$

else

$\mathcal{F}^{i+1} = \mathcal{F}^i$

if $e \in M \;\&\&\; e \notin X^i \;\&\&\; \{e\} \in \mathcal{D} \;\&\&\; \mathcal{R}^i \neq \emptyset$

$X^{i+1} = X^i \cup \{e\}$

$\mathcal{R}^{i+1} = \mathcal{R}^i \backslash \{H \in \mathcal{R}^i \mid H \cdots X^{i+1} \vee H = X^{i+1}\}$

$b = \max\{G \in \mathcal{D} \mid G \subseteq X^i \wedge G \cup \{e\} \in \mathcal{D}\}$

$p = b \cup \{e\}$

else

return $\mathcal{F}^i$

Example

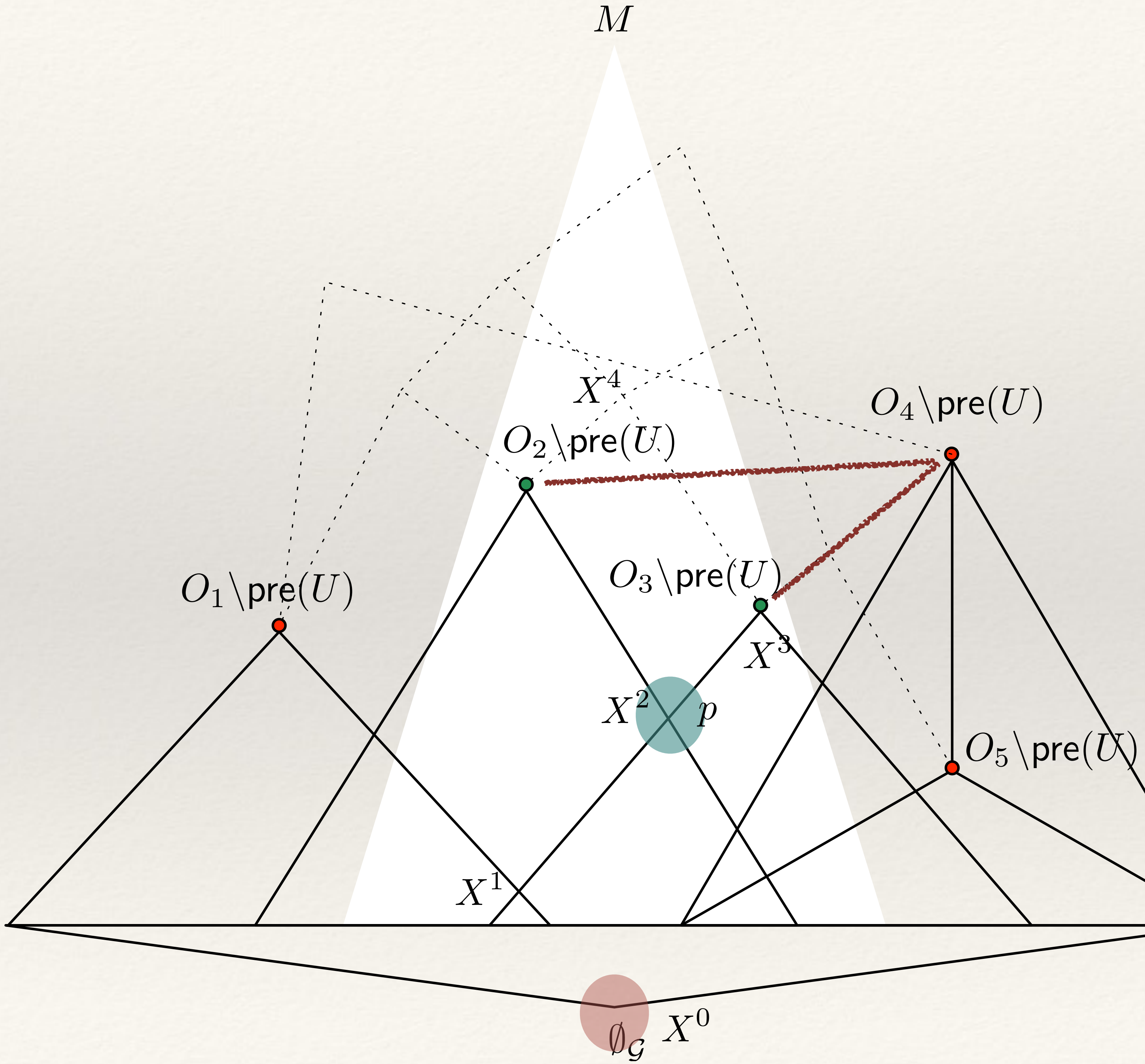
$\mathcal{F}^0 = \emptyset$

$\mathcal{F}^1 = \emptyset$

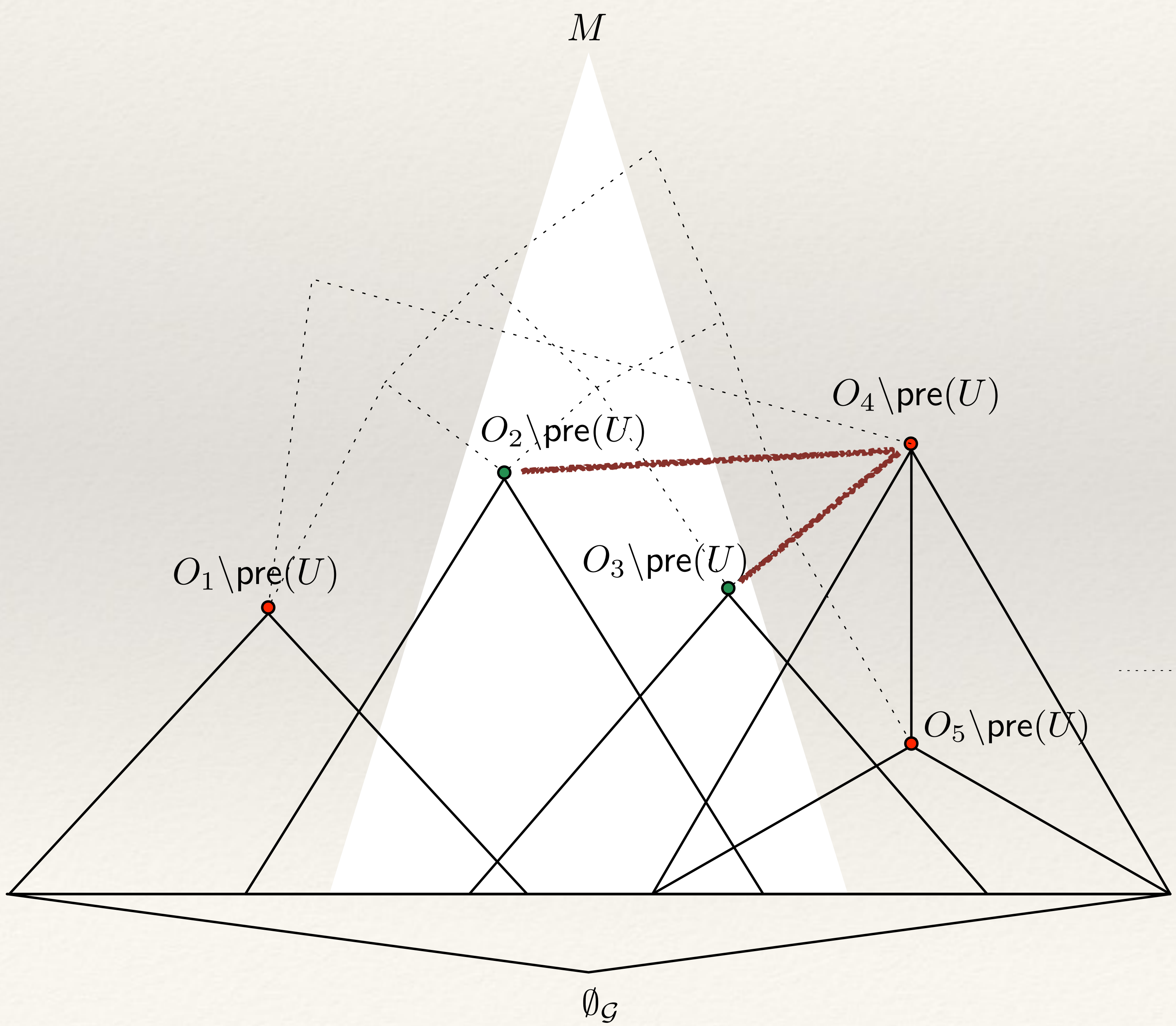
$\mathcal{F}^2 = \emptyset$

