Dissipative conservation laws; shock front stability

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Abstract

Dissipative conservation laws are PDEs of the form
\[ \partial_t u + \text{div} f(u) = \Delta u. \] (1)

Hereabove, the unknown \( u(x, t) \in \mathbb{R} \) depends on a space and a time variables \( x, t \).

The differential operators \( \text{div} \) and \( \Delta \) act only over the \( x \) variable. The function \( f \), called the flux is given and represents the non-linearity of the physical problem modelled by (1). We shall suppose that \( f \) is smooth enough, for instance \( C^\infty \) (way too strong). The Cauchy problem arises when \( x \) runs over the whole space \( \mathbb{R}^d \) and an initial data \( a \) is given :
\[ u(x, 0) = a(x), \quad x \in \mathbb{R}^d. \] (2)

The solution is then searched for \( (x, t) \in \mathbb{R}^d \times (0, +\infty) \).

Shock fronts are travelling waves, that is solutions of the form
\[ u(x, t) = U(x \cdot \nu - ct) \]
where \( \nu \) is some unit vector (the direction of propagation) and \( c \) a constant (the wave velocity).

Working in a coordinate frame that is travelling with the front (Galilean frame), the equation (1) is replaced by an equation of the same form, with the new flux
\[ \tilde{f}(u) = f(u) - cu. \]

This has the advantage of making the front stationary \( (c = 0) \).

Given a stationary shock front \( U(x \cdot \nu) \), a natural question is its stability\(^1\) : given an initial data \( a(x) \) that is a localized disturbance of \( U \), is it true that the corresponding solution of the Cauchy problem converges towards \( U \) as \( t \to +\infty \) ?

The present course covers the following aspects :

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- The well-posedness of the Cauchy problem for (1). Because this involves properties of the heat kernel, we begin by investigating the special case of the heat equation. We prove that the semi-group generated by (1) satisfies the conservation of mass, the maximum principle and is \( L^1 \)-contracting.

- We prove that constants states, which are trivial solutions, are asymptotically \( L^1 \)-stable with respect to zero-mass initial disturbances. That the mass of the disturbance is zero is also a necessary condition. What is remarquable is the absence of a size restriction. The counterpart is the lack of decay rate.

- We prove essentially the same kind of reult for shock fronts. Although the proof of this result involves the previous one, we need an additional tool, taken from dynamical systems.

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\(^1\) This question is related to that of the observability of the front.
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### Notations

- The balls in $\mathbb{R}^d$ are denoted $B(x_0; r)$. If $x_0 = 0$, we just write $B_r$.
- If $p \geq 1$, the norm in $L^p$ is denoted $\| \cdot \|_p$. That in the Sobolev space $W^{s,p}$ is denoted $\| \cdot \|_{s,p}$. The conjugate exponent $p'$ is defined by $\frac{1}{p'} + \frac{1}{p} = 1$.
- If $u(x)$ is a function over an open domain of $\mathbb{R}^d$, we denote $\nabla u$ its gradient and $D^2 u$ its Hessian, that is the symmetric matrix of second order derivatives.
- Inequalities involves constants, notably norms of bounded linear operators. Such constants are denoted $c$, sometimes with subscripts. In different parts of the course, several constants denoted the same way have different meanings. It is understood that when estimating a solution, or an approximate solution, constants don’t depend upon the functions under consideration.
- If $u = u(x,t)$ is a function of the space and time variables, we denote $u(t)$ its restriction at time $t : u(t) = u(\cdot, t)$. It is a function of the space variable only.
- We denote $P_T$ the domain $\mathbb{R}^d \times (0, T)$ and $P = P_\infty$. 
Chapitre 1

The Cauchy problem for the heat equation

A simplistic and fundamental example of DCL is the heat equation

\[
\frac{\partial u}{\partial t} = \Delta u.
\]  

(1.1)

It is a linear equation, whose Cauchy problem can be treated by several approaches, including semi-group theory and Fourier analysis.

1.1 The heat kernel

We define a function

\[
K(x, t) = (2\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{4t}\right), \quad (x, t) \in P
\]

and we denote \(K_t = K(\cdot, t)\). We verify easily the following facts

\[
\frac{\partial}{\partial t} K = \Delta K, \quad K > 0, \quad \int_{\mathbb{R}^d} K_t(x) \, dx = 1
\]

and

\[
\forall \epsilon > 0, \quad \lim_{t \to 0^+} \int_{|x| > \epsilon} K_t(x) \, dx = 0.
\]

If \(a \in L^p(\mathbb{R}^d)\), we define a family of operators \((H_t)_{t>0}\) by convolution:

\[
H_t a = K_t * a.
\]
CHAPITRE 1. THE CAUCHY PROBLEM FOR THE HEAT EQUATION

Since $K_t$ is integrable, the Young inequalities tell us that $H_t \in L(L^p)$ for every $p \in [1, \infty]$, with

$$\|H_t\|_{L(L^p)} \leq \|K_t\|_1 = 1.$$ 

Because $\partial(k*a) = (\partial k)*a$ for every derivation, the function $u(x,t) := (H_t a)(x)$ satisfies

$$(\partial_t - \Delta)u = (\partial_t K - \Delta K) * a = 0 * a = 0,$$

thus is a solution of the heat equation. If in addition $a \in C(\mathbb{R}^d)$, then $u(x,t) \to a(x)$ as $t \to 0+$, locally uniformly. If $a \in L^p(\mathbb{R}^d)$ with $p$ finite, then $\|u(t) - a\|_p \to 0$. Therefore $u$ is a solution of the Cauchy problem. It turns out that it is the unique solution of the Cauchy problem in the appropriate class. For instance:

**Theorem 1.1** If $p \in [1, \infty)$ and $a \in L^p(\mathbb{R}^d)$, the function $u = K *_x a$ is the unique solution of the Cauchy problem for the heat equation in the class $C([0, +\infty); L^p)$.

When $a \in L^\infty$, the initial condition (2) is satisfied in a weaker sense, that is $u(t) \to a$ in the weak*-topology of $L^\infty$; but the solution is still unique.

The operators $(H_t)_{t>0}$, together with $H_0 := \text{id}$, form a strongly continuous semigroup over $L^p$ ($p$ finite), meaning that $H_s \circ H_t = H_{s+t}$ and $t \mapsto H_t a$ is continuous in the norm topology.

1.1.1 The Duhamel formula

TODO

1.2 Regularizing effect

Let $0 < \epsilon < T < \infty$ be given. The heat kernel $K_t$ is a Gaussian, whose space and times derivatives $\partial_t^\ell \nabla^m_x K$ are of the form $R_{\ell,m}(x,t)K_t$ with $R_{\ell,m}$ a polynomial in $x$ and a rational function in $t$ with pole only at $t = 0$. These functions are integrable in $x$, uniformly over $t \in (\epsilon, T)$. Therefore, the Lebesgue theorem tells us that $u = K *_x a$ is a $C^\infty$-function.

We point out that the regularity does not extend to $t = 0$, since otherwise $a$ itself would be a smooth function. Instead, norms of derivatives blow up as $t \to 0+$. For instance, because

$$\partial_t^\ell \nabla^m_x K(x,t) = t^{-\frac{d+m}{2} - \ell} k_{\ell,m}(\frac{x}{\sqrt{t}}),$$
1.3. THE MAXIMUM PRINCIPLE

we have

\[ \| \partial_t \nabla_x^m K (\cdot, t) \|_1 = c_{\ell, m} t^{-\ell - \frac{m}{2}}. \]

This implies the estimate for the solution of the Cauchy problem :

\[ \| \partial_t \nabla_x^m u(t) \|_p \leq c_{\ell, m} t^{-\ell - \frac{m}{2}} \| a \|_p. \]  \hspace{1cm} (1.2)

1.2.1 Dispersive estimates

Let \( 1 \leq p < q \leq +\infty \) be given. Applying all the Young inequality

\[ \| u(t) \|_q \leq \| K_t \|_r \| a \|_p, \quad 1 + \frac{1}{r} = \frac{1}{q} + \frac{1}{p}, \]

and using the straighforward identity

\[ \| K_t \|_r = c_r t^{-d/2r'} \]

where \( c_1 = 1 \) and \( c_\infty = (2\pi t)^{-d/2} \) (this interpolates between the \( L^1 \) and \( L^\infty \) cases), we deduce that

\[ \| u(t) \|_q \leq c_r t^{\frac{d}{2} \left( \frac{1}{q} - \frac{1}{p} \right)} \| a \|_p, \quad q > p, \quad t > 0. \]  \hspace{1cm} (1.3)

1.3 The maximum principle

The maximum principle applies to more general linear parabolic PDEs :

\[ \partial_t u + b(x, t) \cdot \nabla u + h(x, t) u = \text{Tr}(A(x, t)D^2 u), \]  \hspace{1cm} (1.4)

where the symmetric matrix \( A(x, t) \) is positive definite for every \( (x, t) \). The heat equation is the special case where \( b \equiv 0, h \equiv 0 \) and \( A \equiv I_d \).

