If we have \( G_i \) and \( \Lambda \xrightarrow{r} L^+ \),

\[
\begin{align*}
L & \xrightarrow{r} L^+ \\
\text{intuition: } L' & \text{ specifies the subgraph of } L^+ \\
\text{that can already be typed by } G_2
\end{align*}
\]

[expensive instance]

A factorization of \( r \) is an object \( L' \)

arrows \( r': L \rightarrow L' \) and \( r^+: L' \rightarrow L^+ \) s.t.

\( r = r^+ \circ r' \) and an arrow \( x: L' \rightarrow G_2 \)

s.t. \( x \circ r' = h_{nm} \circ m \):

\[
\begin{align*}
L & \xrightarrow{r} L^+ \\
\Lambda & \xrightarrow{\Lambda} L^+ \\
h_{nm} & \circ m = \circ r^+ \circ r' \\
G_2 & \xrightarrow{h_{nm}} G_2^+
\end{align*}
\]

[(this is a bona fide rule application)]

NB: as written, this is not a rule application as \( x \) is not a mono [we will return to this]

However, \( G_i \) is still typed by \( G_2^+ \):

\[
\begin{align*}
G_i & \xrightarrow{h_{nm}} G_2^+ \\
h_{nm} & \circ m = \circ r^+ \circ r' \\
G_2 & \xrightarrow{r^+} G_2^+
\end{align*}
\]

We obtain a canonical re-typing of \( G_i^+ \) by \( G_2^+ \)

\[
\begin{align*}
\Lambda & \xrightarrow{\Lambda} L^+ \\
r & \xrightarrow{r} L^+ \\
m & \xrightarrow{m} G_2^+ \\
h_{nm} & \circ m = \circ r^+ \circ r'
\end{align*}
\]

We obtain a canonical re-typing of \( G_i^+ \) by \( G_2^+ \)

\[ h_{nm}: G_i \rightarrow G_2 \]

\[ L' \]

\[ L^+ \]

\[ x \]

\[ G_2 \]

\[ G_2^+ \]

\[ r^+ \]

\[ \circ r^+ \circ r' \]

\[ \circ x \circ r' \]

[expensive instance]

\[
\begin{align*}
G_i & \xrightarrow{h_{nm}} G_2^+ \\
h_{nm} & \circ m = \circ r^+ \circ r' \\
G_2 & \xrightarrow{r^+} G_2^+
\end{align*}
\]

\[ \Lambda \]

\[ L \]

\[ r \]

\[ x \]

\[ G_i \]

\[ G_2 \]

\[ G_2^+ \]

\[ h_{nm} \]

\[ m \]

\[ r^+ \]

\[ \circ r^+ \circ r' \]

\[ \circ x \circ r' \]

\[ \Lambda \]

\[ L \]

\[ r \]

\[ x \]

\[ G_i \]

\[ G_2 \]

\[ G_2^+ \]

\[ h_{nm} \]

\[ m \]

\[ r^+ \]

\[ \circ r^+ \circ r' \]

\[ \circ x \circ r' \]

\[ \Lambda \]

\[ L \]

\[ r \]

\[ x \]

\[ G_i \]

\[ G_2 \]

\[ G_2^+ \]

\[ h_{nm} \]

\[ m \]

\[ r^+ \]

\[ \circ r^+ \circ r' \]

\[ \circ x \circ r' \]

\[ \Lambda \]

\[ L \]

\[ r \]

\[ x \]

\[ G_i \]

\[ G_2 \]

\[ G_2^+ \]

\[ h_{nm} \]

\[ m \]

\[ r^+ \]

\[ \circ r^+ \circ r' \]

\[ \circ x \circ r' \]

\[ \Lambda \]

\[ L \]

\[ r \]

\[ x \]

\[ G_i \]

\[ G_2 \]

\[ G_2^+ \]

\[ h_{nm} \]

\[ m \]

\[ r^+ \]

\[ \circ r^+ \circ r' \]

\[ \circ x \circ r' \]

\[ \Lambda \]

\[ L \]

\[ r \]

\[ x \]

\[ G_i \]

\[ G_2 \]

\[ G_2^+ \]

\[ h_{nm} \]

\[ m \]

\[ r^+ \]

\[ \circ r^+ \circ r' \]

\[ \circ x \circ r' \]

\[ \Lambda \]

\[ L \]

\[ r \]

\[ x \]

\[ G_i \]

\[ G_2 \]

\[ G_2^+ \]

\[ h_{nm} \]

\[ m \]

\[ r^+ \]

\[ \circ r^+ \circ r' \]

\[ \circ x \circ r' \]
FORWARD PROPAGATION (II)

1. $G_i$ remains unchanged [it already types $L^+]$

2. $L_{n-1} \xrightarrow{r_{n-1}} L^+$ Computes $G_{n-1}^+$ with the 'last stage' of the factorization

