A remark on Y. Brenier's approach to Born–Infeld electro-magnetic fields

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Abstract

We solve a problem that Y. Brenier left open in his analysis [2] of the well-posedness of the Born–Infeld theory (see [1]) for electro-magnetism: Compute the convex hull of the admissible quadruplets (h, B, D, P), where h is the energy density and $P = D \times B$. We discuss the physical relevance of this set.

In [2], Y. Brenier show that the Born–Infeld model of electro-magnetism is hyperbolic symetrizable and therefore the Cauchy problem is locally well-posed within smooth enough fields. The key observation is that the system of PDEs, which consists in six equations in six scalar unknowns $(B, D) \in I\!R^3 \times I\!R^3$, may be enlarged to a system of ten conservation laws in the unknowns $(h, B, D, P) \in I\!R \times I\!R^3 \times I\!R^3 \times I\!R^3$, which admits a strictly convex entropy. See Section 2, and also a variant [6] for more general equations of state, where the enlarged system has nine equations instead.

An important role is played by the set of admissible fields

$$A := \left\{ X = (h, B, D, P)\,;\, P = D \times B \text{ and } h = \sqrt{1 + |B|^2 + |D|^2 + |P|^2} \right\},$$

together with its convex hull $\mathrm{co}(A)$. The latter might be viewed as the set of "weakly admissible" fields. In [2], a partial description of $\mathrm{co}(A)$ was given (Theorem 2.3) :

$$(1) \quad \{X\,;\, 1 + |B| + |D| + |P| \le h\} \subset \mathrm{co}(A) \subset \left\{ X\,;\, \sqrt{1 + |B|^2 + |D|^2 + |P|^2} \le h \right\}.$$

However, the exact description of $\mathrm{co}(A)$ was left open. The purpose of the present note is to fill this gap.

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1 The convex hull of A

Our first result is:

Theorem 1 *One has*

$$\mathrm{co}(A) = \left\{ (h, B, D, P) \in I\!R^{10} \,;\, \sqrt{1 + |B|^2 + |D|^2 + |P|^2 + 2|P + B \times D|} \le h \right\}.$$

Theorem 1 is a consequence of the following

Lemma 1 *Given the set*

$$\mathcal{A} := \left\{ (\tau, b, d, p) \in I\!R \times (I\!R^3)^3 \,;\, \tau p = d \times b,\ \tau > 0 \text{ and } \tau^2 + |b|^2 + |d|^2 + |p|^2 = 1 \right\},$$

one has

$$\text{(2)} \qquad \mathrm{co}(\mathcal{A}) = \left\{ ((\tau, b, d, p) \in I\!R^{10} \,;\, \tau > 0 \text{ and } \tau^2 + |b|^2 + |d|^2 + |p|^2 + 2|\tau p + b \times d| \le 1 \right\}.$$

To pass from Lemma 1 to Theorem 1, we use the well-known "Euler–Lagrange transformation"

$$(h, B, D, P) \mapsto (\tau, b, d, p) := \left(\frac{1}{h}, \frac{B}{h}, \frac{D}{h}, \frac{P}{h} \right),$$

which is known to preserve convexity on $h > 0$ (equivalently $\tau > 0$).

We now prove Lemma 1. Let us denote by $\mathcal{B}$ the set in the right-hand side of Equality (2). Being the intersection of the half-plane $\tau > 0$ and of the circular cylinders

$$\text{(3)} \qquad \tau^2 + |b|^2 + |d|^2 + |p|^2 + 2(\tau p + b \times d) \cdot q \le 1,$$

as q runs over the unit sphere $\mathcal{S}^2$, it is convex ; notice that the right-hand side in (3) is a non-negative quadratic form, which can be written, for instance, as

$$\text{(4)} \qquad Q_q(X) := |\tau q + p|^2 + (b \cdot q)^2 + |d + q \times b|^2.$$

Given a point X in $\mathcal{A}$, multiplying the identity $\tau p = d \times b$ by $2q$, and using $\|X\| = 1$, we see that $Q_q(X) = 1$ for every unit vector q. Hence $\mathcal{A} \subset \mathcal{B}$. Since $\mathcal{B}$ is convex, we deduce

$$\text{(5)} \qquad \mathrm{co}(\mathcal{A}) \subset \mathcal{B}.$$

We next prove that

$$\text{(6)} \qquad \bar{\mathrm{co}}(\mathcal{A}) = \bar{\mathcal{B}},$$

where $\bar{\mathrm{co}}(\mathcal{A})$ denotes the closed convex hull of $\mathcal{A}$. Remark that $\bar{\mathcal{B}}$ is obtained by replacing the inequality $\tau > 0$ by $\tau \ge 0$ in the formula (2). The inclusion $\bar{\mathrm{co}}(\mathcal{A}) \subset \bar{\mathcal{B}}$ proceeds exactly as above. To prove the reverse inclusion, it suffices to show that the set $\mathrm{ext}(\bar{\mathcal{B}})$ of extremal points of $\bar{\mathcal{B}}$, is contained in $\bar{\mathcal{A}}$.

