Failure time in the fiber-bundle model with thermal noise and disorder

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Abstract

The average time for the onset of macroscopic fractures is analytically and numerically investigated in the fiber-bundle model with quenched disorder and thermal noise under a constant load. We find an implicit exact expression for the failure time in the low-temperature limit that is accurately confirmed by direct simulations. The effect of the disorder is to lower the energy barrier.

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I. INTRODUCTION

The onset of fractures in heterogeneous materials has been for a long time the subject of many studies in the engineering community for its obvious technological implications. More recently, the problem has attracted the interest also of the physicist community, because of its non trivial statistical character. Fractures are, indeed, genuine transient phenomena for which it is highly desirable to identify universal laws, but this ambition contrasts with the lack of general tools capable of dealing with non equilibrium phenomena. As a consequence, many simplified models have been proposed in the attempt to capture the relevant dynamical properties, without pretending to accurately reproduce the microscopic details.

Theoretical and experimental investigations of fractures are actually devoted to clarifying many different questions, such as, e.g., the velocity of propagation, the roughness, and the onset of precursors. In this paper we are interested in determining the failure time of a given sample subjected to a constant stress. This the so-called creep-test, widely used by engineers in order to estimate the lifetime τ of a given material as a function of the applied stress. It would be obviously very desirable to construct a theory able to predict the failure time upon the knowledge of a few ingredients and without having to perform experimental tests under different stress conditions.

Several authors [1–4] conjectured that the fracture is a thermal activated process whose effective temperature T_{eff} should coincide with the thermodynamic temperature T. Several experimental observations [5–7] seem to indicate that the activation model proposed by Pomeau predicts correctly the dependence of τ on the applied stress. Conversely, all the experiments [5–7] indicate that the effective temperature in strongly heterogeneous materials can be several orders of magnitude larger than T, or, equivalently, the energy barrier is smaller than what theoretically predicted.

The need to clarify this problem has led Guarino et al. [8,9] to suitably modify the fiber-bundle model, initially introduced [10,11] as a purely deterministic model to describe the behaviour of an ensemble of fibers subjected all to the same load but with different breaking thresholds [12–14]. In the original model, upon increasing from zero the applied stress nothing happens until the weakest fiber breaks. As a consequence of the first failure, the average stress increases and this may induce further failures. In practice, it is only when the applied load is large enough that an avalanche process sets in, giving rise to a complete failure of the system.

In order to account for thermal fluctuations, in Ref. [9,15,16], it was argued that each single fiber can break at any time with a probability per unit time proportional to the probability of a thermal fluctuation above the critical length of the given fiber. As a consequence of thermal fluctuations, the bundle can break for any imposed stress, exactly as expected in real systems.

The modified fiber-bundle model has been then studied both in the homogeneous (same breaking threshold for all fibers) and heterogeneous case, finding that in the former case, the effective temperature coincides with the thermodynamic temperature [18,15,16]. In heterogeneous systems, it has been found that disorder contributes to modifying T, but it has not yet been developed a sufficiently general treatment and the results existing so far do partly conflict with each other and this prevents drawing definite conclusions.

It is precisely the goal of this paper to develop a general approach for dealing with heterogeneous systems in the low temperature limit. We shall show that disorder contributes as a multiplicative correction that can equivalently be interpreted either as an amplification of the temperature or a lowering of the energy barrier. Similar conclusions on the role of disorder in the crack activation processes have been reached by other authors [17].

In the next section, we briefly recall the results obtained in the two previous papers that have dealt with the same model. In section III we derive and solve the dynamical equations that allow us to determine the scaling behavior for the average failure time. The last section is devoted to conclusions and an outline of future perspectives.

