The group of reversible Turing machines and the torsion problem for $\text{Aut}(A^\mathbb{Z})$ and related topological fullgroups

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Motivation

Given a fullshift \((A^\mathbb{Z}, \sigma)\) recall that its automorphism group is given by

\[ \text{Aut}(A^\mathbb{Z}) = \{ \phi : A^\mathbb{Z} \to A^\mathbb{Z} \text{ homeomorphism, } [\sigma, \phi] = \text{id} \} \]
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It is still unknown whether \(\text{Aut}({0, 1}\mathbb{Z}) \cong \text{Aut}({0, 1, 2}\mathbb{Z})\), but we know that

\[
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A simple example with that property:

\[ G_1 = \bigoplus_{i \in \mathbb{N}} \mathbb{Z}/4\mathbb{Z} \quad \text{and} \quad G_2 = \mathbb{Z}/2\mathbb{Z} \bigoplus \bigoplus_{i \in \mathbb{N}} \mathbb{Z}/4\mathbb{Z} \]
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\[ G_1 \hookrightarrow G_2 : (a_1, a_2, \cdots) \rightarrow (0, a_1, a_2, \cdots) \]

However, they are not isomorphic: Each element of \( G_1 \) which has order 2 has square roots, while \((1, 0, 0, 0, \cdots)\) has none in \( G_2 \).

Moral: we should try to understand torsion and roots in \( \text{Aut}\left(\{0, 1\} \mathbb{Z}\right) \).
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Talk highlights

Definition (Torsion problem)

Let \( G = \langle S \mid R \rangle \) be a finitely generated group. The torsion problem of \( G \) is the language \( \text{TP}(G) \) where

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\text{TP}(G) = \{ w \in S^* \mid \exists n \in \mathbb{N} \text{ such that } w^n =_G 1 \}
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For any finite alphabet $|A| \geq 2$, $\text{Aut}(A^\mathbb{Z})$ contains a finitely generated subgroup with undecidable torsion problem.
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Theorem (B, Kari, Salo)
Let \((A^\mathbb{Z}^d, \sigma)\) be a full shift and \( |A| \geq 2 \). The topological fullgroup \([\sigma]\) contains a finitely generated subgroup with undecidable torsion problem if and only if \( d \geq 2 \).
Recall that a Turing machine is defined by a rule:

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Such that if \((x, q) \in \Sigma^\mathbb{Z} \times Q\) and \(\delta_T(x_0, q) = (a, r, d)\) then:

\[ T(x, q) = (\sigma_d(\tilde{x}), q') \]

where \(\sigma : \Sigma^\mathbb{Z} \rightarrow \Sigma^\mathbb{Z}\) is the shift action given by \(\sigma_d(x)_z = x_{z-d}\), \(\tilde{x}_0 = a\) and \(\tilde{x}|_{\mathbb{Z}\setminus\{0\}} = x|_{\mathbb{Z}\setminus\{0\}}\).
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As in cellular automata, the class of CA with radius bounded by some $k \in \mathbb{N}$ is not closed under composition or inverses.
Let’s get rid of these constrains. Given $F, F'$ finite subsets of a group $G$, consider instead of $\delta_T$ a function :

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Let $F = F' = \{0, 1, 2\}^2$, then $f_T(p, q) = (p', q', \vec{d})$ means:

- Turn state $q$ into state $q'$
- Move head by $\vec{d}$. 

Moving head model

\( f_T \) defines naturally an action

\[ T \curvearrowright \Sigma^G \times Q \times \mathbb{Z}^d \]

\( f(\bullet, q_1) = (\circ, q_2, (1, 1)) \)

\( F = \{(0, 0), (1, 0), (1, 1)\} \)
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Let $|\Sigma| = n$ and $|Q| = k$.

