The reverse Hlawka inequality in a Minkowski space

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1 Motivation and statement

In a Euclidian space $E$, the Hlawka inequality (see [2] or [3]) reads

$$\|x + y\| + \|y + z\| + \|z + x\| \leq \|x\| + \|y\| + \|z\| + \|x + y + z\|, \quad \forall x, y, z \in E.$$  

This inequality is sharp in two ways:

- the equality holds if one of the vectors is 0,
- the equality holds if $x, y, z$ are positively colinear.

These equality cases are tightly related to the fact that the norm is positively homogeneous of degree one. Therefore, let us say that a continuous function $f : K \to \mathbb{R}$, defined over a closed convex cone in $\mathbb{R}^n$, which is positively homogeneous of degree one, satisfies a Hlawka-type inequality if

$$(1) \quad f(x + y) + f(y + z) + f(z + x) \leq f(x) + f(y) + f(z) + f(x + y + z), \quad \forall x, y, z \in K.$$  

A necessary condition for this to happen is obtained by taking $z = y$:

$$2f(x + y) \leq f(x) + f(x + 2y), \quad \forall x, y \in K.$$  

This tells us that if $u, v \in K$ are ordered, that is if $v - u \in K$, then the convexity inequality

$$f(u + v) \leq f(u) + f(v)$$

holds true. Actually, we have

**Proposition 1.1** If $f$ is $C^2$ in the interior of $K$ and $f$ satisfies the Hlawka-type inequality, then $f$ is convex.
Proof

It is enough to prove that the Hessian $D^2 f$ is non-negative at interior points. For this, let us denote

$$\phi(x, y, z) = f(x) + f(y) + f(z) + f(x + y + z) - f(x + y) - f(y + z) - f(z + x).$$

By assumption, we have $\phi \geq 0$. Since $\phi(x, x, x) = 0$, the Hessian of $\phi$ at $(x, x, x)$ must be non-negative if $x$ is interior (one finds easily that the gradient is zero). Let us just compute the Hessian with respect to $x$:

$$D^2_x \phi(x, x, x) = D^2 f_x + D^2 f_{3x} - 2D^2 f_{2x}.$$

Because $D^2 f$ is homogeneous of degree $-1$, one deduces

$$0 \leq D^2_x \phi(x, x, x) = \frac{1}{3} D^2 f_x.$$

The Proposition above suggests to investigate which among the convex functions, positively homogenenous of degree one, satisfy a Hlawka inequality. The first natural candidates are norms, where $K = \mathbb{R}^n$, but it is known that the Hlawka inequality is not always true when the norm is not Euclidian.

Other candidates are given in terms of hyperbolic polynomial. Recall that a polynomial $p$ over $\mathbb{R}^n$, homogeneous of degree $d$, is hyperbolic in the direction of some vector $e$ if $p(e) > 0$ and for every $x \in \mathbb{R}^n$, the roots of the univariate polynomial $t \mapsto p(x + te)$ are real. Garling [1] introduced this notion in connection with the well-posedness of the Cauchy problem for hyperbolic differential operators; the vector $e$ is time-like. He proved two important facts:

- The connected component of $e$ in $\{ p > 0 \}$ is a convex cone. Its elements are time-like vectors too.
- Let us denote $K$ the closure of this cone, so that $p \geq 0$ over $K$.

An especially interesting example is that of $p(A) = \det A$ over the space $\text{Sym}_d(\mathbb{R})$ (here $n = \frac{d(d+1)}{2}$), which is hyperbolic in the direction of $I_d$. The future cone $K$ is made of the positive semi-definite matrices, and the concavity property bears the name of Minkovski’s determinantal inequality:

$$(\det A)^{1/d} + (\det B)^{1/d} \leq (\det(A + B))^{1/d}.$$

It is therefore natural to consider $f_p = -p^{1/d}$ where $p$ is a homogeneous hyperbolic polynomial of degree $d$, and ask whether $f_p$ satisfies the Hlawka inequality, that is

