Effectiveness and aperiodicity of subshifts

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Outline

1  Background

2  Effectiveness in groups

3  Aperiodicity
G-Subshifts

- $G$ is a (finitely generated) group.
- $A$ is a finite alphabet.
- $A^G$ is the set of functions from $G$ to $A$.
- $\sigma : G \times A^G \to A^G$ is the left shift action given by:

$$\sigma_g(x)_h = x_{g^{-1}h}$$
G-Subshifts

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**Definition: G-subshift**

$X \subset \mathcal{A}^G$ is a **$G$-subshift** if it invariant under the action of $\sigma$ and closed for the product topology on $\mathcal{A}^G$. 
**G-Subshifts**

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**Definition: G-subshift**

$X \subset \mathcal{A}^G$ is a **G-subshift** if it invariant under the action of $\sigma$ and closed for the product topology on $\mathcal{A}^G$.

Equivalently, $X$ is a G-subshift if it can be defined by a set of forbidden patterns: $\exists \mathcal{F} \subset \bigcup_{F \subset G, |F| < \infty} \mathcal{A}^F$ such that

\[
X = X_{\mathcal{F}} := \{x \in \mathcal{A}^G | \forall P \in \mathcal{F} : P \nsubseteq x\}
\]
**Z-Subshift examples**

**Example: full shift.** Let $\mathcal{A} = \{0, 1\}$ and $\mathcal{F} = \emptyset$. Then $X_\mathcal{F} = \mathcal{A}_\mathbb{Z}$ is the set of all bi-infinite words.
**Z-Subshift examples**

**Example : full shift.** Let $A = \{0, 1\}$ and $F = \emptyset$. Then $X_F = A^\mathbb{Z}$ is the set of all bi-infinite words.

**Example : Fibonacci shift.** Let $A = \{0, 1\}$ and $F = \{11\}$. Then $X_F$ is the set of all bi-infinite words which have no pairs of consecutive 1’s.

$x = \ldots 010100010100100100100 \ldots \in X_F$
### Z-Subshift examples

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$$x = \ldots 010100010100100100100 \ldots \in X_\mathcal{F}$$

**Example : one-or-less subshift**

$$X_{\leq 1} := \{ x \in \{0, 1\}^\mathbb{Z} \mid |\{ n \in \mathbb{Z} : x_n = 1 \}| \leq 1 \}.$$  

Is a $\mathbb{Z}$-subshift as it is defined by the set $\mathcal{F} = \{10^n1 \mid n \in \mathbb{N}_0\}$.

$$x = \ldots 0000000000100000000 \ldots \in X_{\leq 1}$$
Example in $\mathbb{Z}^2$ : Fibonacci shift

Example : Fibonacci shift. $X_{fib}$ is the set of assignments of $\mathbb{Z}^2$ to \{0, 1\} such that there are no two adjacent ones.
Example in $\mathbb{Z}^2$: Fibonacci shift

**Example: Fibonacci shift.** $X_{fib}$ is the set of assignments of $\mathbb{Z}^2$ to \{0, 1\} such that there are no two adjacent ones.

$X_{fib}$ can be represented as a grid where each cell can be assigned a 0 or 1, subject to the condition that no two adjacent cells can both be 1.

The image shows a grid with assignments, where the pattern is repeated periodically, demonstrating the concept of Fibonacci shift in the context of $\mathbb{Z}^2$.
Example: one-or-less subshift

\[ X \leq 1 := \{ x \in \{0, 1\}^\mathbb{Z}^d \mid \| \{ z \in \mathbb{Z}^d : x_z = 1 \} \| \leq 1 \} . \]
Example: $S$-Fibonacci shift for $G = F_2$

$$F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$
Interesting classes

**G-SFTs**

A $G$-subshift $X$ is said to be of finite type (G-SFT) if there exists a finite set of patterns $\mathcal{F}$ such that $X = X_\mathcal{F}$. 

Example: $S$-Fibonacci shift. For every group $G$ generated by a finite set $S$, the $S$-Fibonacci shift is a $G$-SFT.

**Sofic $G$-subshifts**

A $G$-subshift $Y$ over $A$ is said to be a sofic $G$-subshift if there exists a $G$-SFT $X$ and a surjective cellular automaton $\phi: X \to Y$. That is, we have a $G$-SFT where we allow to delete some information.