II. PREVIOUS RESULTS

Initially, the interest has been devoted to studying the behaviour of homogeneous bundles, composed of N fibers. In both Refs. [18,15], it has been found that, in the limit of $N \to \infty$, the average failure time τ is,

$$\tau = \frac{\sqrt{2\pi T}}{\gamma f_0} \exp\left[\frac{(1-f_0)^2}{2T}\right], \tag{1}$$

where T is the temperature scaled to the bond energy at the breaking threshold ($T = \frac{k_B \overline{T}}{Y\ell}$, where Y is the elastic constant, k_B the Boltzmann constant, $\overline{T}$ the absolute temperature and ℓ the critical length), f_0 is the imposed average force (scaled to $Y\ell$), and $1/\gamma$ is the time scale of the thermal fluctuations. Thus, in this specific case, the energy barrier to be overcome in order to break the fiber bundle is $U = Y\ell^2(1-f_0)^2/2$. Moreover, Roux [18] showed that the average failure time of the first fiber is

$$\tau_1 = \sqrt{\frac{2\pi}{T}} \frac{1-f_0}{\gamma N} \exp\left[\frac{(1-f_0)^2}{2T}\right]. \tag{2}$$

Accordingly, we see that the exponential factor is the same in both the expression for τ and τ_1, indicating that the activating energy is the same for both processes.

The disordered case is more easily studied under the assumption of a Gaussian distribution of the breaking thresholds f,

$$P(f) = \frac{1}{\sqrt{2\pi T_d}} \exp\left[-\frac{(f-1)^2}{2T_d}\right] \tag{3}$$

where the variance T_d measures the amount of quenched disorder present in the bundle. In order to be precise, one should restrict the definition of $P(f)$ to positive values, but we shall see in the next section that in the regime we are interested in, this initial anomaly disappears immediately without causing any trouble.

With the above assumption, Roux determined again the average failure time of the first fiber, finding

$$\tau_1 = \sqrt{\frac{2\pi}{T+T_d}} \frac{1-f_0}{N} \exp\left[\frac{(1-f_0)^2}{2(T+T_d)}\right]. \tag{4}$$

Accordingly, he concluded that the effect of disorder is to introduce an additive shift on the effective temperature.

On the other hand, Ciliberto et al. [15,16], performing an analytic approximate calculation, found a multiplicative correction, namely

$$\tau = \tau_0 \exp\left[\frac{(1-f_0)^2}{2T_{eff}}\right], \tag{5}$$

with

$$T_{eff} = \frac{T}{\left(1 - \frac{\sqrt{2\pi T_d}}{2(1-f_0)}\right)^2} \tag{6}$$

Since, one cannot control the accuracy of the approximations involved in the determination of the above formula, it is not possible to discuss a priori its validity in the small temperature limit, when the analogy with activation processes becomes more transparent.

III. MODEL SOLUTION

Let us start by denoting with $f_a(t)$ the force exerted at time t on a fiber whose critical force is f. According to the original formulation of the problem, in the presence of thermal noise, the force applied on each fiber exhibits Gaussian fluctuations around an average value f_a. Therefore, the probability per unit time to break a fiber characterized by a threshold f is proportional to the probability for a fluctuation to overcome the assigned threshold, i.e.

$$G(f - f_a) = \frac{\gamma}{2}\left\{1 - \operatorname{erf}\left[-\frac{(f-f_a)^2}{2T}\right]\right\} \tag{7}$$

where T is the working temperature, while γ is a constant fixing the time scale for the process. In the small temperature limit, we will see that the most relevant contribution to the fiber breakdown occurs in the tail of the distribution, where we can approximate the error function with a Gaussian. Accordingly, we assume that

$$G(f - f_a) = \frac{\gamma}{f - f_a}\sqrt{\frac{T}{2\pi}} \exp\left[-\frac{(f-f_a)^2}{2T}\right]. \tag{8}$$

Let us now introduce the relevant dynamical variable, i.e. the distribution $Q(f,t)$ of unbroken bonds at time t ($Q(f,0)=P(f)$). The fraction of broken bonds is, therefore,

$$\Phi(t) \equiv 1 - \int_{-\infty}^{+\infty} df Q(f,t), \tag{9}$$

and the average force f_a exerted on each fiber at time t is

$$f_a = \frac{f_0}{1-\Phi} \tag{10}$$

where f_0 is the initial average force. The definition of the model is completed by the dynamical equation for $Q(f,t)$

$$\dot{Q}(f,t) = -Q(f,t)G(f-f_a). \tag{11}$$

The difficulty of solving this model arises from the time dependence of f_a which is determined by the integral of Q over all f values.