$(\text{TM}(G, n, k), \circ)$ is the monoid of all such $T$ with the composition operation; $(\text{RTM}(G, n, k), \circ)$ is the group of all such $T$ which are bijective.
Let $Q = \{1, \ldots, k\}$ and $\Sigma = \{0, \ldots, n - 1\}$.

$$\Sigma^G = \{x : G \rightarrow \Sigma\}$$

$$X_k = \{x \in \{0, 1, \ldots, k\}^G \mid 0 \notin \{x_g, x_h\} \implies g = h\}$$

Let $X_{n,k} = \Sigma^G \times X_k$ and $Y = \Sigma^G \times \{0^G\}$. Then:
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Let $X_{n,k} = \Sigma^G \times X_k$ and $Y = \Sigma^G \times \{0^G\}$. Then:

\[
\text{TM}(G, n, k) = \{\phi \in \text{End}(X_{n,k}) \mid \phi|_Y = \text{id}, \phi^{-1}(Y) = Y\}
\]

\[
\text{RTM}(G, n, k) = \{\phi \in \text{Aut}(X_{n,k}) \mid \phi|_Y = \text{id}\}
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$$(\bullet \circ, q_1) = (\circ \bullet, q_2, (1, 1))$$

$$F = \{(0, 0), (1, 0), (1, 1)\}$$
Moving tape model

$f_T$ defines naturally an action

\[ T \circlearrowright \Sigma^G \times Q \]

Let $|\Sigma| = n$ and $|Q| = k$.

$(\text{TM}_{\text{fix}}(G, n, k), \circ)$ is the monoid of all such $T$ with the composition operation; $(\text{RTM}_{\text{fix}}(G, n, k), \circ)$ is the group of all such $T$ which are bijective.
Let $x, y \in \Sigma^G$. $x$ and $y$ are asymptotic, and write $x \sim y$, if they differ in finitely many coordinates. We write $x \sim_F y$ if $x_g = y_g$ for all $g \notin F$, $F$ a finite subset of $G$. 

Let $T : \Sigma^G \times Q \to \Sigma^G \times Q$ be a function. Dynamical definition $T$ is a moving tape Turing machine $\iff T$ is continuous, and for a continuous function $s : \Sigma^G \times Q \to G$ and $F \subset G$ we have $T(x, q) \sim_F s(x, q)$ for all $(x, q) \in \Sigma^G \times Q$. 

$s : \Sigma^G \times Q \to G$ is the shift indicator function.
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Dynamical definition

$T$ is a moving tape Turing machine $\iff T$ is continuous, and for a continuous function $s : \Sigma^G \times Q \to G$ and $F \subset G$ we have $T(x, q)_1 \sim_F \sigma_{s(x,q)}(x)$ for all $(x, q) \in \Sigma^G \times Q$. 

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Moving tape model: dynamical definition

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**Dynamical definition**

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$s : \Sigma^G \times Q \to G$ is the shift indicator function
Equivalence of the models

\[ \text{RTM}_{\text{fix}}(G, 1, k) \cong S_k \text{ and } G \leftrightarrow \text{RTM}(G, 1, k). \]
Equivalence of the models

$\text{RTM}_{\text{fix}}(G, 1, k) \cong S_k$ and $G \hookrightarrow \text{RTM}(G, 1, k)$.

**Proposition**

If $n \geq 2$ then:

\[
\text{TM}_{\text{fix}}(G, n, k) \cong \text{TM}(G, n, k) \\
\text{RTM}_{\text{fix}}(G, n, k) \cong \text{RTM}(G, n, k).
\]
Properties of RTM

Proposition

Let \( T \in TM_{\text{fix}}(G, n, k) \). Then the following are equivalent:

1. \( T \) is injective.
2. \( T \) is surjective.
3. \( T \in RTM_{\text{fix}}(G, n, k) \).
4. \( T \) preserves the uniform measure (\( \mu(T^{-1}(A)) = \mu(A) \) for all Borel sets \( A \)).
5. \( \mu(T(A)) = \mu(A) \) for all Borel sets \( A \).
Proposition

If $n \geq 2$ $\text{RTM}(\mathbb{Z}, n, k)$ is not finitely generated.
Properties of RTM

**Proposition**

If $n \geq 2$ $\text{RTM}(\mathbb{Z}, n, k)$ is not finitely generated.

Proof: We find an epimorphism from $\text{RTM}$ to a non-finitely generated group.