We shall not develop the theory of the Cauchy problem for the general parabolic equation. It will be enough for our purpose to mention that under reasonable boundedness assumptions about the coefficients \( b, h, A \) and some of their derivatives, every solution in some open domain \( D \) is also regular, and even a little more than the coefficients. We shall consider here classical solutions, which have continuous derivatives \( \partial_t u, \nabla u, D^2 u \).

1.3.1 Bounded domains

Consider such a classical solution \( u \) in a domain \( Q_T = D \times (0, T) \) where \( D \) is a bounded domain, and assume that \( u \) is continuous on the closure of \( Q_T \). Because \( \overline{Q_T} \) is compact, \( u \) achieves its supremum at some point \( (x_0, t_0) \). Suppose that \( x_0 \in D \)
and \( t_0 > 0 \), then necessarily \( \nabla u = 0 \), \( \partial_t u \geq 0 \) and \( D^2 u \leq 0 \) at \((x_0, t_0)\). Using the equation (1.4), we deduce the inequality \((hu)(x_0, t_0) \leq 0\). So far, this is not impressive, but suppose know that \( u(\cdot, 0) \) was \(< 0\) over \( \partial D \times [0, T] \). If \( \sup_x u(x, T) \geq 0 \), then we may chose \( T \) as the smallest time at which \( \sup_x u(x, T) = 0 \). Then, in the analysis above, we have \( t_0 = T \) and \( u(x_0, t_0) = 0 \). We therefore obtain the series of inequalities

\[
0 = (hu)(x_0, t_0) = (-\partial_t u - b(x, t) \cdot \nabla u + \text{Tr}(A(x, t)D^2 u))(x_0, t_0) \leq 0,
\]

which implies

\[
\partial_t u(x_0, t_0) = 0, \quad D^2 u(x_0, t_0) = 0_d.
\]

The fact that the PDE implies that the solution must be flat at its maximum (here, when this maximum is non-negative) is always the diagnosis of a maximum principle. A full proof of the MP uses correctors and is rather technical, so we shall admit it:

**Theorem 1.2** Let \( D \) be a bounded domain, \( T \) be positive, and let \( u \) be a classical solution of (1.4). Assume that \( u \leq 0 \) on the bottom and lateral boundaries of \( Q_T \), that is when \( t = 0 \) and when \( x \in \partial D \). Then \( u \leq 0 \) in \( Q_T \).

Remark that we did not use the whole equality (1.4), we infer that the theorem above is valid if we replace it by the differential inequality

\[
\partial_t u + b(x, t) \cdot \nabla u + h(x, t)u \leq \text{Tr}(A(x, t)D^2 u).
\]

For instance, we deduce the following

**Corollary 1.1** In the theorem above, assume instead that \( h \geq 0 \). Let \( m \geq 0 \) be a constant. If \( u \leq m \) on the bottom and lateral boundaries, then \( u \leq m \).

Proof : Apply the theorem to \( u - m \), which satisfies the differential inequality.

### 1.3.2 The case \( D = \mathbb{R}^d \)

When \( D = \mathbb{R}^d \), we lack compactness to argue that \( u \) achieves its maximum. Instead, we may apply Theorem 1.2 to bounded subdomains. If \( u \) doesn’t grow too fast at infinity, we may conclude again. But for this, we have to play with a modified function \( \tilde{u} \) and write the equation that it satisfies. A typical result is the following.

**Theorem 1.3** Let \( u \) be a classical solution of (1.4) in \((0, T) \times \mathbb{R}^d \), continuous and bounded over \( P_T \). Assume that \( b, h \) are bounded.

If \( u(\cdot, 0) \leq 0 \), then \( u \leq 0 \).
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Proof
Let \( \mu, \beta > 0 \) be parameters, to be chosen later. Let us form a function

\[
v(x,t) = u(x,t) \exp \left( -\mu \frac{|x|^2}{T - t} - \beta t \right).
\]

Some calculations yield the following PDE for \( v \):

\[
\partial_t v + \tilde{b} \cdot \nabla v + \tilde{h} v = \Delta v,
\]

where

\[
\tilde{h} = \beta + h + 2\mu \frac{x \cdot b + d}{T - t} + \mu(1 - 4\mu) \frac{|x|^2}{(T - t)^2} \geq \beta + h - |b|^2 + \frac{2\mu d}{T - t} + \mu(1 - 5\mu) \frac{|x|^2}{(T - t)^2}.
\]

We chose \( \mu \in (0, \frac{1}{5}) \) and restrict to the slab \( \mathbb{R}^d \times (0, \frac{T}{2}) \). We have

\[
\tilde{h} \geq \beta + h - |b|^2 - \frac{4\mu d}{T}.
\]

Chosing \( \beta \) large enough, we have \( \tilde{h} \geq 0 \).

We now apply Corollary to \( v \) in the domain \( B_R \times (0, \frac{T}{2}) \) where \( R > 0 \) is given. Denoting \( M \) the supremum of \( u \), the upper bound of \( v \) on the bottom and lateral boundaries is at most \( M \exp(-2\mu R^2/T) \). We deduce that on this domain, we have

\[
v(x,t) \leq M \exp(-2\mu R^2/T).
\]

Keeping \( x \) fixed and letting \( R \to +\infty \), we obtain \( v(x,t) \leq 0 \), which is the same as \( u \leq 0 \). This is true for \( 0 < t < \frac{T}{2} \), but we may play the same game on the next slab. Therefore \( u \leq 0 \) is valid in the whole \( \mathbb{R}^d \times (0, T) \).

This statement is useful in proving the uniqueness of the solution of the Cauchy problem. From linearity, it is enough to prove that if \( u(\cdot, 0) \equiv 0 \), then \( u \equiv 0 \). This is done by applying Theorem 1.3 to \( u \) and to \( -u \). Of course, we need to know that \( u \) is a classical solution, but this true in general for parabolic equations. At least, it is not too difficult to prove that such smooth solutions exist. Therefore, we obtain existence and uniqueness in the same class.
CHAPTER 1. THE CAUCHY PROBLEM FOR THE HEAT EQUATION
Chapitre 2

Dissipative conservation laws

We turn now to the study of the Cauchy problem for (1). Again, we denote by $a$ the initial data, which is taken in $L^\infty$. We look for bounded solutions.

Because constants are killed by derivatives, we may assume (and we do!) that $f(0) = 0$.

2.1 Mild solutions

In order to prove the local-in-time existence, we pretend that the non-linear term is a source term :

$$(\partial_t - \Delta) u = -\text{div} f(u).$$

This suggests to rewrite the Cauchy problem in an integral form, which involves the Duhamel Formula :

$$u(t) = H_ta - \int_0^t H_{t-s}[\text{div} f(u(s))] ds \quad t > 0.$$ 

Using the fact that derivatives commute with the convolution kernel, this amounts to writing a fixed point equation

$$u = N[u],$$

where $N$ is defined by

$$N[v](t) = H_ta - \int_0^t (\nabla K_{t-s}) \ast (f \circ v(s)) ds.$$ 

Hereabove, the symbol $\ast$ stands for the scalar product of convolution products :

$$g \ast h = \sum_{\alpha} g_{\alpha} \ast h_{\alpha}.$$
Definition 2.1 A mild solution of the Cauchy problem (1,2) is a function $u \in L^\infty(P_T)$ satisfying the fixed point equation $N[u] = u$.

Of course, we have to verify that the integral in the definition of $N[v]$ makes sense. This will be done in the next section. Actually, we shall deal only with functions that are smooth for $t > 0$, so that the integral can be understood in the sense of Riemann.