$\mathcal{F}^3 = \{O_3\}$

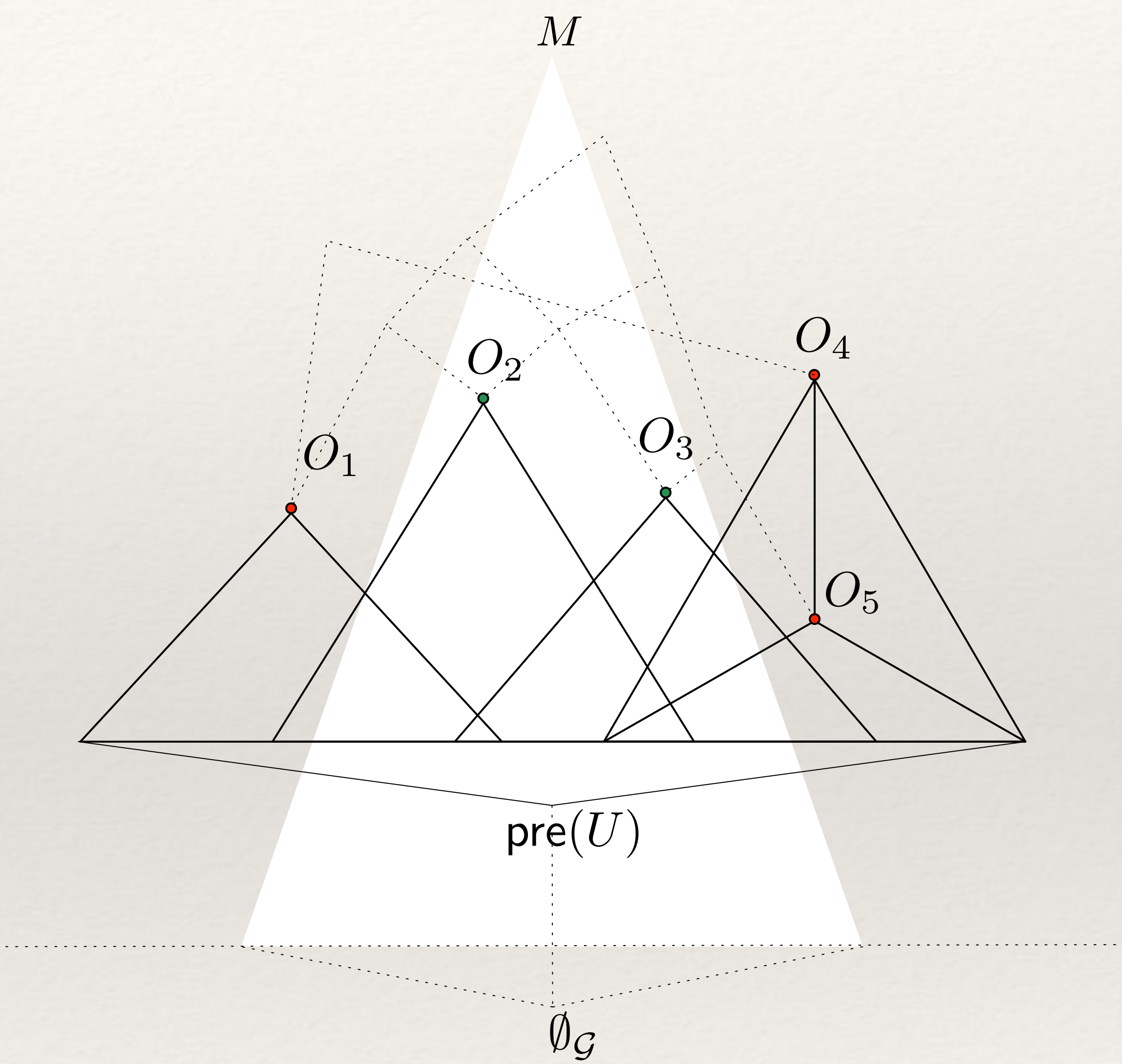
$\mathcal{F}^4 = \{O_3, O_2\}$



Re-rooting



M
$O_1 \backslash \mathsf{pre}(U)$
$O_2 \backslash \mathsf{pre}(U)$
$O_3 \backslash \mathsf{pre}(U)$
$O_4 \backslash \mathsf{pre}(U)$
$O_5 \backslash \mathsf{pre}(U)$
$\emptyset_{\mathcal{G}}$