Let $T \in \text{RTM}_{\text{fix}}(\mathbb{Z}, n, k)$, therefore, it has a shift indicator $s : \Sigma^\mathbb{Z} \times Q \rightarrow \mathbb{Z}$. Define

$$\alpha(T) := E_\mu(s) = \int_{\Sigma^\mathbb{Z} \times Q} s(x, q) d\mu,$$

One can check that $\alpha(T_1 \circ T_2) = \alpha(T_1) + \alpha(T_2)$.

Therefore $\alpha : \text{RTM}(\mathbb{Z}, n, k) \rightarrow \mathbb{Q}$ is an homomorphism.
Now consider the machine $T_{\text{SURF},m}$ where for all $a \in \Sigma$ and $q \in Q$:

$$f(0^m a, q) = (a0^m, q, 1).$$

Otherwise $f(u, q) = (u, q, 0)$.
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$\langle (1/n^m)_{m \in \mathbb{N}} \rangle \subset \alpha(\text{RTM}(\mathbb{Z}, n, k))$ which is thus a non-finitely generated subgroup of $\mathbb{Q}$. 

Properties of RTM
Interesting subgroups of RTM

$\triangleright \ LP(G, n, k) \longrightarrow \text{Local permutations.}$

0 0 1 0 0 1 0 0 1 1 1 1 0 1 0 0 1 1 0 0

$q$

$r$

$T_\pi$

0 0 1 0 0 1 0 1 1 0 0 0 1 1 1 0 0 0 1 0 0
Interesting subgroups of RTM

▷ LP($G, n, k$) $\rightarrow$ Local permutations.
▷ RFA($G, n, k$) $\rightarrow$ Reversible finite-state automata.
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- LP($G, n, k$) $\rightarrow$ Local permutations.
- RFA($G, n, k$) $\rightarrow$ Reversible finite-state automata.
- OB($G, n, k$) $\rightarrow$ Oblivous machines $\langle\text{LP, Shift}\rangle$. 
Interesting subgroups of RTM

- $\text{LP}(G, n, k) \rightarrow$ Local permutations.
- $\text{RFA}(G, n, k) \rightarrow$ Reversible finite-state automata.
- $\text{OB}(G, n, k) \rightarrow$ Oblivous machines $\langle \text{LP}, \text{Shift} \rangle$.
- $\text{EL}(G, n, k) \rightarrow$ Elementary machines $\langle \text{LP}, \text{RFA} \rangle$. 
Res. finite groups are those where every non-identity element can be mapped to a non-identity element by a homomorphism to a finite group.

Amenable groups admit left invariant finitely additive measures.

LEF and LEA stand for locally embeddable into (finite/amenable) groups.

Sofic groups are generalizations of LEF and LEA.
∀ \( n \geq 2 \), \( \text{RTM}(\mathbb{Z}^d, n, k) \) is LEF but neither amenable nor residually finite.
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$\text{LP}(G, n, k)$ is locally finite.

In particular, for $n \geq 2$ \(\text{LP}(G, n, k)\) is amenable and not finitely generated.
Now let’s add the shift. Recall that $\text{OB}(G, n, k) = \langle \text{LP}, \text{Shift} \rangle$. 
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\( \text{OB}(G, n, k) \) is amenable \( \iff \) \( G \) is amenable.

Proof: Use the short exact sequence

\[
1 \rightarrow \text{LP}(G, n, k) \rightarrow \text{OB}(G, n, k) \rightarrow G \rightarrow 1.
\]
Recall that $\text{RFA}(G, n, k)$ is the subgroup of machines which do not modify the tape. Note that if $[[\sigma]]$ is the fullgroup of $(\Sigma^G, \sigma)$ then $[[\sigma]] \cong \text{RFA}(G, n, 1)$.

**Theorem**

For $n \geq 2$, countable and not locally finite $G$ we have that

$$\underbrace{\mathbb{Z}/2\mathbb{Z} \ast \cdots \ast \mathbb{Z}/2\mathbb{Z}}_{m \text{ times}} \hookrightarrow \text{RFA}(G, n, k)$$
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Proof: Blackboard.
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$m$ times

In particular, this means that RFA and RTM are not amenable in this case.