2.2 Local existence of $L^\infty$-solutions

If $R > 0$, we denote $L_R$ the Lipschitz constant of $f$ over the interval $[-R,R]$:

$$L_R = \sup_{|z| \leq R} |f'(z)|.$$

Suppose $v \in L^\infty(P_T)$ with $\| v \|_\infty \leq R$. Because of $f(0) = 0$, we have $\| f(v) \|_\infty \leq L_R \| v \|_\infty$. Applying the regularizing estimate for $H_t$, we find

$$\| N[v](t) \|_\infty \leq \| a \|_\infty + \int_0^t \| \nabla K_{t-s} \|_1 \| f \circ v(s) \|_\infty ds$$

$$\leq \| a \|_\infty + c L_R \| v \|_\infty \int_0^t \frac{ds}{\sqrt{t-s}}$$

$$\leq \| a \|_\infty + 2 c L_R R \sqrt{t}.$$

We deduce that

$$\| N[v] \|_\infty \leq \| a \|_\infty + 2 c L_R R \sqrt{T}. \quad (2.1)$$

Invariant ball : Let us chose $R > \| a \|_\infty$. Then fix $T > 0$ such that

$$T < \left( \frac{R - \| a \|_\infty}{2 c L_R R} \right)^2.$$

Then the estimate (2.1) tells us that

$$(\| v \|_\infty \leq R) \implies (\| N[v] \|_\infty \leq R).$$

In other words, the ball $B_R$ in $L^\infty(P_T)$ is invariant under $N$.

Contraction : We now examine whether $N$ is a contraction in this ball. If $v, w \in B_R$, we write

$$N[v] - N[w] = \int_0^t (\nabla K_{t-s})^\dagger (f \circ w(s) - f \circ v(s)) ds. \quad (2.2)$$
2.3. REGULARITY

Arguing as above, we have
\[ \|N[v] - N[w]\|_\infty \leq 2cL_R\sqrt{T}\|v - w\|_\infty. \]

With the same choice of \( T \) as above, we know that \( 2cL_R\sqrt{T} < 1 \) and therefore \( N \) is a contraction in \( B_R \).

Thanks to Picard’s Theorem, we know that the mild solution \( u \) exists and is unique in the slab \( P_T \). In addition, \( u \) is the limit of approximate solutions \( u^m \) defined inductively by \( u^0 = 0 \) and then \( u^m = N[u^{m-1}] \).

Let us remark that the existence is proved on a time interval \((0, T)\) that depends only upon \( \|a\|_\infty \) (chose \( R = 2\|a\|_\infty \)).

2.3 Regularity

We prove here that the solution constructed above is actually classical, and even of class \( C^\infty \) for \( t > 0 \) is \( f \in C^\infty \). Namely, we obtain the same result as for the heat equation.

Let us first write
\[ \nabla N[v](t) = \nabla H_t - \int_0^t (\nabla K_{t-s}) \ast (\nabla (f \circ v(s))) \, ds \]

and remark that
\[ |\nabla (f \circ v(s))| \leq L_R |\nabla v(s)|. \]

Suppose that \( \sup_{s<T} \sqrt{s} \|v(s)\|_\infty \leq M \). Then
\[ \|\nabla N[v](t)\|_\infty \leq c\|a\|_\infty t^{-1/2} + cML_R \int_0^t \frac{ds}{\sqrt{s(t-s)}} = c\|a\|_\infty t^{-1/2} + 2c\pi ML_R. \]

Let us refine the choice of \( T \) by choosing it such that
\[ T < \frac{1}{(2c\pi L_R)^2}. \] (2.3)

Then there exists \( M > 0 \) such that
\[ c\|a\|_\infty t^{-1/2} + 2c\pi ML_R \leq Mt^{-1/2}, \quad \forall t \in (0, T). \]

This ensures that on the corresponding slab \( P_T \), all the iterates \( u^m \) satisfy the same estimates
\[ \sup_{0<t<\sqrt{T}} \sqrt{t} \|\nabla u^m\|_\infty \leq M, \]
and therefore the limit \( u \) belongs to \( L^\infty(0, T; W^{1,\infty}) \) with the same bound.

We remark again that the new \( T \) depends only upon \( \|a\|_\infty \).

To go further, we may start from an arbitrary time \( \tau \in (0, T) \), for which we know that \( u(t) \in W^{1,\infty} \). We write that our solution satisfies also (semi-group property of mild solutions)

\[
u(t) = H_{t-\tau} u(\tau) - \int_\tau^t (\nabla K_{t-s})^* (f(u(s))) \, ds.
\]

On the one hand, we have

\[
\|\nabla^2 H_{t-\tau} u(\tau)\|_\infty = \|\nabla H_{t-\tau} \nabla u(\tau)\|_\infty \leq \frac{c}{\sqrt{t-\tau}} \|\nabla u(\tau)\|_\infty.
\]

On the other hand

\[
\nabla^2 (f \circ v) = f'(v) \nabla^2 v + f''(v) \nabla v \otimes \nabla v.
\]

Therefore, if \( v = u^m \), we have

\[
\|\nabla v(t)\|_\infty \leq L_R \|\nabla^2 v\|_\infty + \frac{c_R}{t}.
\]

This yields the estimate

\[
\|\nabla^2 N[v](t)\|_\infty \leq \frac{c}{\sqrt{t-\tau}} \|\nabla u(\tau)\|_\infty + c \int_\tau^t \frac{ds}{\sqrt{t-s}} (L_R \|\nabla^2 v(s)\|_\infty + \frac{c_R}{s}).
\]

We guess now an estimate of the form

\[
\sup_{\tau < t < \tau} \sqrt{t-\tau} \|\nabla^2 u^m(t)\| \leq M_\tau < \infty.
\]

For, if \( v \) satisfies it, then we have

\[
\sqrt{t-\tau} \|\nabla^2 N[v](t)\|_\infty \leq c \|\nabla u(\tau)\|_\infty + 2c\pi L_R \sqrt{T} M_\tau + c c_R \sqrt{T} \log \frac{T}{\tau}.
\]

Once again, we may chose \( M_\tau \) large enough that the right-hand side is less than \( M_\tau \), because of (2.3). With this \( M_\tau \), we get inductively the claim, and passing to the limit, we obtain

\[
\sup_{\tau < t < \tau} \sqrt{t-\tau} \|\nabla^2 u(t)\| \leq M_\tau.
\]

Because \( \tau > 0 \) is arbitrary, we find that \( u \in L^\infty_{loc}(0, T; W^{2,\infty}) \).

A similar analysis can be made for every space derivative, with the same time \( T \) given by (2.3). We obtain that for \( t \in (\tau, T) \) (\( \tau > 0 \)), every derivative \( \nabla^m u \) is a bounded function. Then, because of the equation itself, we get the same property for \( \partial_t u \), and successively for every \( \partial_t^\ell \nabla^m u \). Finally, we have proved that \( u \in C^\infty(P_T) \).
2.4 Local is global

So far, we have proved that if $a \in L^\infty$, then there exists a $T = T(\|a\|_\infty)$, given by (2.3), such that the Cauchy problem admits a unique mild solution in the slab $P_T$, which is $C^\infty$ for $t > 0$. We point out that the map $\rho \mapsto T(\rho)$ is non-increasing.

Because of the regularity, we may rewrite the equation as

$$(\partial_t + f'(u) \cdot \nabla - \Delta)u = 0$$

and apply the maximum principle. Let $m_+$ be the essential supremum of $a$. Then $u - m_+$ satisfies the same equation

$$(\partial_t + f'(u) \cdot \nabla - \Delta)(u - m_+) = 0$$

and is $\leq 0$ at initial time. We deduce from Theorem 1.3 that $u - m_+ \leq 0$. The same argument works for $m_- - u$ where $m_-$ is the infimum of $a$. In other words, we have

$$\inf a \leq u(x,t) \leq \sup a, \quad (x,t) \in P_T.$$

We may now solve the Cauchy problem with initial time $\tau \in (0,T)$ and initial data $\tilde{a} = u(\tau)$. We know that it admits a unique mild solution $\tilde{u}$ over a time interval $(\tau, \tau + \tilde{T})$, where $\tilde{T} = T(\|u(\tau)\|_\infty)$. Because of $\|u(\tau)\|_\infty \leq \|a\|_\infty$, we have $\tilde{T} \geq T$. By uniqueness, $\tilde{u}$ coincides with $u$ over $(\tau,T)$, and the concatenation of $u$ and $\tilde{u}$ provides a mild solution over $(0, \tau + T)$. Finally, we have a mild solution over $(0,2T)$.

Repeating the argument above, we obtain a mild solution over the time interval $(0,kT)$ for every positive integer $k$. The mild solution is therefore defined over $P = \mathbb{R}^d \times (0, +\infty)$.

**Theorem 2.1** Let the initial data $a$ be given in $L^\infty$. Then the mild solution of (1,2) exists and is unique for all $t > 0$. It is $C^\infty$ for $t > 0$.

The map $t \mapsto \sup_x u(x,t)$ is non-increasing. The map $t \mapsto \inf_x u(x,t)$ is non-decreasing.