Example: $X_{\leq 1}$ is a sofic $G$-subshift if $G$ is $\mathbb{Z}^d$ or a finitely generated free group $F_k$. 

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Interesting classes

**G-SFTs**

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**Example : S-Fibonacci shift.** For every group $G$ generated by a finite set $S$ the $S$-Fibonacci shift is a $G$-SFT.
Interesting classes

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A $G$-subshift $X$ is said to be of **finite type** ($G$-SFT) if there exists a finite set of patterns $\mathcal{F}$ such that $X = X_\mathcal{F}$.

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**Example**: $X_{\leq 1}$ is a sofic $G$-subshift if $G$ is $\mathbb{Z}^d$ or a finitely generated free group $F_k$. 
**Remark:** These classes are interesting from a computational perspective because they can be defined with a finite amount of data. How far can we take this idea?
Interesting classes

*Remark:* These classes are interesting from a computational perspective because they can be defined with a *finite amount of data*. How far can we take this idea?

**Definition: Effectiveness in \( \mathbb{Z} \)**

A \( \mathbb{Z} \)-subshift \( X \subset \mathcal{A}^\mathbb{Z} \) is said to be *effective* if there is a recognizable set \( \mathcal{F} \subset \mathcal{A}^* \) such that \( X = X_{\mathcal{F}} \).
Interesting classes

Remark: These classes are interesting from a computational perspective because they can be defined with a finite amount of data. How far can we take this idea?

Definition: Effectiveness in $\mathbb{Z}$

A $\mathbb{Z}$-subshift $X \subset \mathcal{A}^\mathbb{Z}$ is said to be effective if there is a recognizable set $\mathcal{F} \subset \mathcal{A}^*$ such that $X = X_\mathcal{F}$.

Question: How can the idea of effectiveness be translated into general groups?
Outline

1 Background

2 Effectiveness in groups

3 Aperiodicity
First approach: $\mathbb{Z}$-effectiveness

Let $G$ be a finitely generated group and $S \subset G$ a finite generator.
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Let $G$ be a finitely generated group and $S \subseteq G$ a finite generator.

**Definition: $\mathbb{Z}$-effectiveness**

A $G$-subshift $X \subset \mathcal{A}^G$ is **$\mathbb{Z}$-effective** if there is a Turing machine which enumerates a set of pattern codings such that the set of consistent pattern codings defines a set $\mathcal{F}$ such that $X = X_\mathcal{F}$. 
First approach: $\mathbb{Z}$-effectiveness

Let $G$ be a finitely generated group and $S \subset G$ a finite generator.

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**Question**: Is it always possible to recognize if a pattern coding is inconsistent?
Example: the Baumslag-Solitar group $BS(1, 2)$

Consider the group $BS(1, 2) = \langle a, b \mid ab = ba^2 \rangle$. 

$\epsilon, 0$ 

$\langle a, b \rangle$ 

$\langle ab, 0 \rangle$ 

$\langle ba, 0 \rangle$ 

$\langle ba, 1 \rangle$ 

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- The pattern coding

$$
\begin{align*}
(\epsilon, 0) & \quad (b, 1) & \quad (a, 1) \\
(ab, 0) & \quad (ba^2, 0) & \quad (ba, 1)
\end{align*}
$$
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The pattern coding

$$(\epsilon, 0) \quad (b, 1) \quad (a, 1)$$
$$(ab, 0) \quad (ba^2, 0) \quad (ba, 1)$$

is consistent and defines the pattern

$$\Pi_{1_{G}} = 0 \quad \Pi_{a} = 1$$
$$\Pi_{b} = 1 \quad \Pi_{ba} = 1 \quad \Pi_{ba^2} = \Pi_{ab} = 0$$
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$\begin{align*}
(\epsilon, 0) & \quad (b, 1) & \quad (a, 1) \\
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\end{align*}$

is consistent and defines the pattern

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\Pi_{1G} &= 0 \\
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\Pi_b &= 1 \\
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\end{align*}$

- The pattern coding

$\begin{align*}
(\epsilon, 0) & \quad (a^2, 1) & \quad (bab^{-1}a, 1) \\
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\end{align*}$
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Consider the group $BS(1,2) = \langle a, b \mid ab = ba^2 \rangle$.