Obviously, $\mathrm{ext}(\bar{\mathcal{B}})$ is contained in the boundary of $\bar{\mathcal{B}}$, meaning that an extremal point satisfies either $\tau = 0$ or

$$\tau^2 + |b|^2 + |d|^2 + |p|^2 + 2|\tau p + b \times d| = 1. \tag{7}$$

However, since an extremal point of $\bar{\mathcal{B}}$ that belongs to the tangent plane $\tau = 0$ must be extremal with respect to the face $\bar{\mathcal{B}} \cap \{\tau = 0\}$, it must belong to the relative boundary of the latter. Therefore, every extremal point of $\bar{\mathcal{B}}$ does satisfy (7).

Giving a point X that satisfies $\tau > 0$ together with (7), and assuming moreover that $\tau p + b \times d \neq 0$, we denote by q the unit vector

$$q := \frac{\tau p + b \times d}{|\tau p + b \times d|}.$$

We shall show that such an X cannot be extremal, for there exists a non-trivial segment in $\partial\bar{\mathcal{B}}$, containing X in its relative interior. This segment is parametrized by $\lambda \mapsto X(\lambda) := X + \lambda\tilde{X}$, where $\tilde{X}$ is chosen in the following way. It is a non-trivial solution $(\tilde{\tau}, \tilde{b}, \tilde{d}, \tilde{p})$ that solves the under-determined (ten scalar unknowns, nine equations) homogeneous linear system

$$\left\{\begin{array}{rcl} \tilde{\tau} q + \tilde{p} & = & 0, \\ \tilde{d} + q \times \tilde{b} & = & 0, \\ \tilde{b} \cdot q & = & 0, \\ \tau\tilde{p} + \tilde{\tau} p + \tilde{b} \times d + b \times \tilde{d} & \parallel & q. \end{array}\right.$$

Remark that the second and third equations imply

$$\tilde{b} \times \tilde{d} \parallel q.$$

One immediately checks that $\tilde{X}$ belongs to $\ker Q_q$, implying

$$Q_q(X(\lambda)) \equiv 1, \tag{8}$$

Besides, the vector $\tau(\lambda)p(\lambda) + b(\lambda) \times d(\lambda)$ remains colinear to q, since on the one hand its linear part $\tau\tilde{p} + \tilde{\tau}p + \tilde{b} \times d + b \times \tilde{d}$, and on the other hand the quadratic part $\tilde{\tau}\tilde{p} + \tilde{b} \times \tilde{d}$, are parallel to q by construction. For small values of λ, vectors $\tau(\lambda)p(\lambda) + b(\lambda) \times d(\lambda)$ and q have the same sense, and therefore (8) means that $X(\lambda)$ satifies (7). We obtain $X(\lambda) \in \bar{\mathcal{B}}$, proving that X was not extremal.

We perform a similar analysis for points X that satisfy $\tau = 0$ together with (7). Again, we assume that $\tau p + b \times d \neq 0$, meaning here that $b \times d \neq 0$. Using the same notation q, we have $q \parallel b \times d$. This implies that the linear homogeneous system

$$\tilde{b} \cdot q = 0, \quad \tilde{b} \times d + b \times (\tilde{b} \times q) \parallel q$$

is not Cramer, although it consists in three equations in three unknowns. As a matter of fact, it has the solution $\tilde{b} = b$. Hence we define $\tilde{X}$ through

$$\tilde{\tau} = 0, \quad \tilde{b} = b, \quad \tilde{d} = b \times q, \quad \tilde{p} = 0.$$

As before, we check that $X(\lambda)$ belongs to $\bar{\mathcal{B}}$ for small values of λ, and we conclude that X was not extremal.

The previous paragraphs tell that extremal points not only satisfy (7) and $\tau \geq 0$, but also fit the identity $\tau p + b \times d = 0$. This immediately implies that $\|X\|^2 = 1$. In other words, they belong to $\bar{\mathcal{A}}$. This completes the proof of (6).