**Theorem**

For $n \geq 2$, infinite and residually finite $G$ we have that $\text{RFA}(G, n, k)$ is residually finite but *not finitely generated*. 
Some properties: $\text{EL}(\mathbb{Z}^d, n, k)$ and $\text{RTM}(\mathbb{Z}^d, n, k)$

$\text{EL}(\mathbb{Z}^d, n, k) = \langle \text{LP}(\mathbb{Z}^d, n, k), \text{RFA}(\mathbb{Z}^d, n, k) \rangle$ is the subgroup of elementary Turing machines.
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$EL(\mathbb{Z}^d, n, k) = \langle LP(\mathbb{Z}^d, n, k), RFA(\mathbb{Z}^d, n, k) \rangle$ is the subgroup of elementary Turing machines.
Example: Langton’s ant $\in EL(\mathbb{Z}^2, 2, 4)$. 
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Question: Is \( EL(\mathbb{Z}^d, n, k) = \text{RTM}(\mathbb{Z}^d, n, k) \)?
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Question: Is $\text{EL}(\mathbb{Z}^d, n, k) = \text{RTM}(\mathbb{Z}^d, n, k)$?

For $n \geq 2$, $\alpha(\text{EL}(\mathbb{Z}^d, n, k)) = \alpha(\text{RFA}(\mathbb{Z}^d, n, k))$ has bounded denominator. In particular $\text{EL} \nsubseteq \text{RTM}$. 
Computability properties

Given a finite rules : $f, f'$ :

- It is decidable (in any model) whether $T_f = T_{f'}$.
- We can effectively calculate a rule for $T_f \circ T_{f'}$.
- It is decidable whether $T_f$ is reversible.
- If it is, we can effectively compute a rule for $T_f^{-1}$. 

$RTM(\mathbb{Z}, d, n, k)$ is a recursively presented group with decidable word problem.

What can we say about the torsion ($\exists n$ such that $T_n = 1$)?
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$RTM(\mathbb{Z}^d, n, k)$ is a recursively presented group with decidable word problem.

What can we say about the torsion ($\exists n$ such that $T^n = 1$) problem?
Back to the target: $TP(\text{Aut}(A^\mathbb{Z}))$ is undecidable.

We want to prove that the torsion problem is undecidable for a f.g. subgroup of $\text{Aut}(A^\mathbb{Z})$. The sketch is as follows:

1. The torsion problem for reversible classical Turing machines is undecidable [Kari, Ollinger 2008].
We want to prove that the torsion problem is undecidable for a f.g. subgroup of $\text{Aut}(A^\mathbb{Z})$. The sketch is as follows:

1. The torsion problem for reversible classical Turing machines is undecidable [Kari, Ollinger 2008].
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2. Classical Turing machines embed into $\text{EL}(\mathbb{Z}, n, k)$.
3. $\text{EL}(\mathbb{Z}, n, k)$ is finitely generated.
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1. The torsion problem for reversible classical Turing machines is undecidable [Kari, Ollinger 2008].
2. Classical Turing machines embed into $\text{EL}(\mathbb{Z}, n, k)$.
3. $\text{EL}(\mathbb{Z}, n, k)$ is finitely generated.
4. There exists a "torsion preserving function" from $\text{EL}(\mathbb{Z}, n, k)$ to $\text{Aut}(A^\mathbb{Z})$

Classical $\hookrightarrow$ EL $\hookrightarrow$ Aut($A^\mathbb{Z}$)
$\text{OB}(\mathbb{Z}^d, n, k)$ is finitely generated.

This proof is inspired both on the existence of strongly universal reversible gates for permutations of $\Sigma^m$ and the Juschenko Monod proof for the fullgroup of minimal actions. A controlled swap is a transposition $(s, t)$ where $s, t$ have Hamming distance 1 in $Q \times \Sigma^m$.

**Theorem**

The group generated by the applications of controlled swaps of $Q \times \Sigma^4$ at arbitrary positions generates $\text{Sym}(Q \times \Sigma^m)$ if $|\Sigma|$ is odd and $\text{Alt}(Q \times \Sigma^m)$ if it’s even.