- The pattern coding

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(\epsilon, 0) \quad (b, 1) \quad (a, 1) \\
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is **consistent** and defines the pattern

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\Pi_{1g} = 0 \quad \Pi_a = 1 \\
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$$

- The pattern coding

$$
(\epsilon, 0) \quad (a^2, 1) \quad (bab^{-1}a, 1) \\
(a, 1) \quad (ba, 1) \quad (abab^{-1}, 0)
$$

is **inconsistent** since $abab^{-1}$ and $bab^{-1}a$ represent the same element.

$$abab^{-1} = ba^3b^{-1} = ba(b^{-1}b)a^{-1}b^{-1} = bab^{-1}abb^{-1} = bab^{-1}a$$
Limitations of \( \mathbb{Z} \)-effectiveness

**Definition : Word problem**

Let \( S \subset G \) be a finite generator of \( G \). The **word problem** of \( G \) asks whether two words on \( S \cup S^{-1} \) are equivalent in \( G \). Formally:

\[
WP(G) = \left\{ w \in (S \cup S^{-1})^* \mid w =_G 1_G \right\}.
\]
Limitations of $\mathbb{Z}$-effectiveness

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$$ WP(G) = \left\{ w \in (S \cup S^{-1})^* \mid w =_G 1 \right\}. $$

**Example : Decidable word problem.** The word problem for $\mathbb{Z}^2 \simeq \langle a, b \mid ab = ba \rangle$ is:

$$ WP(\mathbb{Z}^2) = \left\{ w \in \{a, b, a^{-1}, b^{-1}\}^* \mid |w|_a = |w|_{a^{-1}} \land |w|_b = |w|_{b^{-1}} \right\}. $$
Limitations of \( \mathbb{Z} \)-effectiveness

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\]

**Example : Undecidable word problem.** If \( f : \mathbb{N} \to \{0, 1\} \) is non-computable the group \( G = \langle a, b, c, d \mid ab^n = c^n d, n \in f^{-1}(1) \rangle \) has undecidable word problem.
Limitations of \( \mathbb{Z} \)-effectiveness

*Remark*: If \( G \) is not recursively presented, it is not possible to recognize whether a pattern coding is consistent!
Limitations of $\mathbb{Z}$-effectiveness

Remark: If $G$ is not recursively presented, it is not possible to recognize whether a pattern coding is consistent!

Remark: Even if $G$ is finitely presented, there are simple subshifts which are not $\mathbb{Z}$-effective!
Limitations of $\mathbb{Z}$-effectiveness

**Remark**: If $G$ is not recursively presented, it is not possible to recognize whether a pattern coding is consistent!

**Remark**: Even if $G$ is finitely presented, there are simple subshifts which are not $\mathbb{Z}$-effective!

**Remark** [Theorem: Novikov(55), Boone(58)]

There are finitely presented groups with undecidable word problem!

**Theorem**

For a recursively presented group the one-or-less subshift:

$$X_{\leq 1} := \{ x \in \{0, 1\}^G | |\{g \in G : x_g = 1\}| \leq 1 \}.$$  

is not $\mathbb{Z}$-effective if $WP(G)$ is undecidable.
Another approach: Don’t codify anything!
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**Definition: G-machine**

A **G-machine** is a Turing machine whose tape has been replaced by the group $G$. The transition function is

$$
\delta : Q \times \Sigma \rightarrow Q \times \Sigma \times (S \cup S^{-1} \cup \{1_G\})
$$

where $S$ is a finite set of generators of $G$. 

Remark: Computation is over patterns instead of words.
Another approach: Don’t codify anything!