**The non-linear semi-group**

The map $S_t : a \mapsto u(t)$ is non-linear because the equation itself is not linear. However, it has the semi-group property

$$S_t \circ S_s = S_{t+s}.$$
2.5 The difference of two solutions

In Section 2.2, we estimated the difference $N[v] - N[w]$ in the $L^\infty$-norm. We know play the same game in the $L^1$-norm. For this, we still assume that $\|v\|_\infty, \|w\|_\infty \leq R$, but we also suppose that $v - w \in L^\infty(0, T; L^1(\mathbb{R}^d))$. This kind of assumption turns out to be essential in the sequel: neither $v(t)$ nor $w(t)$ will be integrable but their difference will be.

We start again from (2.2). Because $\|f(w(s)) - f(v(s))\|_1 \leq L_R \|w(s) - v(s)\|_1$, we have

$$\|N[v](t) - N[w](t)\|_1 \leq \|v(0) - w(0)\|_1 + 2cL_R \sqrt{T} \sup_s \|v(s) - w(s)\|_1.$$ 

We now chose a time $T$ such that $\rho := 2cL_R \sqrt{T} < 1$.

Let $a_1, a_2$ be two bounded initial data and denote $u_1, u_2$ the mild solutions of the corresponding Cauchy problems. Applying the above inequality to the iterates $u^m_j$, we obtain inductively

$$\sup_{0 \leq t \leq T} \|u^m_1 - u^m_2\|_1 \leq \frac{1}{1 - \rho} \|a_1 - a_2\|_1.$$ 

Passing to the limit as $m \to +\infty$ and using Fatou’s Lemma, we get

$$\sup_{0 \leq t \leq T} \|u_1 - u_2\|_1 \leq \frac{1}{1 - \rho} \|a_1 - a_2\|_1. \quad (2.4)$$

In particular, $u_1(t) - u_2(t) \in L^1$ for every $t \in (0, T)$.

Because $T$ depends only upon $\|a\|_\infty$, and because $u$ is bounded for $t > 0$, we may apply the argument above from an initial time $\tau$ and to initial data $u_j(\tau)$, to get that $u_1(t) - u_2(t) \in L^1$ for $t \in (\tau, \tau + T)$ with the same $T$ as above. Induction gives immediately the same property for every $t > 0$.

Likewise, we obtain an estimate for the $L^1$-norm of $\nabla(u_1 - u_2)$:

$$\sup_{0 < t < T} \sqrt{T} \|u_1(t) - u_2(t)\|_1 \leq c\|a_1 - a_2\|_1. \quad (2.5)$$

**Remark 2.1** The same analysis works with $L^p$ instead of $L^1$. We find that if $a_1 - a_2 \in L^p$, then $u_1(t) - u_2(t) \in L^p$ for every $t > 0$. Actually, we have $u_1 - u_2 \in C([0, +\infty); L^p)$ if $1 \leq p < \infty$, because the heat semi-group is strongly continuous over $L^p$. This continuity fails if $p = \infty$.

**Remark 2.2** Taking $a_2 \equiv 0$, we find that if $a \in L^1 \cap L^\infty$, then the mild solution of the Cauchy problem takes values in $L^1 \cap L^\infty$ at every $t \geq 0$. 
2.5. THE DIFFERENCE OF TWO SOLUTIONS

2.5.1 Mass conservation

Let $\phi \in \mathcal{D}(\mathbb{R}^d)$ be non-negative, with support in $B_2$, such that $\phi \equiv 1$ over $B_1$. If $r > 0$, we define $\phi_r(x) = \phi(x/r)$.

We continue with two mild solutions $u_j$ with initial data $a_j$ such that $a_1 - a_2 \in L^1$. We define $w = u_1 - u_2$, which satisfies $w \in C_t(L^1)$ and

$$\partial_t w = \Delta w + \text{div}(f(u_2) - f(u_1)).$$

Because the $u_j$’s are smooth, we may write

$$\partial_t (\phi_r w) = \phi_r (\Delta w + \text{div}(f(u_2) - f(u_1)))$$

$$= \text{div}\{\phi_r (\nabla w + f(u_2) - f(u_1))\} - \nabla \phi_r \cdot (\nabla w + f(u_2) - f(u_1)).$$

Integrating over $\mathbb{R}^d$ (mind that every product has compact support), we obtain

$$\frac{d}{dt} \int \phi_r w(t) \, dx = \int \nabla \phi_r \cdot (-\nabla w + f(u_1) - f(u_2)) \, dx.$$

With $\nabla \phi_r = \frac{1}{r^d} (\nabla \phi)_r$, this yields

$$\left| \int \phi_r (w(t) - w(0)) \, dx \right| \leq \frac{1}{r^d} \| \nabla \phi \|_\infty \int_0^t (\| \nabla w(s) \|_1 + L_R \| w(s) \|_1) \, ds.$$

Because of (2.4,2.5), the integral above is convergent. Letting $r \to +\infty$ and using dominated convergence, we obtain in the limit

$$\int w(t) \, dx = \int w(0) \, dx, \quad \forall t \in (0, T).$$

Once again, we may argue by induction over intervals $\left(\frac{k}{2} T, \frac{k+1}{2} T\right)$ to prove that this equality holds true for every $t > 0$. Finally, we have proved:

**Theorem 2.2 (Conservation of mass.)** Let $a_1, a_2$ be bounded functions. Let $u_1, u_2$ be the corresponding mild solutions of the Cauchy problem.

If $a_1 - a_2 \in L^1$, then $u_1 - u_2 \in C_t(L^1)$ and we have

$$\int_{\mathbb{R}^d} (u_1(x,t) - u_2(x,t)) \, dx = \int_{\mathbb{R}^d} (a_1(x) - a_2(x)) \, dx. \quad (2.6)$$
2.5.2 Comparison (maximum principle)

We again deal with two solutions \( u_1, u_2 \) of the Cauchy problem and \( w = u_2 - u_1 \). From the Taylor formula, we have

\[
f(u_2) - f(u_1) = A(u_1, u_2)w, \quad A(u, v) = \int_0^1 f'(\theta u + (1-\theta)v) \, d\theta.
\]

Making the difference of the PDEs satisfied by each \( u_j \), we get

\[
\partial_t w + \text{div}(wb(x, t)) = \Delta w,
\]

where \( b := A(u_1, u_2) \) is a smooth, bounded vector field. Equivalently,

\[
\partial_t w + b \cdot \nabla w + (\text{div} \, b)w = \Delta w,
\]

to which we may apply the maximum principle, provided \( w(0) \leq 0 \). This leads us to the following statement.

**Proposition 2.1** If \( a_{1,2} \in L^\infty \) satisfy \( a_1 \leq a_2 \), then the corresponding mild solutions of the Cauchy problem for (1) satisfy \( u_1 \leq u_2 \) everywhere.

2.5.3 Contraction in \( L^1 \)

Assume again that \( a_1 - a_2 \) is integrable and form

\[
a^- = \inf(a_1, a_2), \quad a^+ = \sup(a_1, a_2).
\]

We have

\[
a^- \leq a_1, a_2 \leq a^+, \quad a^+ - a^- = |a_2 - a_1|.
\]

We denote \( u_{1,2} \) and \( u^\pm \) the mild solutions associated with these initial data. Because of Proposition 2.1, we have

\[
u^- \leq u_1, u_2 \leq u^+.
\]

This implies

\[
\|u_1(t) - u_2(t)\|_1 \leq \int (u^+ - u^-) \, dx.
\]

Because of the mass conservation, the later equals \( \int (a^+ - a^-) \, dx \). Combining all this, we obtain

\[
\|u_1(t) - u_2(t)\|_1 \leq \|a_1 - a_2\|_1.
\]

Finally, the semi-group property tells us that this remains true if we replace \( a_j \) by \( u_j(\tau) \) if \( t > \tau \). Whence the statement
2.6. THE CO-PROPERTIES

Proposition 2.2 \((L^1\text{-contraction.})\) Let \(a_1, a_2\) be bounded functions. Let \(u_1, u_2\) be the corresponding mild solutions of the Cauchy problem.

If \(a_1 - a_2 \in L^1\), then \(u_1 - u_2 \in C_t(L^1)\) and the map

\[ t \mapsto \|u_1(t) - u_2(t)\|_1 \]

is non-increasing over \([0, +\infty)\).

We make two important remarks.

Remark 2.3 Despite our terminology, the property that we have proved is more general than contraction in \(L^1\), because the data \(a_j\) need not be individually in \(L^1\).

Remark 2.4 We point out that the contraction property is true only in \(L^1\). It fails in every other \(L^p\)-space, unless the equation is linear.