M
O_1
O_2
O_3
O_4
O_5
$\mathsf{pre}(U)$
$\emptyset_{\mathcal{G}}$

Abstracting to finite domains

Abstract graphs

$$\text{CGraph} \xrightarrow{[\text{-}]} \text{AGraph}$$

Obj: concrete site graphs

$$[G] \triangleq \{H \in \mathcal{G} \mid \exists \phi \in \mathsf{Iso}(\mathcal{G}).\phi H = G\}$$

Morphisms: inclusion

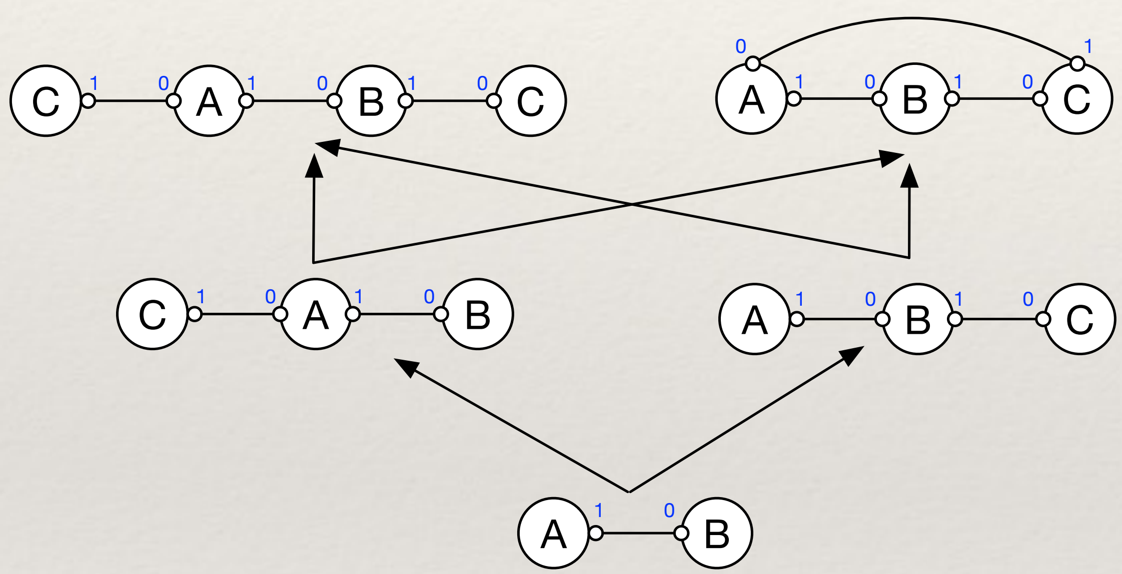
$$[f] = \begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\phi"'] \arrow[dr, phantom, "="] & H \arrow[d, "\psi"] \\ G' \arrow[r, "f'"] & H' \end{tikzcd}$$

Property 6. $[\_]$ *defines a functor from* CGraph *to* AGraph.

Property 7. AGraph *describes a poset with initial element* $\mathbf{0}_{\mathsf{Graph}} = [\emptyset]$.