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**Remark**: Computation is over patterns instead of words.
Example: Transition in a $F_2$-machine

$$\delta(q_1, \bullet) = (q_2, \bullet, s_1)$$
**G-effectiveness**

**Definition:**
- A set of patterns \( \mathcal{P} \) is said to be **recognizable** if there is a \( G \)-machine which accepts if and only if \( P \in \mathcal{P} \).
- A set of patterns \( \mathcal{P} \) is said to be **decidable** if there is a \( G \)-machine which accepts if \( P \in \mathcal{P} \) and rejects otherwise.
**G-effectiveness**

**Definition:**
- A set of patterns $\mathcal{P}$ is said to be **recognizable** if there is a $G$-machine which accepts if and only if $P \in \mathcal{P}$.
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**G-effectiveness**

A $G$-subshift $X \subset A^G$ is **$G$-effective** if there exists a set of forbidden patterns $\mathcal{F}$ such that $X = X_\mathcal{F}$ and $\mathcal{F}$ is $G$-recognizable.
What can we say about $G$-effectiveness?

Remark: The one-or-less subshift $X_{\leq 1}$ is $G$-effective for every finitely generated group $G$. 
What can we say about \( G \)-effectiveness?

**Remark:** The one-or-less subshift \( X_{\leq 1} \) is \( G \)-effective for every finitely generated group \( G \).

**Theorem**

Let \( G \) be an infinite, finitely generated group, then every \( \mathbb{Z} \)-effective subshift is \( G \)-effective.
What can we say about $G$-effectiveness?

*Remark*: The one-or-less subshift $X_{\leq 1}$ is $G$-effective for every finitely generated group $G$.

**Theorem**

*Let $G$ be an infinite, finitely generated group, then every $\mathbb{Z}$-effective subshift is $G$-effective.*

**Theorem**

*Let $G$ be finitely generated group with decidable word problem then every $G$-effective subshift is $\mathbb{Z}$-effective.*
Some results about $G$-effectiveness?

Theorem

A subshift is $G$-effective if and only if it satisfies the conditions of $\mathbb{Z}$-effectiveness with a Turing machine which has access to an oracle of $WP(G)$.
Some results about $G$-effectiveness?

**Theorem**

A subshift is $G$-effective if and only if it satisfies the conditions of $\mathbb{Z}$-effectiveness with a Turing machine which has access to an oracle of $WP(G)$.

- We have also shown that the class of $G$-effective subshifts contains every $G$-SFT, every sofic and every $\mathbb{Z}$-effective $G$-subshift.
Outline

1. Background
2. Effectiveness in groups
3. Aperiodicity
Aperiodicity in a subshift

Definition: Strongly aperiodic

A $G$-subshift $X$ is said to be \textit{strongly aperiodic} if

$$\forall x \in X, \quad \text{stab}_\sigma(x) := \{g \in G \mid gx = x\} = \{1_G\}$$
**Aperodicity in a subshift**

**Definition: Strongly aperiodic**

A $G$-subshift $X$ is said to be *strongly aperiodic* if

$$\forall x \in X, \ stab_\sigma(x) := \{g \in G \mid gx = x\} = \{1_G\}$$

**Example in** $G = \mathbb{Z}$. Let $A = \{0, 1, 2\}$ and $\mathcal{F} = \{ww \mid w \in A^*\}$. Then $X_\mathcal{F}$ is strongly aperiodic.
Some known facts

- $\mathbb{Z}$-SFTs are never strongly aperiodic.
- There are strongly aperiodic $\mathbb{Z}^2$-SFTs. (1964 Berger, 1971 Robinson, 1996 Kari)
- There are weakly aperiodic SFTs in Baumslag Solitar groups (2013 Aubrun-Kari)
- There are strongly aperiodic SFTs in the Heisenberg group (2014 Sahin-Schraudner)
- The existence of a strongly aperiodic $G$-SFT implies $G$ is one ended (2014 Cohen)
- A finitely presented group which admits a strongly aperiodic SFT has decidable word problem (2015 Jeandel)
The Robinson tiling
Our result

Theorem:
For every infinite and finitely generated group $G$ there exists a strongly aperiodic $G$-effective subshift.
Our result

**Theorem:**
For every infinite and finitely generated group $G$ there exists a strongly aperiodic $G$-effective subshift.