**Corollary** : $[\text{Sym}(Q \times \Sigma^m)]_{m+1} \subset \langle [\text{Sym}(Q \times \Sigma^4)]_{m+1} \rangle$. 
OB(\mathbb{Z}^d, n, k) is finitely generated.

Using this result, a generating set can be constructed:

- $A_1 =$ Shifts $T_{e_i}$ for \( \{e_i\}_{i \leq d} \) a base of $\mathbb{Z}^d$.
- $A_2 =$ All $T_\pi \in \text{LP}(\mathbb{Z}^d, n, k)$ of fixed support $E \subset \mathbb{Z}^d$ of size 4.
- $A_3 =$ The swaps of symbols in positions $(\vec{0}, e_i)$. 
EL(\mathbb{Z}, n, k) is finitely generated.

EL(\mathbb{Z}, n, k) = \langle LP(\mathbb{Z}, n, k), RFA(\mathbb{Z}, n, k) \rangle = \langle OB(\mathbb{Z}, n, k), RFA(\mathbb{Z}, n, k) \rangle
EL($\mathbb{Z}, n, k$) is finitely generated.

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We can show that $RFA(\mathbb{Z}, n, k)$ is generated by orbitwise shifts and controlled position swaps.

1. $f$ is orbitwise shift is $\forall x \in X \exists k \in \mathbb{Z}$ such that $f(\sigma^n(x)) = \sigma^{n+k}(x)$.

2. $f$ is controlled position swap if for some $u, v \in \Sigma^*$, $f(xu.avy) = xua.vy$ and $f(xua.vy) = xu.avy$. 
EL(\mathbb{Z}, n, k) is finitely generated.

\[ \text{EL}(\mathbb{Z}, n, k) = \langle \text{LP}(\mathbb{Z}, n, k), \text{RFA}(\mathbb{Z}, n, k) \rangle = \langle \text{OB}(\mathbb{Z}, n, k), \text{RFA}(\mathbb{Z}, n, k) \rangle \]

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   f(xu.avy) = xua.vy \text{ and } f(xua.vy) = xu.avy.
   \]

In the fullshift, orbitwise shifts are precisely the shifts. So we only need to implement controlled position position swaps [technical].
Definition

Let $G$ and $H$ be groups. We say a function $\phi : G \to H$ is a **blurphism** if the following holds: If $F \subseteq G^*$ is finite, then the group $\langle w \mid w \in F \rangle \leq G$ is infinite if and only if the group $\langle \phi(w_1)\phi(w_2)\cdots\phi(w_{|w|}) \mid w \in F \rangle$ is infinite.

Lemma

If $G$ has a finitely generated subgroup $G'$ with generating set $B$ with undecidable torsion problem and there is a computable blurphism $\phi : G \to H$, then the subgroup $H'$ of $H$ generated by $\phi(b)$ where $b \in B$ has undecidable torsion problem.

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Construction of the blurphism

- Let $A = \{\Sigma^2 \times (\{\leftarrow, \rightarrow\} \cup \{Q \times \{\uparrow, \downarrow\}\})\}$. 

Parse the third layer into zones $(\leftarrow^* (q, a) \rightarrow^* |\leftarrow^* \rightarrow^*)^*$. Define $\phi$ to act as a conveyor belt. $\phi$ is a computable blurphism. Therefore $\phi(EL(Z, n, k))$ is a finitely generated subgroup of $\text{Aut}(A_Z)$ with undecidable torsion problem. As $\text{Aut}(A_Z) \rightarrow \text{Aut}(\{0, 1\}^Z)$ the same is valid for any automorphism group of a fullshift.
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- Define $\phi$ to act as a conveyor belt [Blackboard]
- $\phi$ is a computable blurphism.

Therefore $\phi(\text{EL}(\mathbb{Z}, n, k))$ is a finitely generated subgroup of $\text{Aut}(A^\mathbb{Z})$ with undecidable torsion problem. As $\text{Aut}(A^\mathbb{Z}) \hookrightarrow \text{Aut}(\{0, 1\}^\mathbb{Z})$ the same is valid for any automorphism group of a fullshift.
The torsion problem for RFA

RFA(\(\mathbb{Z}, n, k\)) has decidable torsion problem.