Before passing to the analytical calculations, let us discuss the numerical integration of Eq. (11). We find it convenient to introduce the variable

$$S(f,t) = \frac{Q(f,t)}{P(f)} \tag{12}$$

representing the fraction of unbroken bonds at time t per class of fibers with thresholds between f and $f+df$. In fact, the evolution of S provides an insightful representation of the fracture process. As one might have expected, we see in Fig. 1 that the breakdown of the bundle starts from the weaker fibers to progressively affect the more robust ones. Less obvious, is that the fracture appears to proceed as a moving front with constant shape. Moreover, the temporal spacing of the various fronts reported in Fig. 1 reveals a progressive slowing down of the evolution. This latter feature will turn out to be the crucial point for understanding the scaling properties of the whole process.

The increasing slowness of the bond breakdown is better revealed by looking at the time derivative of Φ. The monotonous decrease of $\dot{\Phi}$ preceding the final macroscopic fracture (see Fig. 2) indicates that one cannot estimate the average breaking time τ by limiting oneself to follow the initial stages of the process.

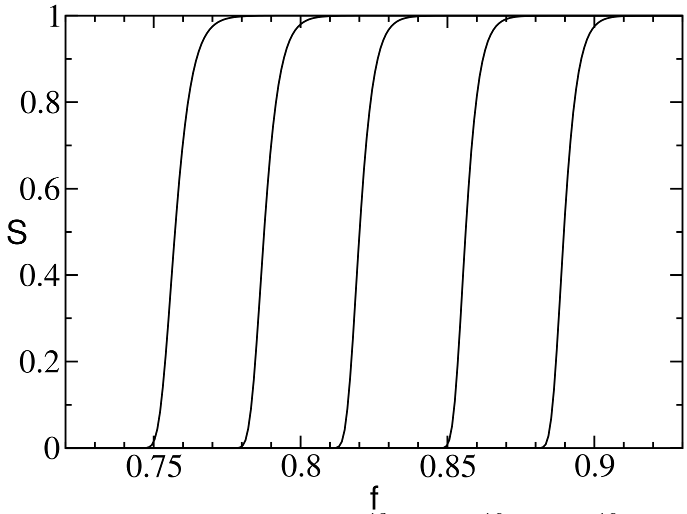

FIG. 1. Position of the front S at times $6.6\,10^{13}$, $6.6\,10^{16}$, $5.5\,10^{19}$, $6.6\,10^{13}$, $8.5\,10^{21}$, $3.3\,10^{23}$ (from left to right) in a simulation with $T = 10^{-3}$, $T_d = 10^{-2}$ and $\gamma = 1$.

A yet clearer description of the breakdown process is obtained by formally interpreting $S(f)$ as the integral of some probability distribution $R(f)$ (i.e. $dS/df = R(f)$). This allows also a straightforward identification of the step region, where the ongoing breakdowns are concentrated at a given time. More interesting, we find that the shape of $R(f)$ is independent of T in the slowest evolution region (i.e., where most of the time is spent before the final breakdown). This can be appreciated in Fig. 3, where we have plotted $R(f)$ for two different temperature values, after shifting the distribution around the average value and scaling f to have unit r.m.s.. Besides observing the independence of the shape on the temperature, notice also the strong similarity with distributions obtained for extreme-value statistics [19–21]. This is certainly not a surprise, since the tail of $R(f)$ consists of events that, over time, proved to be anomalous. The accuracy of the data allows us to show that in this case the Gumbel's distribution [20,21] fits precisely the $R(f)$. This specific shape of the $R(f)$ and its scale invariance deserve further investigations, but here we are more interested in describing the temporal evolution of the fracture process. To this goal, it is more important to notice that the standard deviation σ_r of $R(f)$ goes to 0 linearly with T. This can be clearly seen in Fig. 4, where we have reported σ_r versus T for $T_d = 10^{-2}$ (there, it can also be seen that the