2.6 The co-properties

We summarize the results obtained in the previous section as a set of Co-properties, meaning that every description begins with the letters Co.

Conservation of mass. If \(a_1 - a_2 \in L^1\), then \(u_1 - u_2 \in C_t(L^1)\) and

\[ \int_{\mathbb{R}^d} (u_1(x,t) - u_2(x,t)) \, dx = \int_{\mathbb{R}^d} (a_1 - a_2) \, dx. \]

Contraction. Under the same assumption, \(S_t\) is a contraction for the \(L^1\)-distance:

\[ t \mapsto \|u_1(t) - u_2(t)\|_1 \] is non-increasing.

Comparison. If \(a_1 \leq a_2\), then \(u_1 \leq u_2\).

Constants. The constants are solutions of (1).

As we have seen above, not all the four properties are independent from each other: mass conservation and comparison imply \(L^1\)-contraction. Likewise, mass conservation plus \(L^1\)-contraction implies comparison, at least when \(a_1 - a_2 \in L^1\), for, if \(a_2 \geq a_1\), then

\[ \|u_2(t) - u_1(t)\|_1 \leq \|a_2 - a_1\|_1 = \int (a_2 - a_1) \, dx = \int (u_2(t) - u_1(t)) \, dx, \]

which implies \(u_2(t) - u_1(t) \geq 0\).

Remark also that the \(L^1\)-contraction is non-strict. As a matter of fact, if \(a_2 \geq a_1\), then \(t \mapsto \|u_2(t) - u_1(t)\|_1\) is constant, because of the comparison principle and the mass conservation. This will make the \(L^1\)-stability analysis of special solutions very delicate.
2.7 \( L^1 \)-stability of patterns

Let \( U \in L^\infty \) be a steady solution of (1). If \( a \in U + L^1 \cap L^\infty \) is an initial data, we know that \( u(t) - U \in L^1 \) and that the function

\[ t \mapsto \|u(t) - U\|_1 \]

is non-increasing. In particular, it admits a limit as \( t \to +\infty \), which we denote \( \ell(a) \).

A general asymptotic stability question asks whether a small enough initial disturbance \( a - U \) ensures that \( u(t) \) tends to \( U \) with respect to the \( L^1 \)-distance, that is if \( \ell(a) = 0 \). However, because we deal with conservation laws, this notion is too strong: we know that

\[ \int (u(t) - U) \, dx = \int (a - U) \, dx, \]

which implies

\[ \ell(a) \geq \left| \int \mathbb{R}^d (a - U) \, dx \right|. \tag{2.7} \]

This leads us to a restricted stability notion, which is more appropriate to the context of conservation laws, in which we denote \( L^1_0(\mathbb{R}^d) \) the hyperplane of functions \( h \in L^1 \) such that

\[ \int \mathbb{R}^d h(x) \, dx = 0. \]

**Definition 2.2** The steady solution \( U \in L^\infty \) is \( L^1 \)-asymptotically stable if there exists an \( \epsilon > 0 \) such that if \( a \in U + L^\infty \cap L^1_0 \), and if \( \|a - U\|_1 < \epsilon \), then \( \ell(a) = 0 \).

We say that \( U \) is unconditionally \( L^1 \)-asymptotically stable if \( \ell(a) = 0 \) for every \( a \in U + L^\infty \cap L^1_0 \).

The set of initial data \( a \in U + L^1_0 \cap L^\infty \) for which \( u(t) \) tends to \( U \) in the sense above, that is the set of equation

\[ \ell(a) = 0 \]

is called the attraction basin of \( U \).

Because of this definition, and because the solution \( u(t) \) takes values in \( U + L^1 \cap L^\infty \), we shall equip this affine subspace with the topology defined by the distance

\[ d(v, w) = \|v - w\|_1. \]

It is true that under the assumption \( a - U \in L^1_0 \), our solution takes values in the smaller subspace \( U + L^1_0 \cap L^\infty \), but we shall need, in some part of our analysis, to
compare $u(t)$ with elements $v \in U + L^1 \cap L^\infty$ for which $\int (v - U) \, dx \neq 0$. This is typically the case when we apply the comparison principle, since two ordered functions $w \geq v$ satisfy $\int (w - v) \, dx \neq 0$, unless $w \equiv v$.

**Proposition 2.3** The functional $\ell$ is 1-Lipschitz: If $a, b \in U + L^1 \cap L^\infty$, then

$$|\ell(b) - \ell(a)| \leq \|b - a\|_1.$$

**Proof**

Let $u$ and $v$ be the corresponding solutions of the Cauchy problem. We have

$$\|v(t) - U\|_1 \leq \|v(t) - u(t)\|_1 + \|u(t) - U\|_1 \leq \|b - a\|_1 + \|u(t) - U\|_1,$$

because of contraction. Passing to the limit as $t \to +\infty$, we obtain

$$\ell(v) \leq \|b - a\|_1 + \ell(a).$$

Then switch $(a, u)$ and $(b, v)$.

**Corollary 2.1** The attraction basin of $U$ is closed in $(U + L^1_0 \cap L^\infty, d)$.

### 2.7.1 Travelling waves

A travelling wave is a solution of (1) of the form

$$u(x,t) = U(x - \sigma t \nu),$$

where $U$ is bounded, $\sigma$ is a constant (the wave speed) and $\nu$ a unit vector (the direction of propagation). If $\sigma = 0$, we just have a steady solution.

Travelling waves can always be transformed so as to become steady, by introducing a moving frame of coordinates $(y, t)$ where $y = x - \sigma t \nu$. Then (1) is transformed into an equation of the same form but with a different flux $g(u) = f(u) - \sigma u \nu$:

$$\partial_t v + \text{div}_y(g(v)) = \Delta_y v.$$  \hfill (2.8)

We say that the travelling wave $u$ is $L^1$-asymptotically stable if $U$ is an $L^1$-asymptotically stable steady solution of (2.8).
2.8 Dispersion effect

Recall that for the heat equation, an $L^1$-initial data yields a solution that decays in $L^p$ for every $p > 1$, as some rate depending upon $p$ and the dimension $d$. The same phenomenon occurs in the nonlinear case, with the amazing fact that the estimates don’t depend upon the flux $f$ at all!

Because we need only this case in the sequel, we restrict to the situation where $p = 2$ and leave the general case as an exercise for the reader. We start from the identity

$$\partial_t \frac{u^2}{2} + \text{div} g(u) + |\nabla u|^2 = \text{div}(u \nabla u),$$

where $g'(z) = z f'(z)$. Integrating in space, we obtain formally

$$\frac{d}{dt} \|u\|_2^2 + 2 \|\nabla u\|_2^2 \leq 0.$$  

This is where we get rid of $f$ or $g$. We know invoke the Gagliardo–Nirenberg inequality, which is an extension of Sobolev inequality:

$$\|z\|_2 \leq c \|\nabla z\|_2^{\frac{d}{2}} \|z\|_1^{\frac{2}{2d}}.$$  

Apply this inequality to $u(t)$. When the initial data is integrable, the contraction gives

$$\|u(t)\|_2 \leq c \|\nabla u(t)\|_2^{\frac{d}{2}} \|a\|_1^{\frac{2}{2d}},$$

from which we deduce a differential inequality

$$\frac{d}{dt} \|u\|_2^2 + c \|u\|_2^{2+4/d} \|a\|_1^{-4/d} \leq 0,$$

for some constant $c = c_d > 0$. This amounts to saying

$$\frac{d}{dt} \left( \frac{\|a\|_1}{\|u\|_2} \right)^{4/d} \geq 0,$$

which implies an inequality

$$\|u(t)\|_2 \leq c \frac{\|a\|_1}{d^{4/d}}.$$

(2.9)

We remark that the larger the dimension, the faster the decay.

---

1. To justify the calculation, first prove that $u \in C_t(L^2)$ by working at the level of approximate solutions $u^m$. Then use test functions $\phi_r$ as in the proof of mass conservation.
Chapitre 3

$L^1$-stability of constants

We consider the case of constants $U \equiv \alpha$, which are a special case of standing solutions. This case is so special that working in a moving frame as above does not change the solution itself. But since it allows us to replace the flux $f$ by a flux $f(u) - u\vec{q}$ for some arbitrary vector $\vec{q}$, we may assume, without loss of generality, that

$$f(\alpha) = f'(\alpha) = 0.$$  

Of course, we may also shift the $u$-axis, so that $\alpha = 0$. Then, on the interval $[-R, R]$, we have

$$|f(u)| \leq M_R u^2.$$  

(3.1)

3.1 The linear case

If $f$ is linear, the flatness tells us that $f \equiv 0$ and the equation is just the heat equation (1.1). The solution of the Cauchy problem is $u(t) = H_t a = K_t * a$.