Property 8. AGraph *has pullbacks and pullback complements.*

Properties

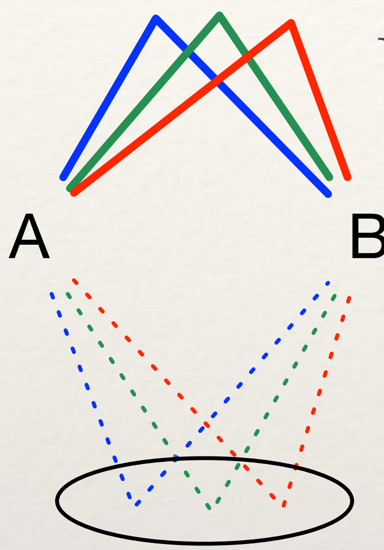


AGraph has no pushout

$$\begin{tikzcd}
 & & & D & & & \\
 & D_0 & \dots & D_i \arrow[u, dotted, "!\mathfrak{h}"] & \dots & D_k & \\
A \arrow[urrr, bend left, "\mathfrak{g}"] \arrow[ur, "\mathfrak{g}_0"] \arrow[urrr, "\mathfrak{g}_i"] \arrow[urrrrr, "\mathfrak{g}_k"] & & & & & & B \arrow[ulll, bend right, "\mathfrak{g}'"'] \arrow[ulllll, "\mathfrak{g}'_0"] \arrow[ulll, "\mathfrak{g}'_i"] \arrow[ul, "\mathfrak{g}'_k"'] \\
 & & & C \arrow[ulll, "\mathfrak{f}"] \arrow[urrr, "\mathfrak{f}'"'] & & &
\end{tikzcd}$$