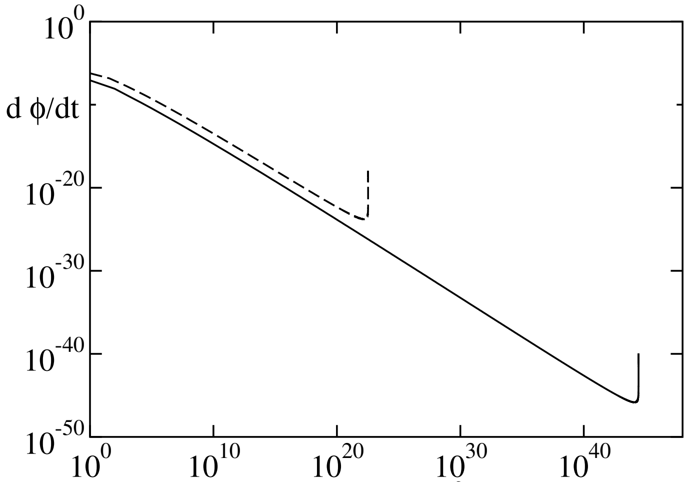

FIG. 2. Time derivative of Φ versus time for $T_d = 10^{-2}$ and two different values of the temperature, $T = 5.10^{-4}$ (solid curve), $T = 10^{-3}$ (dashed curve).

proportionality constant is approximately equal to 4). Notice that the linear dependence on T is quite a fast decrease, as thermal fluctuations are on the order of $\sqrt{T}$. This suggests that a good (asymptotically exact for $T \to 0$) approximation consists in assuming a Heaviside shape for $S(f)$. Such an approximation has also the advantage of parametrizing an a priori infinite-dimensional object such as $S(f)$ with a single variable: the position of the step f_s. Equipped with such an assumption, $Q(f,t)$ can be approximated with a Gaussian truncated below some threshold $f = f_s$. Notice that this differs from the hypothesis formulated in Ref. [15,16], where it was assumed that $Q(f,t)$ remains unchanged for $f > 1$ while it decreases linearly to 0 for $f < 1$ with a slope to be determined self-consistently.

By integrating Eq. (11) over f, we obtain the one-dimensional differential equation

$$\dot{\Phi} = \frac{\gamma}{2\pi}\sqrt{\frac{T}{T_d}} \int_{f_s}^{\infty} \frac{df}{f - f_a} \exp\left[-\frac{(1-f)^2}{2T_d}\right] \exp\left[-\frac{(f-f_a)^2}{2T}\right] \tag{13}$$

where the dependence on Φ in the r.h.s. is contained in f_a (see Eq. (10)) and in f_s through the following obvious equation

$$\Phi = \frac{1}{\sqrt{2\pi T_d}} \int_{-\infty}^{f_s} df \exp\left[-\frac{(1-f)^2}{2T_d}\right] \tag{14}$$

Upon suitably rewriting the product of two Gaussians in Eq. (13), we obtain

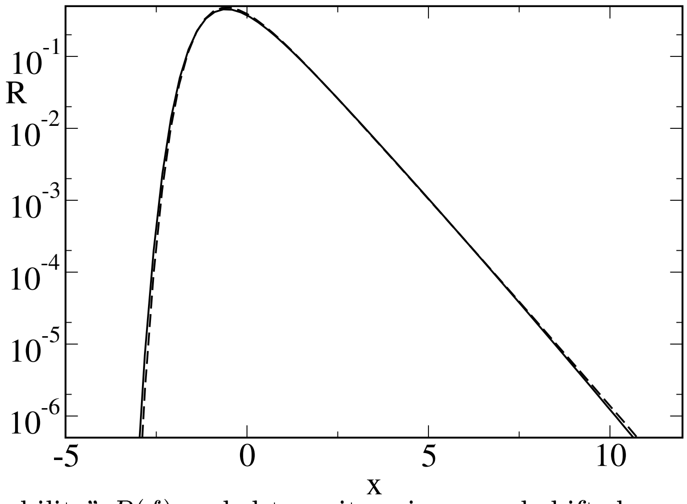

FIG. 3. The "probability" $R(f)$ scaled to unit variance and shifted around the center of mass for the same values as in the previous figure and the same notations.