Suppose first that $a$ is of the form $a = \partial_\beta z$ where $z \in \mathcal{D}(\mathbb{R}^d)$. Then

$$\|u(t)\|_1 = \|(\partial_\beta K_t) * z\|_1 \leq \frac{c}{\sqrt{t}} \|z\|_1 \to 0,$$

so that $\ell(a) = 0$. By linearity we have again $\ell(a) = 0$ if $a = \text{div} \vec{z}$ and each component $z_\beta$ belongs to $\mathcal{D}(\mathbb{R}^d)$. Such initial data form a vector space $X_0$. Because of Proposition 2.3, $\ell$ vanishes identically over the closure $\overline{X_0}$.

Lemma 3.1 The closure $\overline{X_0}$ in $L^1$ is nothing but $L^1_0$. That in $L^1 \cap L^\infty$ is just $L^1_0 \cap L^\infty$.

Proof
We prove only the first assertion, using Hahn–Banach. Linear forms over $L^1$ are maps $v \mapsto \int_{\mathbb{R}^d} \phi v \, dx$ where $v \in L^\infty$. We examine those forms which vanish identically over $X_0$. This means
\[
\int \phi \text{div} \vec{z} \, dx = 0, \quad \forall \vec{z} \in D^d.
\]
In distribution terms, this rewrites as $\langle \nabla \phi, \vec{z} \rangle = 0$ for every test vector field $\vec{z}$; in other words $\nabla \phi = 0$, which is known to be equivalent to $\phi = \text{cst.}$

**Corollary 3.1** If $a \in L^1_0 \cap L^\infty$, then $\|H_t a\|_1 \to 0$ as $t \to +\infty$.

**Remark 3.1** We warn the reader that there is no decay rate at all: the operator norm of $H_t$ over $L^1_0$ equals 1 for every $t > 0$. Only when $a$ has some finite moment, it is possible to give a decay, as above in the case $a \in \text{div} W^{1,1}$.

### 3.2 The stability in space dimension $d \geq 2$

We go back to the semi-linear case, where we recall that $f(u) = O(u^2)$ at the origin.

Let us estimate the norm of the integral term in Duhamel’s formula:
\[
\left\| \int_0^t (\nabla K_{t-s}) * f(u(s)) \, ds \right\|_1 \leq c \int_0^t \|f(u(s))\|_1 \frac{ds}{\sqrt{t-s}} \\
\leq c M_R \int_0^t \|u(s)\|_2^2 \frac{ds}{\sqrt{t-s}}
\]

Applying the dispersion inequality (2.9), we infer an upper bound
\[
c M_R \|a\|_1^2 \int_0^t \frac{ds}{\sqrt{(t-s)s^d}},
\]
which is useless, because the integral diverges (in $s = 0$). But it will work if we write the Duhamel formula while taking an initial time $\tau > 0$. We shall have
\[
\left\| \int_\tau^t (\nabla K_{t-s}) * f(u(s)) \, ds \right\|_1 \leq c M_R \|a\|_1^2 \int_\tau^t \frac{ds}{\sqrt{(t-s)s^d}} \\
\leq c M_R \|a\|_1^2 \frac{1}{2} \int_\tau^{t_1} \frac{ds}{\sqrt{(1-s)s^d}} \\
\sim c M_R \|a\|_1^2 \frac{1}{2} \frac{t^{1-d/2}}{t^{1/2}} \to 0 \text{ as } t \to +\infty,
\]
3.2. THE STABILITY IN SPACE DIMENSION \( D \geq 2 \)

if \( d \geq 3 \). For \( d = 2 \), we have instead a bound \( t^{-1/2} \log \frac{t}{\tau} \), which tends to zero as well.

From the fixed point equation

\[
    u(t) = H_{t-\tau}u(\tau) + \int_{\tau}^{t} (\nabla K_{t-s}) * f(u(s)) \, ds,
\]

we therefore deduce

\[
    \lim_{t \to +\infty} \|u(t) - H_{t-\tau}u(\tau)\|_1 = 0.
\]

Finally, we get

\[
    \|u(t) - H_{t}a\|_1 \leq \|u(t) - H_{t-\tau}u(\tau)\|_1 + \|H_{t-\tau}(u(\tau) - H_{\tau}a)\|_1 \to 0,
\]

where we use the mass conservation

\[
    \int_{\mathbb{R}^d} (u(\tau) - H_{\tau}a) \, ds = 0
\]

and apply Corollary 3.1.

Denote \( m = \int a \, dx \). Remark that, likewise

\[
    H_{t}a - mK_{t} = H_{t-\tau}(H_{\tau}a - mK_{\tau}) \quad \text{and} \quad \int (H_{\tau}a - mK_{\tau}) \, dx = 0,
\]

we have also

\[
    \|H_{t}a - mK_{t}\|_1 \to 0,
\]

whence a definitive result:

**Theorem 3.1 \((d \geq 2)\)** Suppose \( d \geq 2 \). Given an initial data \( a \in L^{1} \cap L^{\infty} \), the mild solution satisfies

\[
    \lim_{t \to +\infty} \|u(t) - mK_{t}\|_1,
\]

where

\[
    m = \int_{\mathbb{R}^d} a(x) \, dx.
\]

This result is stronger than the one searched for at the beginning, because it does not assume \( a \in L^{1}_0 \). Its corollary is the formula

\[
    \ell(a) = \left| \int_{\mathbb{R}^d} a(x) \, dx \right|, \quad d \geq 2.
\]
3.3 The stability in one space dimension

The situation is slightly different in one space dimension \(d = 1\), because the integral term does not decay to zero in the \(L^1\) norm when the initial mass is non-zero. In other words, \(u(t)\) is not asymptotic to \(mK_t\) in this topology. Yet we are able to treat fully the case \(a \in L^1_0 \cap L^\infty\).

To begin with, we fix an arbitrary \(R > 0\) and consider those \(a\) such that \(\|a\|_\infty \leq R\). We estimate the integral term, now

\[
\left\| \int_0^t (\nabla K_{t-s} \ast f(u(s))) \, ds \right\|_1 \leq cM_R \|a\|_1^2 \int_0^t \frac{ds}{\sqrt(t - s)s} = c\pi M_R \|a\|_1^2.
\]

We infer

\[
\|u(t)\|_1 \leq \|H_t a\|_1 + c\pi M_R \|a\|_1^2.
\]

Using now Corollary 3.1, we end up with

\[
\ell(a) \leq c\pi M_R \|a\|_1^2.
\]

Let us now observe that \(\ell(a) = \ell(u(t))\) for every time \(t\). We thus have

\[
\ell(a) \leq c\pi M_R \|u(t)\|_1^2, \quad t > 0.
\]

Passing to the limit, there remains the constraint

\[
\ell(a) \leq c\pi M_R \ell(a)^2 \quad (3.2)
\]

This tells us that \(\ell\) takes values in \(\{0\} \cup \left[\frac{1}{c\pi M_R}, +\infty\right)\). But since \(L^1_0 \cap B_R(L^\infty)\) is convex, thus connected (for the \(L^1\)-norm), and because \(\ell\) is continuous, its range is connected. Because it contains 0, we deduce that \(\ell \equiv 0\) over \(L^1_0 \cap B_R(L^\infty)\). Finally, \(R\) being arbitrary, we have

**Theorem 3.2** \((d = 1)\) Let \(a \in L^1_0 \cap L^\infty\) be an initial data. Then the mild solution of the Cauchy problem for (1) satisfies

\[
\lim_{t \to +\infty} \|u(t)\|_1 = 0.
\]

**Remark 3.2** This proves a posteriori that if \(a \in L^1_0 \cap L^\infty\), then the integral term tends to zero in \(L^1\) as \(t \to +\infty\). If instead the mass \(\int a\) is non-zero, then this becomes false, unless \(f''(0)\) vanishes.
Chapitre 4

$L^1$-stability of shock fronts

This chapter is entirely devoted to the 1-d case. Shock fronts are travelling waves whose values at both ends $\pm \infty$ are distinct.