$$p(x)^{1/d} + p(y)^{1/d} + p(z)^{1/d} + p(x + y + z)^{1/d} \leq p(x + y)^{1/d} + p(y + z)^{1/d} + p(z + x)^{1/d},$$

$$\forall x, y, z \in K.$$
The following example shows that this turns out to be false in general. Take again for $p$ the
determinant over symmetric matrices, where $d \geq 3$. One can write $I_d = P + Q + R$ as the
sum of non-trivial mutually orthogonal projectors. Then $\det P = \cdots = \det(Q + R) = 0$, but
$\det(P + Q + R) = 1$, so that (2) is violated. This flaw looks to be caused by the fact that the
boundary of $K$ has flat parts.

The counter-example given above leaves open the case $d = 2$, where the determinant is a
non-degenerate quadratic form. In degree 2, the determinant becomes actually a paradigm,
because of the following observations:

- The Hlawka inequality involves only three vectors. By restricting to the space spanned
  by $x, y$ and $z$, it is therefore enough to consider forms in 2 or 3 space variables.

- A quadratic form $q$ is hyperbolic if and only if its signature is $(1, n-1)$; in other words,
  when $(\mathbb{R}^n, q)$ is a Minkowski space. In particular, there is only one hyperbolic quadratic
  form in $\mathbb{R}^n$, up to a change of variable.

Our main result is

**Theorem 1.1** The reverse Hlawka inequality is true in Minkowski spaces: if $q$ is a quadratic
form on $\mathbb{R}^n$, with signature $(1, n-1)$, then the “length” $\ell = \sqrt{q}$ satisfies

$$
\ell(x) + \ell(y) + \ell(z) + \ell(x + y + z) \leq \ell(x + y) + \ell(y + z) + \ell(z + x)
$$

for every vectors $x, y, z$ in the future cone.

According to the observations made above, it is enough to consider the cases

- $n = 2$ and $q(x) = x_1 x_2$,
- $n = 3$ and $q(A) = \det A$, with $\mathbb{R}^3 \sim \text{Sym}_2(\mathbb{R})$.

We treat the first case in Section 2. We prove in Section 3 that it implies the second case.

2 The two-dimensional case

We consider the form $q(x) = x_1 x_2$ whose future cone is $K = (\mathbb{R}^+)^2$. The corresponding bilinear
form is

$$
x \cdot y = \frac{1}{2}(x_1 y_2 + x_2 y_1).
$$

Let $g$ denote $\sqrt{q}$ (the opposite of $f$). One seeks for the inequality

(3) \quad \quad g(x) + g(y) + g(z) + g(x + y + z) \leq g(x + y) + g(y + z) + g(z + x), \quad \forall x, y, z \in K

If $n = 2$, then $q$ can be written as $q(x) = x_1 x_2$ in suitable coordinates, and we have
$K = (\mathbb{R}^+)^2$.  