$$\dot{\Phi} = \frac{\gamma}{2\pi}\sqrt{\frac{T}{T_d}}\exp\left[-\frac{(1-f_a)^2}{2(T+T_d)}\right]\int_{f_s}^{\infty}\frac{df}{f-f_a}\exp\left[-\frac{(f-f_b)^2}{2T_b}\right] \tag{15}$$

where

$$f_b = \frac{T+f_aT_d}{T+T_d} \tag{16}$$

and

$$T_b = \frac{TT_d}{T+T_d}. \tag{17}$$

Several observations are now in order. The dependence on the temperature is very different in the two exponentials. The variance in the term out of the integral is the sum of the true- and of the disorder-temperature. This is the contribution that was already singled out by Roux in Ref. [18]. The second term, instead, exhibits a dependence as if the two temperatures were in parallel. Now, it is important to establish which term is the leading one in determining the relevant time scale. As long as $f_b > f_s$, the exponential integral is of order 1 and the evolution is controlled by the first term. However, this is not what happens (at least except for the very first and last stages) in the limit of very small T. To discuss this point, we must keep in mind all the various f's that are involved in the process at a

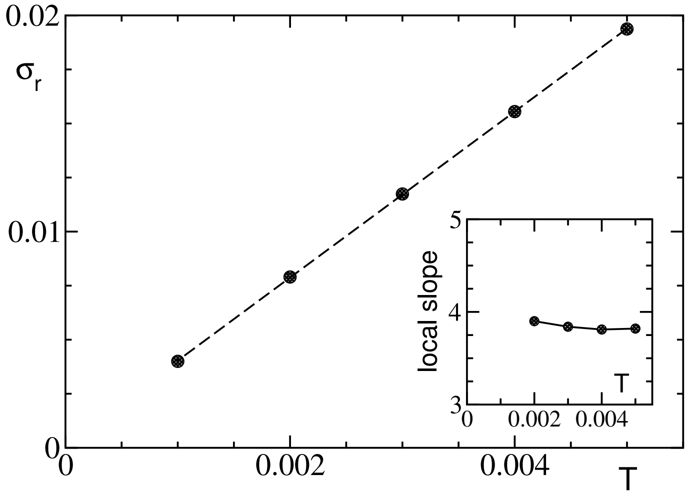

FIG. 4. The standard deviation σ_r of $R(f)$, computed when the time derivative of Φ is minimum, versus the temperature T for fixed disorder $T_d = 10^{-2}$. In the inset, we can appreciate the small deviations from a purely linear behaviour.

generic time t, starting from $f_a(t)$, the average force applied to each unbroken fiber, going to $f_s(t)$, the threshold of the weaker fiber to break, and to $f_b(t)$ the most numerous fibers to break (if still alive).

If T is very small, it is by far easier to break the few fibers whose threshold is just above the applied force f_a than the very many fibers with high threshold. This implies that in the very beginning of the fracture process, it is generated a gap between the force needed to break the weakest fibers and the average applied force, leading to a picture analogous to that for the homogeneous case. Under such conditions, $f_b < f_s$ and the integral is dominated by the amplitude of the integrand at the left extremum f_s of the integration domain. Upon computing the leading contribution to the integral in Eq. (15), we can write

$$\dot{\Phi} = \frac{\gamma\sqrt{TT_d}}{2\pi(f_s - f_a)(f_s - f_b)} \exp\left[-\frac{(1-f_a)^2}{2(T+T_d)} + \frac{(f_s - f_b)^2}{2T_b}\right]. \tag{18}$$