4.1 Shock fronts

Let $v(x, t) = U(x - \sigma t)$ be a travelling wave of velocity $\sigma$. We call $U$ the profile of the wave. The differential equation for $v$ becomes

$$-\sigma U' + (f(U))' = U'',$$

which can be integrated once as

$$U' = f(U) - \sigma U - q =: g(U)$$

with $q$ a constant. This is an autonomous ODE, scalar and first-order. Maximal solutions are monotonous and their extreme values are either zeroes of $g$, or infinite. Bounded maximal solutions are therefore heteroclinic, joining two consecutive zeroes $u_\pm = U(\pm \infty)$ of $g$. We have

$$q = f(u_\pm) - \sigma u_\pm,$$

from which we derive the Rankine–Hugoniot condition

$$\sigma = \frac{f(u_+)}{u_+} - \frac{f(u_-)}{u_-}. \quad (4.1)$$

If $u_- < u_+$, then $g > 0$ over $(u_-, u_+)$, while if $u_+ < u_-$, then $g < 0$ over $(u_+, u_-)$. This is translated into the Oleinik condition:

If $u_- < u_+$, the graph of $f$ is above its chord over $(u_-, u_+)$. 27
If \( u_- > u_+ \), the graph of \( f \) is below its chord over \((u_+, u_-)\).

We speak of the triplet \((u_-, u_+; \sigma)\) as a \textit{shock}. This is reminiscent to the fact that the function

\[
V(x, t) = \begin{cases} 
  u_-, & \text{if } x < \sigma t, \\
  u_+, & \text{if } x > \sigma t 
\end{cases}
\]

is a discontinuous solution (a shock) of the first-order conservation law

\[
\partial_t V + \partial_x f(V) = 0.
\]

Of course, any translated function \( \tau_h U := U(\cdot - h) \) is again a shock profile, for the same shock data. Importantly, we have \( \tau_h U - U \in L^1 \) for every \( h \in \mathbb{R} \), because this difference has a constant sign (monotonicity of \( U \)), and since

\[
\int_A^B (U(x-h) - U(x)) \, dx = \left( \int_{A-h}^{B-h} U(x) \, dx \right) - \left( \int_A^B U(x) \, dx \right) = h(u_- - u_+).
\]

In particular, we have an exact formula

\[
\int_{\mathbb{R}} (U - \tau_h U) \, dx = h(u_+ - u_-). \tag{4.2}
\]

### 4.2 The stability problem

As mentioned above, the stability of a travelling wave is studied in the moving frame associated with it. Therefore we are led to the study of \textit{steady} shock fronts \( U(x) \). We denote \( u_\pm \) the limits at \( \pm \infty \), which are distinct and satisfy

\[
f(u_+) = f(u_-).
\]

Let \( a \in L^\infty \) be an initial data and \( u(t) \) the solution of the Cauchy problem for (1). We have seen that in order that \( \|u(t) - U\|_1 \) tend to zero, it is necessary that \( a - U \in L^1_0 \). This condition is not so restrictive here because if we only know that \( a - U \in L^1 \), then we may select

\[
h = \frac{1}{u_+ - u_-} \int_{\mathbb{R}} (U - a) \, dx
\]

so as to have \( a - \tau_h U \in L^1_0 \). Therefore we always have an \( L^1 \)-target at our disposal.
4.3. A DYNAMICAL SYSTEM APPROACH

We therefore assume that

$$a \in U + L_0^1 \cap L^\infty. \quad (4.3)$$

As above, we denote

$$\ell(a) = \lim_{t \to +\infty} \|u(t) - U\|_1 = \inf_t \|u(t) - U\|_1,$$

which is a 1-Lipschitz functional in the affine space $U + L^1 \cap L^\infty$.

4.3 A dynamical system approach

Let us begin with special initial data, those which are clamped by translated profiles: we assume that there exist $h, k \in \mathbb{R}$ such that

$$U(x - h) = \tau_h U(x) \leq a(x) \leq \tau_k U(x) = U(x - k). \quad (4.4)$$

4.3.1 Precompactness of the trajectory

Because of the comparison principle, and since $\tau_h U, \tau_k U$ are solutions, we deduce

$$\tau_h U \leq u(t) \leq \tau_k U, \quad t > 0. \quad (4.5)$$

Subtracting $U$, we obtain

$$\tau_h U - U =: V_h \leq u(t) - U \leq V_k := \tau_k U - U,$$

where we know $V_h, V_k \in L^1$. Because $V_h, V_k$ don’t depend upon $t$, the family $(u(t) - U)_{t \geq 0}$ is tight in $L^1$.

Next, we remark that $\tau_\varepsilon u = u(\cdot - \varepsilon, \cdot)$ is the solution of the Cauchy problem associated with the data $\tau_\varepsilon a$. From the $L^1$-contraction, we deduce

$$\|\tau_\varepsilon u(t) - a\|_1 \leq \|\tau_\varepsilon a - a\|_1 =: \omega_\varepsilon(\varepsilon).$$

Because $a = U + b$ with $b \in L^1$, we have

$$\omega_\varepsilon(\varepsilon) \leq \omega_U(\varepsilon) + \omega_b(\varepsilon) = |\varepsilon| \cdot \|u_+ - u_-\| + \omega_b(\varepsilon) \to 0+, \quad \text{as } \varepsilon \to 0.$$

We therefore have

$$\lim_{\varepsilon \to 0} \|\tau_\varepsilon u(t) - u(t)\|_1 = 0, \quad \text{uniformly in } t.$$
and likewise

$$\lim_{\epsilon \to 0} \| \tau_{\epsilon} (u(t) - U) - (u(t) - U) \|_1 = 0, \quad \text{uniformly in } t. \quad (4.6)$$

Finally, the trajectory $t \mapsto u(t) - U$ is bounded in $L^1$ because of contraction:

$$\| u(t) - U \|_1 \leq \| a - U \|_1.$$  

This, together with (4.6) and the tightness, imply (Theorem of Fréchet–Kolmogorov) the precompactness of this trajectory. By translation, the trajectory $(u(t))_{t \geq 0}$ is precompact in $U + L^1 \cap L^\infty$, for the $L^1$-distance.

### 4.3.2 The omega-limit set

We are now in position to introduce a tool from dynamical system theory, the *omega-limit set* of the trajectory. Its precise definition is

$$\Omega(a) = \bigcap_{\tau \geq 0} \{ u(t) \mid t \geq \tau \} = \bigcap_{\tau} B_{\tau}.$$  

We have seen that for every $\tau$, $B_{\tau}$ is a non-void compact set. Because $\tau \mapsto B_{\tau}$ is non-increasing, the intersection $\Omega(a)$ is therefore non-void and compact. It is the set of cluster points of $u(t)$ as $t \to +\infty$. To prove that $u(t)$ converges as $t \to +\infty$ amounts to prove that $\Omega(a)$ is a singleton.

Obviously, we have $\Omega(a) = \Omega(S_t a)$, from which we deduce that $\Omega(a)$ is invariant under the semi-group $S_t$, both forward and backward:

$$S_t \Omega(a) = \Omega(a), \quad \forall t > 0.$$  

Because of the regularizing property, this implies

$$\Omega(a) \subset C^\infty(\mathbb{R}).$$  

### 4.4 The Lasalle’s invariance principle

**Definition 4.1** Let $J : U + L^1 \cap L^\infty \to \mathbb{R}$ be a continuous function. We say that $J$ is a Lyapunov function for (1) if, for every $\alpha \in U + L^1 \cap L^\infty$, the map $t \mapsto J(S_t \alpha)$ is non-increasing.
4.4. THE LASALLE’S INVARIANCE PRINCIPLE

4.4.1 Examples

A trivial, though important example is that of the mass defect

\[ M(\alpha) = \int_{\mathbb{R}} (\alpha - U) \, dx, \]

which is constant along trajectories.

A more subtle example is obtained from the contraction property: if \( h \in \mathbb{R}, \) then we know that \( \tau_h U \) is a steady solution that belongs to \( U + L^1 \cap L^\infty. \) Hence

\[ J_h(\alpha) := \| \alpha - \tau_h U \|_1 \]

is a Lyapunov function.

The decay of \( J_h \) along trajectories is a manifestation of the dissipation. This justifies our terminology of dissipative conservation laws.