At last, it is not difficult to see that $\mathrm{co}(\mathcal{A})$ is exactly the intersection of $\bar{\mathrm{co}}(\mathcal{A})$ with the open half-space $\tau > 0$. First of all, the inclusion

$$\mathrm{co}(\mathcal{A}) \subset \bar{\mathrm{co}}(\mathcal{A}) \cup \{\tau > 0\} = \mathcal{B}$$

is trivial. Next, replacing the constraint $\tau > 0$ by $\tau \geq \epsilon$, where $\epsilon > 0$, we obtain as above that

$$\mathrm{co}(\mathcal{A}_\epsilon) = \mathcal{B}_\epsilon.$$

This implies that

$$\mathcal{B} = \bigcup_{\epsilon > 0} \mathrm{co}(\mathcal{A}_\epsilon) \subset \mathrm{co}(\mathcal{A}),$$

and therefore (2) holds. This ends the proof of Lemma 1, hence that of Theorem 1.

2 Positive invariance and well-posedness

Lack of positive invariance. Although it was not explicitly raised in [2], the question of the positive invariance of $\mathrm{co}(A)$ is essential, for in systems with a physical contents, the evolution usually preserves the set of admissible fields. For instance the positivity of the mass density and of the internal energy, in gas dynamics. Here, the components of $u = (h, B, D, P)^T$ are independent of each other at the level of the "enlarged system", which reads

$$\begin{aligned} \partial_t h + \mathrm{div} P &= 0, \\ \partial_t B + \mathrm{curl} \frac{D - P \times B}{h} &= 0, \\ \partial_t D + \mathrm{curl} \frac{-B - P \times D}{h} &= 0, \\ \partial_t P + \mathrm{Div} \frac{P \otimes P - B \otimes B - D \otimes D - I_3}{h} &= 0. \end{aligned}$$

Let us write for short

$$\partial_t u + \sum_{\alpha=1}^{3} \partial_{x_\alpha} f^\alpha(u) = 0.$$

As is well-known since the fundamental paper [3], the invariance of a convex domain defined by an inequality $G(u) \leq 0$ amounts to saying that $dG(u)$ is a common left eigenvector of the matrices $\mathrm{d}f^\alpha(u)$, whenever $G(u) = 0$. We prove here that this property *does not hold* for the system above. This sheds a doubt on the physical meaning of the convex hull of A.

Because of rotational invariance, it is enough to limit our analysis to $\alpha = 1$. This amounts to studying plane waves in the direction $x_1 =: x$. For such waves, we have $\partial_t B_1 = \partial_t D_1 = 0$. Therefore, we may ignore the variables B_1 and D_1, taking them as constants. Then the enlarged system reduces to a system of eight equations in the variable $u := (h, B_2, B_3, D_2, D_3, P)^T$. The flux of this system is

$$f^1(u) = \frac{1}{h} \begin{pmatrix} hP_1 \\ -D_3 + B_2 P_1 - B_1 P_2 \\ D_2 + B_3 P_1 - B_1 P_3 \\ B_3 + D_2 P_1 - D_1 P_2 \\ -B_2 + D_3 P_1 - D_1 P_3 \\ P_1 P - B_1 B - D_1 D - \vec{e}^1 \end{pmatrix}.$$

The set $\mathrm{co}(A)$ is defined by $G \leq 0$, where

$$G(u) := \frac{1}{2} \max\{-h, 1 + |B|^2 + |D|^2 + |P|^2 + 2|P - D \times B| - h^2\}.$$

We ask therefore whether $\mathrm{d}G(u)$ is an eigenform for $\mathrm{d}f^1(u)$ when $G(u) = 0$. The spectrum of $\mathrm{d}f^1(u)$ has been computed in [2] ; its eigenvalues are

$$\lambda_- = \frac{P_1 - Z}{h}, \quad \lambda_0 = \frac{P_1}{h}, \quad \lambda_+ = \frac{P_1 + Z}{h},$$

with

$$Z := \sqrt{1 + B_1^2 + D_1^2}.$$

Hence there remains to check whether the expression $\mathrm{d}G(u)(\mathrm{d}f^1(u) - \lambda_\epsilon(u))$ vanishes or not as $G(u) = 0$, with ϵ one of the signs $0, \pm$. Componentwise, the have to check the equalities

$$\mathrm{d}G(u) \frac{\partial f^1}{\partial u_l}(u) = \lambda_\epsilon(u) \frac{\partial G}{\partial u_l}(u) \quad ?$$