An upper bound to τ can be obtained by determining the maximum of

$$\tau(\Phi) = 1/\dot{\Phi}(\Phi). \tag{19}$$

Such an estimate would be exact only in the case of a constant derivative: although we have seen in Fig. 2 that this is not the case, it is nevertheless true that most of the time is spent near the minimum of the derivative, so that we can expect that the above estimate is rather accurate. With no pretense of estimating prefactors, let us pay attention only at the exponential factors in the above equation. In the small T limit, the first contribution is negligible, and thus we write

$$\ln \tau \approx \frac{(f_s^* - f_a^*)^2}{2T} \equiv \frac{U}{T} \tag{20}$$

where f_s^* and f_a^* are the f_s- and f_a-values yielding the minimum $\dot{\Phi}$. $U \equiv (f_s^* - f_a*)^2/2$ can be interpreted as the effective energy barrier to be overcome in the activation process to give rise to the final breakdown. It is instructive to notice that U is smaller than the height in the homogeneous case ($(1-f_0)^2/2$) for two reasons: (i) f_0 increases to f_a^* as a consequence of the initial "easy" ruptures that occur on short time scales; (ii) the most populated class of thresholds "f=1" decreases to f_s*, the critical force above which the process starts accelerating giving eventually rise to an avalanche. The Φ^* value corresponding to the maximum of $\tau(\Phi)$ (and, in turn, the values f_s^* and f_a^*) can be determined from the zero of the derivative of $\tau(\Phi)$. From Eq. (20), taking into account Eq. (14), one obtains

$$(1-\Phi^*)^2 = \frac{f_0}{\sqrt{2\pi T_d}} \exp\left[\frac{(1-f_s^*)^2}{2T_d}\right] . \tag{21}$$

Eq. (21), together with Eq. (14), determines the critical value Φ^* and thus the effective height U of the energy barrier. In Fig. 5 we have plotted U versus T_d. As expected, in the limit $T_d \to 0$, U converges to 1/8, the height in the absence of disorder for $f_0 = 1/2$ (the value fixed in our numerical simulations). The decrease of U with T_d confirms that the presence of disorder helps the fracture process, making it more probable. In the limit of small T_d, Φ^* tends to 0 and one can perform a perturbative calculation, obtaining

$$U = \frac{(1-f_0)^2}{2} - (1-f_0)\sqrt{T_d}\left\{\left[2\ln\left(\frac{f_0}{\sqrt{2\pi T_d}}\right)\right]^{1/2} + \left[2\ln\left(\frac{f_0}{\sqrt{2\pi T_d}}\right)\right]^{-1/2}\right\} . \tag{22}$$

The two terms contributing to the deviation from the homogeneous case arise, respectively, from the decrease of f_s^* below 1 and the increase of f_a above f_0^*. Both corrections are approximately of the same order, i.e. $\sqrt{T_d}$. It is only by looking at the logarithmic correction

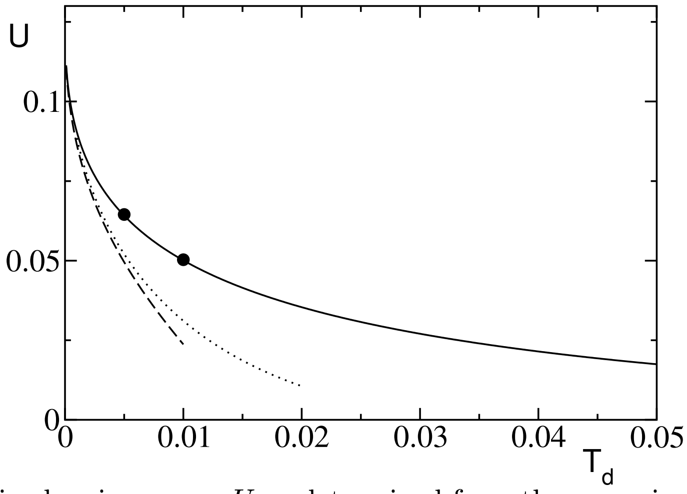

FIG. 5. The effective barrier energy U as determined from the numerical solution of the equation (21) (solid line). The two circles refer to the extrapolated value of U from direct numerical simulations.The dashed line refers to the perturbative formula (22), while the dotted line corresponds to the approximated solution (5).

that we can conclude that the former contribution is the largest one. It is presumably the presence of such corrections that makes the validity range of this perturbative calculation so small, as it can be seen by looking at the dashed line in Fig. 5. In the same figure, we have reported also the analytic solution (5) obtained in Ref. [15]: its closeness to the perturbative solution suggests that the result is rather robust against approximations made on the shape of $Q(f, t)$.