**Corollary:**
For a recursively presented group, there exists a $\mathbb{Z}$-effective strongly aperiodic subshift if and only if $WP(G)$ is decidable.
An ingredient for the proof

**Definition**

Let \((X, d)\) be a metric space. We say \(F \subseteq G\) is **r-covering** if for each \(x \in G\) there is \(y \in F\) such that \(d(x, y) \leq r\). We say \(F\) is **s-separating** if for each \(x \neq y \in F\) then \(d(x, y) > s\).
An ingredient for the proof

Definition
Let \((X, d)\) be a metric space. We say \(F \subseteq G\) is \(r\)-covering if for each \(x \in G\) there is \(y \in F\) such that \(d(x, y) \leq r\). We say \(F\) is \(s\)-separating if for each \(x \neq y \in F\) then \(d(x, y) > s\).

Proposition
If \(X\) is countable, then for any \(r \in \mathbb{R}\) there exists \(Y \subseteq X\) such that \(Y\) is both \(r\)-separating and \(r\)-covering.
Example: 2-covering and 2-separating set in \( \text{PSL}(\mathbb{Z}, 2) \)
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Proof

- First we create a layer with a hierarchical structure.
- \( Y \subset (S \cup S^{-1} \cup \{1_G\})^G \)
- The points \( y \in Y \) codify forests with a property:
Proof

First we create a layer with a hierarchical structure.

\[ Y \subset (S \cup S^{-1} \cup \{1_G\})^G \]

The points \( y \in Y \) codify forests with a property:

**Property**

For every \( n \in \mathbb{N} \), \( G \) can be partitioned in sets \( (C_i)_{i \in I} \) such that

\[ \exists g_i \in C_i \text{ such that } B(g_i, n) \subset C_i \subset B(g_i, 5^n) \]

And for each \( C_i \) there is either a single \( h \in C_i \) with \( x_h = 1_G \) and for every other \( g \in C_i \) then \( gx_g \in C_i \) or \( \forall g \in C_i \) \( x_g \neq 1_G \) and there is a single \( h \in C_i \) such that \( hx_h \not\in C_i \).
Proof

- First we create a layer with a hierarchical structure.
- \( Y \subset (S \cup S^{-1} \cup \{1_G\})^G \)
- The points \( y \in Y \) codify forests with a property:

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**Remark:** This property can be easily verified with a TM with access to \( WP(G) \).
Cluster structure
Cluster structure
Cluster structure
Cluster structure
Second layer

*Remark*: We are not done yet!

*Example*: $G = \mathbb{Z}$.

$$y = \ldots, +1, +1, +1, +1, +1, +1, \ldots \in Y$$
Second layer

Remark: We are not done yet!

Example: \( G = \mathbb{Z} \).

\[
y = \ldots, +1, +1, +1, +1, +1, +1, \ldots \in Y
\]

Consider an infinite word \( \mathcal{W} \) without squares, such as the one produced by \( \phi : \{0, 1, 2\} \rightarrow \{0, 1, 2\}^* \) given by:

\[
\phi(k) = \begin{cases} 
01210, & \text{if } k = 0 \\
12021, & \text{if } k = 1 \\
20102, & \text{if } k = 2 
\end{cases}
\]
Remark: We are not done yet!
Example: \( G = \mathbb{Z} \).

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y = \ldots, +1, +1, +1, +1, +1, +1, \ldots \in Y
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\end{cases}
\]

We consider \( X \subset ((S \cup S^{-1} \cup \{1_G\}) \times \{0, 1, 2\})^G \) such that for \( x \in X \) then \( \pi_1(x) \in Y \) and every path in \( \pi_1(x) \) contains a subword of \( \mathcal{W} \) in the second layer.
Final argument

The existence of $h \neq 1_G$ such that $h \in \text{stab}_\sigma(x)$ creates a square word.
Corollary:

For a recursively presented group, there exists a $\mathbb{Z}$-effective strongly aperiodic subshift if and only if $WP(G)$ is decidable.
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Proof: As $WP(G)$ is decidable, every $G$-effective subshift is $\mathbb{Z}$-effective and thus our construction shows the existence. Jeandel’s result gives the other direction.
Current work

- Use simulation theorems with our construction to produce strongly aperiodic SFTs in some classes of groups.
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- Use simulation theorems with our construction to produce strongly aperiodic SFTs in some classes of groups.
- Apply the idea of clusters to generate entropies in amenable groups.
Merci beaucoup pour votre attention !

Avez-vous des questions ?