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Because both sides of (3) are non-negative, and because of the identity
\[ q(x) + q(y) + q(z) + q(x + y + z) = q(x + y) + q(y + z) + q(z + x), \]
the inequality is equivalent to
\[
(4) \quad (g(x) + g(y) + g(z))g(x + y + z) + g(x)g(y)g(z) + g(z)g(x) \leq g(x + y)g(y + z) + g(y + z)g(z + x) + g(z + x)g(x + y), \quad \forall x, y, z \in K.
\]
The latter can be written as
\[ \theta(x, y, z) + \theta(z, x, y) + \theta(y, z, x) \leq 0, \]
where
\[ \theta(x, y, z) := g(x)g(x + y + z) + g(y)g(z) - g(x + y)g(x + z). \]
It is therefore enough to prove that
(5) \[ \theta(x, y, z) \leq 0, \quad \forall x, y, z \in K. \]
Because \(g\) is non-negative, (5) is equivalent to
\[ (g(x)g(x + y + z) + g(y)g(z)) \leq (g(x + y)g(x + z))^2, \]
that is
\[ 2g(x)g(y)g(z)g(x + y + z) \leq \pi(x, y, z) := q(x + y)q(x + z) - q(x)q(x + y + z) - q(y)q(z). \]
One verifies
\[ \pi(x, y, z) = 4(x \cdot y)(x \cdot z) + 2(x \cdot y)q(z) + 2(x \cdot z)q(y) - 2(y \cdot z)q(x), \]
and
\[ \pi = x_1^2 y_2 z_2 + x_2^2 y_1 z_1 + 2(x \cdot y)q(z) + 2(x \cdot z)q(y), \]
which is obviously non-negative for \(x, y, z \in K\). From this, we infer that (6) holds true if and only if
\[ 4q(x)q(y)q(z)q(x + y + z) \leq \pi(x, y, z)^2, \quad \forall x, y, z \in K \]
The latter inequality turns out to hold true in an even more generality, because of the identity
\[ \pi(x, y, z)^2 - 4q(x)q(y)q(z)q(x + y + z) = Q^2 \geq 0, \]
where \(Q := x_1 y_2 z_2 (x_1 + y_1 + z_1) - x_2 y_1 z_1 (x_2 + y_2 + z_2).\) Actually, one has
\[ \pi + Q = 2x_1 y_2 z_2 (x + y + z)_1, \quad \pi - Q = 2x_2 y_1 z_1 (x + y + z)_2. \]
The correctness of (7) is that of (5), which implies the correctness of (4), which amounts to the truth of (3). This ends the proof of the case.
3 The end of the proof

We now turn to the three-dimensional case where $K = \text{Sym}^+_2$ and $q(A) = \det A$. Again, we write $g = \sqrt{q}$. By a continuity argument, we may assume that the three elements, denoted here $A, B, C$, are positive definite. Then, defining $A' = C^{-1/2}AC^{-1/2}$ and $B' = C^{-1/2}BC^{-1/2}$, we see that (3) amounts to

$$g(I_2 + A' + B') + g(A') + g(B') + 1 \leq g(I_2 + A') + g(I_2 + B') + g(A' + B').$$

In other words, it is enough to consider the case where $C = I_2$.

Let us denote $a_1 \leq a_2$ and $b_1 \leq b_2$ the eigenvalues of $A$ and $B$, and $\lambda, \mu$ those of $A + B$. We know $\lambda + \mu = T := \Tr A + \Tr B$. By Weyl’s inequalities, we have

$$a_1 + b_1 \leq \lambda, \mu \leq a_2 + b_2.$$

We therefore have the constraints $\bar{s} := (a_1 + b_1)(a_2 + b_2) \leq \lambda \mu \leq T^2/4$. Let us estimate

$$\sqrt{\det(I_2 + A + B)} - \sqrt{\det(A + B)} = \sqrt{1 + T + \lambda \mu} - \sqrt{\lambda \mu}.$$

Because the function $s \mapsto \sqrt{1 + T + s} - \sqrt{s}$ is monotone decreasing, its maximum under the conditions $\bar{s} \leq s \leq T^2/4$ is achieved at $\bar{s}$. We deduce

$$\sqrt{\det(I_2 + A + B)} - \sqrt{\det(A + B)} \leq \sqrt{(1 + a_1 + b_1)(1 + a_2 + b_2)} - \sqrt{(a_1 + b_1)(a_2 + b_2)}.$$

Since

$$g(I_2 + A) + g(I_2 + B) - g(A) - g(B) = \sqrt{(1 + a_1)(1 + a_2)} + \sqrt{(1 + b_1)(1 + b_2)} - \sqrt{a_1a_2} - \sqrt{b_1b_2},$$

there remains to prove

$$\sqrt{(1 + a_1 + b_1)(1 + a_2 + b_2)} + \sqrt{a_1a_2} + \sqrt{b_1b_2} + 1 \leq \sqrt{(1 + a_1)(1 + a_2)} + \sqrt{(1 + b_1)(1 + b_2)} + \sqrt{(a_1 + b_1)(a_2 + b_2)},$$

which is a consequence of the two-D case studied in Section 2.

References