Proof: As \(\mathbb{Z}\) is two-ended, any non-torsion machine must shift to the left or right in at least a periodic configuration.
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Theorem

RFA(\mathbb{Z}^d, n, k) has a finitely generated subgroup with undecidable torsion problem for \( d, n \geq 2 \).
The torsion problem for RFA

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**Theorem**

RFA(\(\mathbb{Z}^d, n, k\)) has a finitely generated subgroup with undecidable torsion problem for \(d, n \geq 2\).

Proof: Reduction to the snake tiling problem, which reduces to the domino problem for \(\mathbb{Z}^d\).
The snake problem

Can we tile the plane in a way which produces a bi-infinite path?
The snake problem

**Theorem (Kari)**

The snake tiling problem is undecidable.

The proof uses a plane filling curve generated by a substitution.
The snake problem

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For every instance of the snake tiling problem, one can construct $T \in \text{RFA}$ which walks the path of the snake, and turns back if it encounters a problem.
The torsion problem for RFA: Cheating version

We’ll first do it by cheating: Arbitrary alphabet \( \tau \) as an instance of the snake tiling problem and at least two states \( L, R \).

- Let \( t \) be the tile at \((0, 0)\). If \( t = \epsilon \), do nothing.
- Otherwise:
  - If the state is \( L \). Check the tile in the direction \( \text{left}(t) \). If it matches correctly with \( t \) move the head to that position, otherwise switch the state to \( R \).
  - If the state is \( R \). Check the tile in the direction \( \text{right}(t) \). If it matches correctly with \( t \) move the head to that position, otherwise switch the state to \( L \)
The torsion problem for RFA: The real deal

We are going to code everything in a binary alphabet and use no states.

\[
\begin{array}{ccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & b_1 & b_2 & b_3 & 0 & 1 \\
1 & 0 & r_1 & r_2 & b_4 & 0 & 1 \\
1 & 0 & l_1 & l_2 & b_5 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]
Consider the group spanned by the following machines:

1. \( \{ T\vec{v} \}_{\vec{v} \in D} \) that walks in the direction \( \vec{v} \in D \) independently of the configuration.

2. \( T_{\text{walk}} \) that walks along the direction codified by \( l_1, l_2 \) or \( r_1, r_2 \) depending on the direction bit.

3. \( \{ g_c \}_{c \in C} \) that flips the direction bit if the current pattern is \( c \in C \),

4. \( \{ h_c \}_{c \in C} \) that flips the auxiliary bit if the current pattern is \( c \in C \),

5. \( \{ g_{+,c} \}_{c \in C} \) that adds the auxiliary bit to the direction bit if the current pattern is \( c \in C \), and

6. \( \{ h_{+,c} \}_{c \in C} \) that adds the direction bit to the auxiliary bit if the current pattern is \( c \in C \),
The torsion problem for RFA: The real deal

The previous group spans the machines $g_p$ and $h_p$ for patterns $p$ composed of fragments of $c$ in compatible positions.
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$$g_p^* = \left( T_{-7\bar{v}} \circ g_{+,c} \circ T_{7\bar{v}} \circ h_{p^*\{\bar{v}\}} \right)^2.$$

$$h_p^* = \left( T_{-7\bar{v}} \circ h_{+,c} \circ T_{7\bar{v}} \circ g_{p^*\{\bar{v}\}} \right)^2.$$

Finally, we use these machines to code the first ones.
The torsion problem for RFA: The real deal

\[ \mathcal{M}(t) = \]

[Diagram of a grid with arrows indicating torsion movements]
The torsion problem for RFA: The real deal

\[ T^* = (T_{\text{walk}})^M \circ \prod_{p^* \in M} g_{p^*} \circ \prod_{c \in C} g_c \]

Acts as the first machine, but using these coded macrotiles.
The torsion problem for RFA: The real deal

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**Corollary**

Let \( d \geq 2 \) and \( \sigma \) be the shift action of \( \mathbb{Z}^d \) over a full shift \( A^{\mathbb{Z}^d} \) where \( |A| \geq 2 \). Then the full group \( [[\sigma]] \) contains a finitely generated subgroup with undecidable torsion problem.
Thank you for your attention!