3. $L \xrightarrow{r_{n-1}} L^+$ and $L \xrightarrow{r} L^+$

4. Retyping works as before and we obtain

$v_i$ $\rightarrow$ $v_i^+$ $\rightarrow$ $v_i^{++}$ $\rightarrow$ $v_i^{+++}$ $\rightarrow$ $v_i^{++++}$

5. \[ G_i \xrightarrow{r} G_i^+ \]

6. \[ G_i^+ \xrightarrow{r} G_i^{++} \]

7. \[ G_i^{++} \xrightarrow{r} G_i^{+++} \]

8. \[ G_i^{+++} \xrightarrow{r} G_i^{++++} \]

This generalizes immediately to any hierarchy that is a phytree, i.e., a DAG whose underlying undirected graph is acyclic.

is not a phytree but the same idea still works: we factorize $r$ like

$L \xrightarrow{r_i} L_2 \xrightarrow{r_3} L_5 \xrightarrow{r_4} L^+$

and rewrite

- $G_i$ with $r_i$
- $G_i$ with $r_i \cdot r_i$
- $G_i$ with $r$

NB: unlike $G_i$, when there are $G_2$, $G_3$, $G_4$, two independent factorizations of $r$ — one for $G_2$, the other for $G_3$ — the (undirected) cyclic case imposes the constraint that the factorizing $r_2$ and $r_3$ are compatible — just as in the case of...
**Backward Propagation**

- If we have \( G_i \) and \( L \), and \( L \leftarrow L' \),

  \[ \begin{array}{c}
  \xrightarrow{\text{restrictive instance}} \\
  \end{array} \]

  we take the pull-back

  \[ \begin{array}{c}
  L_1 \xrightarrow{\text{PB}} L \\
  \xrightarrow{\text{m}} \\
  L_1 \xrightarrow{\text{m}} G_1 \\
  \end{array} \]

  and define a factorization of \( r \) as:

  \[ \begin{array}{c}
  r \leftarrow L_1 \xleftarrow{L} L' \\
  \xrightarrow{\text{m}} \\
  r' = \hat{r} \leftarrow \hat{L}_1 \xleftarrow{\hat{L}} \hat{L}' \\
  \end{array} \]  

  \[ \text{NB: } L_1 \text{ is the largest sub-graph of } G_i \]

  \[ \text{whose typing is modified by } r \]

\[ \text{[Note that } G_i \text{ is still typed by } G_i \text{]} \]

(i) Compute the pull-back

\[ \begin{array}{c}
  \xrightarrow{\text{in}} \\
  \xrightarrow{\text{PB}} \\
  \xrightarrow{\text{m}} \\
  L_1 \xrightarrow{\text{m}} L_1' \\
  \text{to define the rule lifting:} \\
  L_1' \leftarrow L_1' \\
  \end{array} \]

(ii) Compute the pull-back complement

\[ \begin{array}{c}
  \xrightarrow{\text{m}} \\
  \xrightarrow{\text{m}} \\
  \xrightarrow{\text{m}} \\
  L_1 \xrightarrow{\text{m}} L_1' \\
  \text{to } G_i \leftarrow G_i' \\
  \end{array} \]

\[ \text{[Note that } G_i \text{ is still typed by } G_i \text{]} \]

(iii) Compute the pull-back complement

\[ \begin{array}{c}
  \xrightarrow{\text{m}} \\
  \xrightarrow{\text{m}} \\
  \xrightarrow{\text{m}} \\
  L \xrightarrow{\text{m}} L \xrightarrow{\text{m}} L' \\
  \text{to } G_i \leftarrow G_i' \\
  \end{array} \]

\[ \text{can we still type } G_i \text{ by } G_i' ? \]

\[ \text{by the pasting lemma for PBs, the outer rectangle of} \]

\[ \begin{array}{c}
  \xrightarrow{\text{m}} \\
  \xrightarrow{\text{m}} \\
  \xrightarrow{\text{m}} \\
  L \xrightarrow{\text{m}} L_1 \xrightarrow{\text{m}} L_1' \\
  \text{is a PB;} \text{ moreover,} \\
  \lambda_{h_2} \circ r_1 = r_1 \circ \lambda_{h_2} \\
  \lambda_{h_2} = r_1 \circ \lambda_{h_2} \quad \text{by (5)} \\
  \end{array} \]

\[ \text{So, by the UP of (8),} \]

\[ \begin{array}{c}
  \xrightarrow{\text{m}} \\
  \xrightarrow{\text{m}} \\
  \xrightarrow{\text{m}} \\
  L \xrightarrow{\text{m}} L_1 \xrightarrow{\text{m}} L_1' \\
  \text{We obtain a canonical retyping of} \\
  G_i \text{ by } G_i' \\
  \text{by (8)} \\
  \end{array} \]

\[ \text{We obtain a canonical retyping of} \]

\[ \lambda_{h_2} : G_i \rightarrow G_i' \]