Exercise 1  NP-completeness

Show that the following problems are NP-complete:

1. **CLIQUE**: a couple \((G, k)\) is in CLIQUE when \(G\) is an undirected graph, \(k\) an integer, and \(G\) has a \(k\)-clique, i.e. a subset of \(k\) vertices all pairwise connected.

2. **0\slash 1 IPROG** (integer programming): a given set of linear inequalities with rational coefficients over variables \(u_1, \ldots, u_n\) is in 0\slash 1 IPROG, if there exists an assignment of each variable in \(\{0, 1\}\) which satisfies it.

Exercise 2  I had a dream

If \(P = NP\), then show that

1. there is a polynomial time algorithm that given a formula in CNF (conjunctive normal form), returns an assignment of its variables satisfying it, and 0 otherwise.

2. \(EXP = NEXP\).

Exercise 3  Mahaney’s theorem (1982)

1. **Définition.** A language \(L \subseteq \Sigma^*\) is said sparse (“creux”) if there exists a polynomial \(p\) such that this cardinal inequality holds: \(|L \cap \Sigma^n| \leq p(n)|.

   1. Be \(L\) a sparse language, what can you say about \(|L \cap \Sigma^n|\)?

   2. We are going to show that if a sparse language \(L\) is NP-complete, then \(P = NP\). Be such an \(L\), and \(X\) in NP defined as:

\[
x \in X \iff \exists w \in \Sigma^{p(|x|)}, (x, w) \in A
\]

where \(p\) is a polynomial function and \(A \in P\). We want to show that \(X\) is decidable in polynomial time. Be \(G(A) = \{(x, w) : \exists y \in \Sigma^{p(|x|)}, y \geq w \land (x, y) \in A\}\). What very simple fact can you say about \(G(A)\)?

3. Reduce \(G(A)\) to \(L\), and show that \(X\) is decidable in polynomial time.

Exercise 4  Tally languages (Book’s theorem, 1974)

A language is said to be tally (or unary), if it is included in a unary alphabet \(\{a\}^*\) for a fixed symbol \(a\) (typically, \(a = 0\) or \(a = 1\)). Given a language \(L \subseteq \Sigma\) with \(\Sigma\) identified to \(\{1, \ldots, k\}\), we define \(Tally(u_0 \cdots u_n) = \sum_{i=0}^n u_i k^i\).

1. Prove that: if \(g(n) \geq 2^n\), and \(L \in \text{DTIME}(g(n))\) (resp \(\text{NDTIME}(g(n))\)) then \(\exists c_1, c_2 > 0\) such that \(Tally(L) \in \text{DTIME}(c_1 g(c_2 \log(n)))\) (resp \(\text{NDTIME}(c_1 g(c_2 \log(n)))\)).

2. Let \(g(n) \geq n\). If \(L_1 \in \text{Tally} \cap \text{DTIME}(g)\) (resp. \(\text{NDTIME}(g)\)), then prove that \(\forall L_2, \text{Tally}(L_2) = L_1, \exists c_1, c_2 > 0, L_2 \in \text{DTIME}(c_1 g(2^{c_2 n}))\) (resp \(\text{NDTIME}(c_1 g(2^{c_2 n}))\)).

3. Show the following equivalence:

\[
\text{NP} \cap \text{Tally} = \text{P} \cap \text{Tally} \iff \forall c, \exists d > 0, \text{NDTIME}(2^{cn}) \subseteq \text{DTIME}(2^{dn}) \iff \text{NEXP} = \text{EXP}
\]

4. If \(\text{EXP} \neq \text{NEXP}\), what can you say about \(P\) and \(NP\)?