We conclude the analysis, by comparing these theoretical predictions with the outcome of numerical simulations performed both by integrating the one-dimensional Eq. (13) and the original model. In Fig. 6, we have plotted the rupture time versus $1/T$ for two different values of the disorder temperature T_d. The rather clean linear behaviour confirms the scaling behaviour expected for an activation process. In fact, it is necessary to look at the local logarithmic derivative of τ (which corresponds to U) to see deviations from linearity (see the inset) and even this analysis indicates that deviations from linearity vanish for $T \to 0$. By comparing the full circles with the solid line, we can instead appreciate the validity of the truncated-Gaussian approximation, since the circles refer to the integration of the full

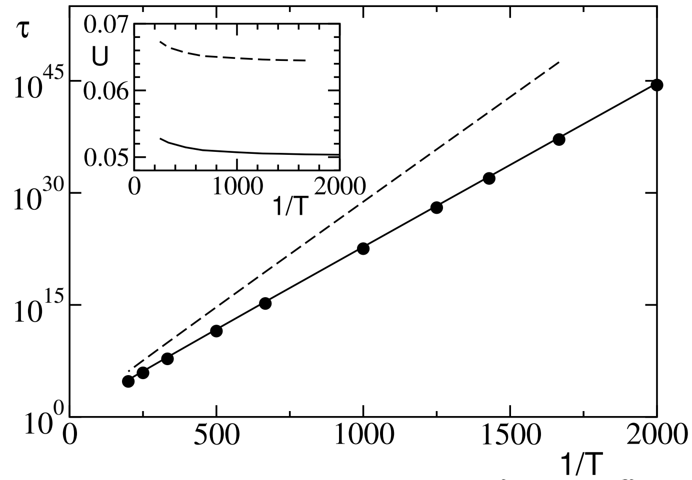

FIG. 6. The rupture time versus the inverse temperature for two different values of the disorder temperature as determined from the integration of the simplified one-dimensional equation (13): the solid and dashed lines refer to $T_d = 10^{-2}$, and $T_d = 510^{-3}$. Circles correspond to the integration of the full equation. In the inset we report the energy barrier U determined as the local logarithmic derivative of τ with respect to $1/T$

model, while the solid line arises from the one-dimensional approximation.

We are now in the position to compare the value of U, extrapolated from numerical simulations, with the theoretical prediction plotted in Fig. 5. The fact that the two circles fall precisely on top of the theoretical curve further confirm the validity of the whole approach.

IV. CONCLUSIONS AND PERSPECTIVES

The analytical treatment developed in this paper confirms the claim that the presence of disorder contributes to increasing the effective temperature of a sample subject to a constant load. Equivalently, but perhaps more physically, one can state that disorder renormalizes the barrier height to be overcome in order to give rise to a macroscopic failure of the fiber bundle. This scenario can be understood by noticing that the fracture evolves through a sequence of many irreversible processes. After the failure of the weakest fibers, the system cannot any longer come back to its initial state, while, at the same time, the energy barrier has lowered. A correct estimation of the time scale for observing the onset of a macroscopic failure is obtained by determining the time scale for the slowest of such intermediate steps.

From the way this result has been obtained, there is no reason to suspect that it follows from some peculiarity of the fiber-bundle model with quenched noise. As, indeed, suggested by experimental results, it is natural to conjecture that the presence of noise lowers the energy barrier also in more realistic set-ups. It becomes desirable now to implement more general tools to go beyond mean field models.

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