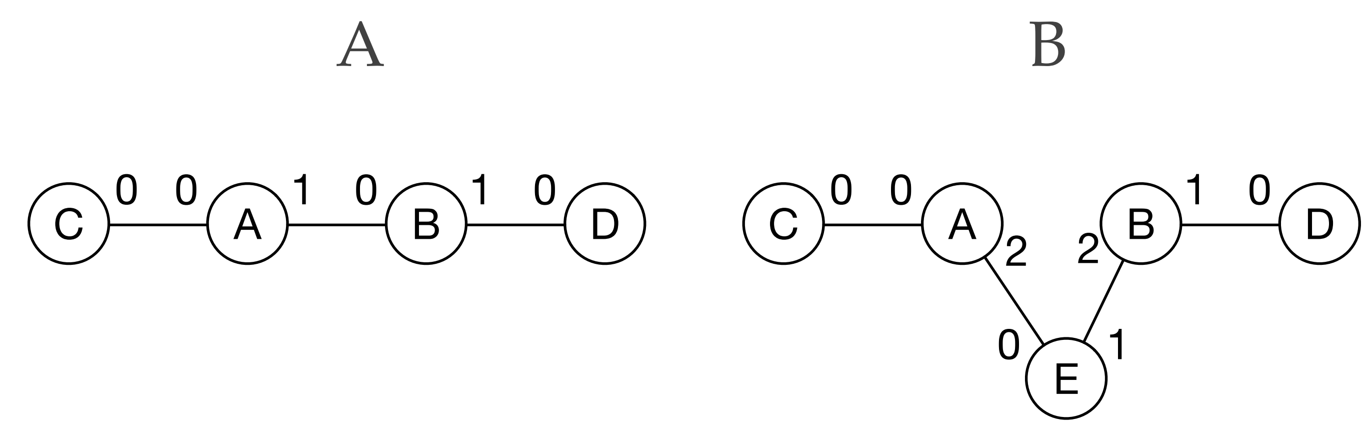
but multi-pushouts…

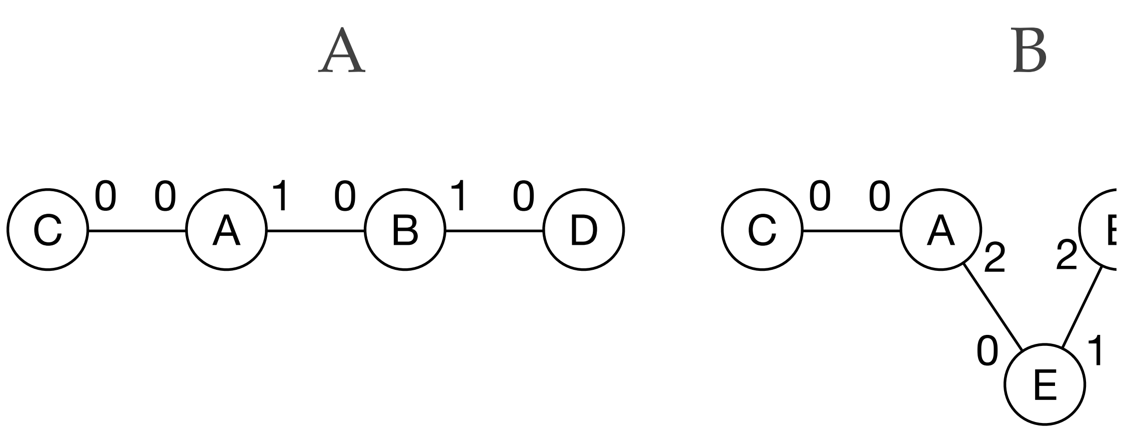
Abstract operations



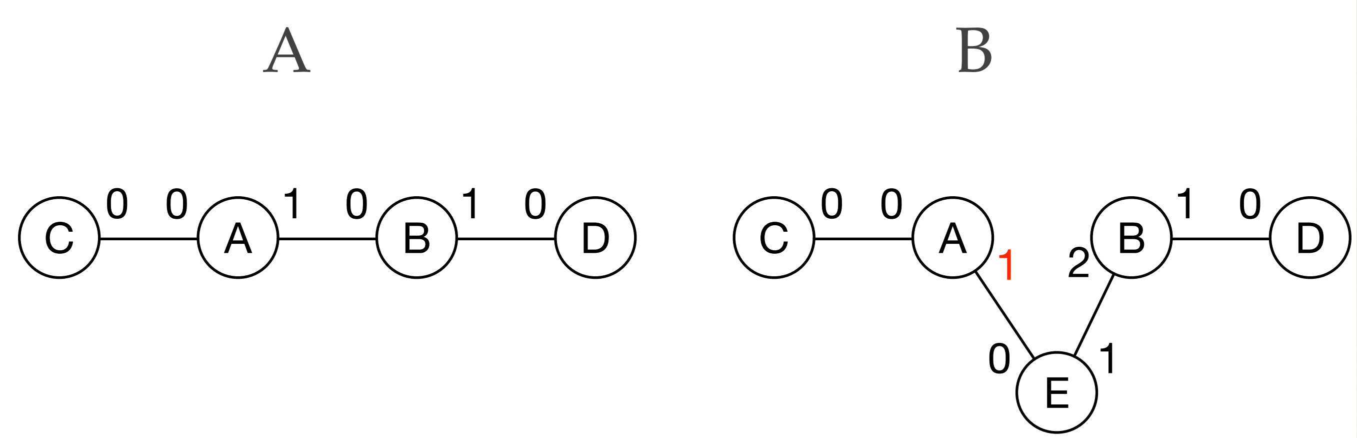
Using pullbacks

$A \cap^{\#} B$



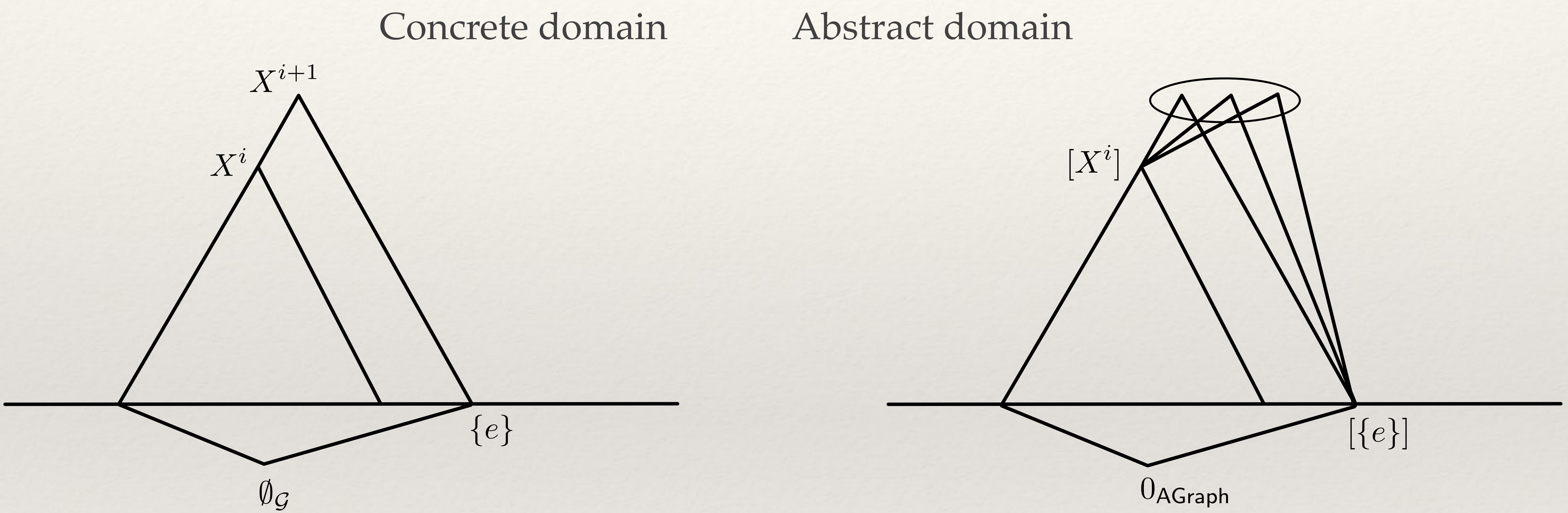


$A \backslash^{\#} B?$



$\exists C. A \cdots^{\#}_{C} B?$

Concrete vs. abstract domain



Because of multi-pushout, transitions in the concrete domain do not have a unique abstract counterpart.

Concretization

Transition in the concrete domain

$$
\begin{tikzcd}
G \arrow[r, "\subset"] \arrow[d, "{[-]}"'] & G \cup \{e\} \arrow[d, dotted, "{[-]}"] \\
A \arrow[r, blue, "{+[\{e\}]}"] \arrow[dr, blue, "{+[\{e\}]}"'] & B_i \arrow[d, dotted, no head] \\
& B_k
\end{tikzcd}
$$

Transitions in the abstract domain

Only one makes the diagram commute

Abstract algorithm (neg. case)

$\mathcal{H}_U^- = \{H \mid H = O \backslash \mathsf{pre}(U) \wedge O \cap N_U \neq \emptyset\}$

$\mathcal{R}^0 = \mathcal{H}_U^- \qquad X^0 = \emptyset_{\mathcal{G}} \qquad \mathcal{F}^0 = \emptyset \qquad p = X^0$

while true

if $p \in \mathcal{H}_U^-$

$\mathcal{F}^{i+1} = \mathcal{F}^i \cup \{p\}$

else

$\mathcal{F}^{i+1} = \mathcal{F}^i$