Since f^1 is linear with respect to each u_l for $l \geq 2$, the calculations of such expression is fairly easy once we know explicitly $\mathrm{d}G(u)$. Since h cannot vanish, we have

$$\mathrm{d}G(u) = (-h, (B + D \times q)_2, (B + D \times q)_3, (D - B \times q)_2, (D - B \times q)_3, P + q),$$

where

$$q := \frac{P - D \times B}{|P - D \times B|}.$$

Let us take for instance $l = 2$. Then

$$\frac{\partial f^1}{\partial B_2} = \frac{1}{h} \begin{pmatrix} 0 \\ P_1 \\ 0 \\ 0 \\ -1 \\ -B_1 \vec{e}^2 \end{pmatrix}.$$

Thus

$$h\left(\mathrm{d}G(u)\frac{\partial f^1}{\partial B_2} - \lambda_\epsilon(u)\frac{\partial G}{\partial B_2}\right) = -\epsilon Z\frac{\partial G}{\partial B_2} + (B\times q - D)_3 + B_1(P_2+q_2).$$

Remark that, given the triplet (B, D, P), we always may fix the value of h so that G vanishes, without changing the value of the right-hand side above. Hence the constraint $G(u) = 0$ is harmless, and we ask whether the expression

$$J_\epsilon := -\epsilon Z(B + D\times q)_2 + (B\times q - D)_3 + B_1(P_2+q_2)$$

vanishes identically or not as (B, D, P) runs over $I\!R^9$. The answer is obviously *no*, as J_ϵ behaves like B_1P_2 when P_2 tends to infinity.

Ill-posedness. The lack of positive invariance yields a serious weakness of the Cauchy problem for the Born–Infeld model. As a matter of fact, if we consider bounded initial data (B^0, D^0) for Born–Infeld, with arbitrarily large total variation, we may as well consider weak-star limits in the Young sense. This amounts to considering arbitrary data $u^0 := (h^0, B^0, D^0, P^0)$ for the enlarged Brenier's system, with the restriction that they take values in $\mathrm{co}(A)$. If the Young limit of bounded sequences of solutions is the solution associated with the Young limit of initial data (a strong kind of well-posedness that we shall call *weak-star well-posedness*), then the solution u of Brenier's system would stay in $\mathrm{co}(A)$. Since we showed above that this is not the case in general, this means either that the solutions of Born–Infeld model do not remain bounded uniformly in terms of $\|B^0, D^0\|_\infty$ (for instance, they may blow-up arbitrarily soon), or that Brenier's enlarged system is not weakly-star well-posed. The latter fact seems in contradiction with the situation for planar waves, as described[1] in Proposition 4.3 of [2]. Since the lack of positive invariance is already present at this level, we anticipate that the Cauchy problem for Born–Infeld suffers from a lack of global existence. This is confirmed by a direct study of smooth and piecewise smooth solution by W. Neves and the author [5]. Notice however that for small and smooth initial data, the Cauchy problem admits a unique global smooth solution [4]. Likewise, the Riemann problem with small data is solvable, since it falls in the category studied by Lax. Therefore the lack of global existence should presumably concern large data.

More general models. In [6], more general constitutive laws were studied, where the Maxwell system reads

$$\partial_t B + \mathrm{curl}\frac{\partial W}{\partial D} = 0, \quad \partial_t D - \mathrm{curl}\frac{\partial W}{\partial B} = 0,$$

and $W : I\!R^3 \times I\!R^3 \to I\!R$ is the given energy density. The Born–Infeld model corresponds to the special case

$$W_{BI}(B, D) = \sqrt{1 + |B|^2 + |D|^2 + |P|^2}.$$

[1]However, weak stability might be lost when the inequalities in the assumptions of Proposition 4.3 are violated.

The enlarged system then consisted of nine scalar evolution equations, in the nine unknowns (B, D, P), where P stands for the relaxation of the expression $D \times B$. The same problems arise in this context: Describe the convex hull of the set A' of admissible fields $(B, D, P : D \times B)$; check whether this convex hull is positively invariant or not.

There is no need to make a new analysis to compute $\mathrm{co}(A')$, as it is nothing but the projection on $\mathbb{R}^9$ of $\mathrm{co}(A)$. Therefore we have the trivial result

$$\mathrm{co}(A') = \mathbb{R}^9.$$

In particular, the positive invariance holds true in this situation.

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