4.4.2 The value of a Lyapunov function over \( \Omega(a) \)

If \( J \) is a Lyapunov function, then it achieves a constant value over \( \Omega(a), \) namely

\[ J(b) = \lim_{t \to +\infty} J(u(t)), \quad \forall b \in \Omega(a). \]

This is a very strong property, because of the invariance of \( \Omega(a) \) under the semigroup: It tells us that if \( b \in \Omega(a), \) then \( J \) is constant along the trajectory \( (S_t b)_{t \geq 0}. \) In other words, the dissipative system is not that dissipative along a trajectory lying in \( \Omega(a). \)

4.4.3 Strict \( L^1 \)-contraction

**Lemma 4.1** Let \( b_1, b_2 \in L^\infty \cap C^\infty \) be such that \( b_2 - b_1 \in L^1. \) Let \( w_1, w_2 \) be the corresponding solutions of the Cauchy problem. Finally, suppose that \( b_2(x_0) = b_1(x_0). \)

If \( b_2(x_0) \neq b_1(x_0), \) then \( \| u_2(t) - u_1(t) \|_1 < \| b_2 - b_1 \|_1 \) for every \( t > 0. \)

**Proof**

We recall the proof of Proposition 2.2: define \( b^- = \min\{b_1, b_2\}, b^+ = \max\{b_1, b_2\}, \) \( u^\pm \) the corresponding solutions and write the maximum principle

\[ u^- \leq u_{1,2} \leq u^+. \]

With the conservation of mass, deduce that

\[ \| u_2(t) - u_1(t) \|_1 \leq \int_{\mathbb{R}} (u^+ - u^-) \, dx = \int_{\mathbb{R}} (b^+ - b^-) \, dx = \| b_2 - b_1 \|_1. \]
If we had \( \| u_2(T) - u_1(T) \|_1 = \| b_2 - b_1 \|_1 \) at some time \( T > 0 \), then the same would hold at every \( t \in (0, T) \) and we should have \( | u_2 - u_1 | \equiv u^+ - u^- \), that is
\[
  u^- \equiv \min \{ u_1, u_2 \}, \quad u^+ \equiv \max \{ u_1, u_2 \}, \quad t \in [0, T]. \tag{4.7}
\]

This is impossible because \( u^\pm \) are \( C^\infty \) for \( t > 0 \), whereas \( \min \{ u_1, u_2 \} \) and \( \max \{ u_1, u_2 \} \) are not. To see this, apply the Implicit Function Theorem: there exists a smooth curve \( t \mapsto X(t) \) such that \( X(0) = x_0 \) and \( u_2(X(t), t) = u_1(X(t), t) \).

Let \( b \in \Omega(a) \) be given. Define \( y = \phi(x) \), then \( \epsilon = x - y \). Then \( U(y) = b(x) \), that is \( \tau_U(x) = b(x) \).

Let \( x \in \mathbb{R} \) be given. Define \( y = \phi(x) \), then \( \epsilon = x - y \). Then \( U(y) = b(x) \), that is \( \tau_U(x) = b(x) \). Because \( b \in \Omega(a) \), we know that \( t \mapsto \| \tau_U - v(t) \|_1 \) is constant.

Apply Lemma 4.1 to \( \tau_U \) and \( b \) : we obtain that \( (\tau_U)'(x) = b'(x) \). In other words,
\[
  U'(y) = b'(x). \tag{4.8}
\]

Deriving \( b \equiv U \circ \phi \), we also get \( b'(z) = \phi'(z)U'(\phi(z)) \). In particular \( b'(x) = \phi'(x)U'(y) \).

Eliminating with (4.8), there remains \( (\phi'(x) - 1)U'(y) = 0 \). But since \( U' \) does not vanish, this is \( \phi'(x) = 1 \).

From \( \phi'(x) = 1 \), we infer \( \phi(x) = x - \eta \), so that \( b = \tau_U \). Finally, passing to the limit in \( \int (u(t) - U) \, dx \equiv \int (a - U) \, dx = 0 \), we have \( \int (b - U) \, dx = 0 \). This restricts the value of the shift to \( \eta = 0 \).

We have proved therefore that \( \Omega(a) \subset \{ U \} \). Because \( \Omega(a) \) is non-void, this reads \( \Omega(a) = \{ U \} \), which tells us that \( u(t) \to U \).
Proposition 4.1 Let $U$ be a steady front and $a \in U + L^1 \cap L^\infty$ an initial data. Assume that $$\tau_h U \leq a \leq \tau_k U$$ for some shifts $h, k$, and that $$\int_{\mathbb{R}} (a - U) \, dx = 0.$$ Then the solution $u(t) = S_t a$ satisfies $$\lim_{t \to +\infty} \|u(t) - U\|_1 = 0.$$ According to Corollary 2.1, the attraction basin $A$ of $U$ is closed for the distance $$d(a_1, a_2) := \|a_2 - a_1\|_1.$$ It therefore contains the closure of the data set described in Proposition 4.1. This closure is $$\mathcal{B} := \{a \in U + L^1 \cap L^\infty \mid a(x) \in [u_-, u_+] \text{ a.e.} \},$$ where, again, the interval could be $[u_+, u_-]$.

4.5 Unconditional stability

We now overcome the restriction that $a$ takes values in $[u_-, u_+]$. Observing that $$\int_{\mathbb{R}} (a(x) - u_+)_+ \, dx < \infty, \quad \text{while} \quad \int_{\mathbb{R}} (a(x) - u_+)_- \, dx = \infty,$$ we may find an auxiliary initial data $b \in u_+ + L^1 \cap L^\infty$ with the following properties (TODO : make a picture) $$a \leq b \quad \text{and} \quad \int_{\mathbb{R}} (b(x) - u_+) \, dx = 0.$$ From Theorem 3.2, we have $$\int_{\mathbb{R}} |S_t b - u_+| \, dx \to 0,$$ and in particular $$\int_{\mathbb{R}} (S_t b - u_+)_+ \, dx \to 0.$$
Because of the comparison principle, $S_t a \leq S_t b$ and thus
\[
\int_{\mathbb{R}} (S_t a - u_+) \, dx \to 0.
\]
Likewise, we have
\[
\int_{\mathbb{R}} (S_t a - u_-) \, dx \to 0.
\]
Both properties tell us together that
\[
\lim_{t \to +\infty} d(S_t a; \mathcal{B}) = 0.
\]
From the continuity of $\ell$, its constancy along a trajectory and the fact that $\ell \equiv 0$ over $\mathcal{B}$, we deduce
\[
\ell(a) = \lim_{t \to +\infty} \ell(S_t a) \leq \sup \{ \ell(b) \mid b \in \mathcal{B} \} = 0.
\]
Whence our final result:

**Theorem 4.1 (Main theorem.)** Let $U$ be a steady shock front for (1). Let $a \in L^\infty$ be an initial data such that $a \in U + L^1_0$. Then
\[
\lim_{t \to +\infty} \| S_t a - U \|_1 = 0.
\]
This implies, when the front is a travelling wave and the defect mass is non-zero:

**Corollary 4.1** Let $U$ be a shock front of velocity $\sigma$. Let $a \in L^\infty$ be an initial data such that $a \in U + L^1$. Then
\[
\lim_{t \to +\infty} \| S_t a - \tau_h aU \|_1 = 0,
\]
where $h$ is defined by
\[
h = \frac{1}{u_+ - u_-} \int_{\mathbb{R}} (U - a) \, dx.
\]

### 4.6 Further developments and references

In several space dimensions, the mass of the initial disturbance cannot be absorbed by shifts of the front, because $\tau_h U - U$ is not integrable, unless $h$ is orthogonal to the direction of propagation, in which case $\tau_h U \equiv U$. A non-zero mass $\int (a - U) \, dx$ is responsible for a distortion of the front (J. Goodman, J. Miller, *Long-time behaviour of scalar viscous shock fronts in two dimensions*. J. Dynam. Diff. Eq., 11 (1999), pp
This distortion is governed asymptotically by a linear diffusion equation in the transverse coordinates.


There is a huge literature about the stability of shock fronts in stronger topologies, originating in D. H. Sattinger’s paper On the stability of waves of nonlinear parabolic systems. Advances in Math., 22 (1976), pp 312–355. There are mainly two strategies, either based upon spectral analysis or upon relative entropy estimates. However, none of these strategies can get rid of a smallness assumption about the initial disturbance. Only our strategy, based on $L^1$-contraction, can be successful for arbitrary data.

The analysis presented here applies to every conservation law that displays the four co-properties. This was explained in details in our review paper $L^1$-stability of nonlinear waves in scalar conservation laws. Handbook of Differential Equations. Evolutionary equations, vol. I. C. Dafermos, E. Feireisl editors. Elsevier, North-Holland (2004), pp 473–553. Here is a list of such equations:

— the inviscid conservation law,
— the Jin–Xin relaxation of an inviscid conservation law,
— the Rosenau model, where an inviscid conservation law is coupled to a linear elliptic equation,
— the approximation of an inviscid conservation law by a monotone difference scheme.

Because these models display different kinds of dissipation, the proof of the stability of constants differ from one model to the other. In particular, it does not hold for an inviscid conservation law; in this latter situation, the stability of shock waves is unconditional only when the shock is non-characteristic, that is satisfies the strict Lax shock condition.