if $e \in M \;\&\&\; e \notin X^i \;\&\&\; \{e\} \in \mathcal{D} \;\&\&\; \mathcal{R}^i \neq \emptyset$

$X^{i+1} = X^i \cup \{e\}$

$\mathcal{R}^{i+1} = \mathcal{R}^i \backslash \{H \in \mathcal{R}^i \mid H \cdots X^{i+1} \vee H = X^{i+1}\}$

$b = \mathsf{max}\{G \in \mathcal{D} \mid G \subseteq X^i \wedge G \cup \{e\} \in \mathcal{D}\}$

$p = b \cup \{e\}$

else

return $\mathcal{F}^i$

$\mathcal{A}_U^- = \{A \mid A \in [O] \backslash^{\#} [pre(U)] \wedge [O] \cap^{\#} [N_U] \neq \{0_{\mathsf{AGraph}}\}\}$

$\mathcal{R}^0 = \mathcal{A}_U^- \qquad X^0 = \emptyset_{\mathcal{G}} \qquad \mathcal{F}^0 = \emptyset \qquad p = [X^0]$

while true

if $p \in \mathcal{A}_U^-$

$\mathcal{F}^{i+1} = \mathcal{F}^i \cup \{p\}$

else

$\mathcal{F}^{i+1} = \mathcal{F}^i$

if $e \in M \;\&\&\; e \notin X^i \;\&\&\; [\{e\}] \in \mathcal{D}^{\#} \;\&\&\; \mathcal{R}^i \neq \emptyset$

$X^{i+1} = X^i \cup \{e\}$

$\mathcal{R}^{i+1} = \mathcal{R}^i \backslash \{A \in \mathcal{R}^i \mid A \cdots^{\#} [X^{i+1}] \vee A = [X^{i+1}]\}$

$b = \mathsf{max}\{[G] \in \mathcal{D}^{\#} \mid [G] < [X^i] \wedge [G \cup \{e\}] \in \mathcal{D}^{\#}\}$

$p = b + [\{e\}]$

Unique!

else

return $\mathcal{F}^i$

A compact representation

Mixture