Spectral stability of periodic solutions of viscous conservation laws: Large wavelength analysis

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Abstract

We complete and unify the works by Oh & Zumbrun [13] and by the author [16], about the spectral stability of traveling waves that are spatially periodic, in systems of $n$ conservation laws. Our context is one-dimensional. These systems are of order larger than one, in general. For instance, they could be viscous approximation of first-order systems that are not everywhere hyperbolic. However, modelling considerations often lead to higher-order terms, like capillarity in fluid dynamics; our framework remains valid in this more general setting.

We make generic assumptions, saying in particular that the set of periodic traveling waves is a manifold of maximal dimension, under the restrictions given by the conserved quantities.

The spectral stability of a periodic traveling wave is studied through Floquet’s theory. Following Gardner [4], we introduce an Evans function $D(\lambda, \theta)$, with $\lambda$ the Laplace frequency and $\theta$ the phase shift. The large wavelength analysis is the description of the zero set of $D$ around the origin.

Our main result tells that this zero set is described, at the leading order, by a characteristic equation

$$\det(\lambda I_N - i\theta \partial F(U)/\partial U) = 0.$$ 

This formula involves a flux $F$, which enters in a first-order system of conservation laws

$$\partial_t U + \partial_x F(U) = 0,$$

describing the slow modulation of the periodic traveling waves. Its size $N$ is in practice larger than $n$. The important consequence is that hyperbolicity of the latter system is a necessary condition for spectral stability of periodic traveling waves.

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Finally, we show that a similar treatment works for Coupled Map Lattices obtained by discretizing systems of conservation laws.

Let
\[ \partial_t u + \partial_x f(u) = \partial_x (B(u) \partial_x u) \tag{1} \]
be a viscous system of conservation laws. The phase space is an open domain \( \mathcal{U} \subseteq \mathbb{R}^n \).

As in [13], we are interested in the spectral stability of space-periodic, time-steady (or traveling) solutions of (1). Such traveling waves of velocity \( s \) are associated to the periodic solutions of the profile equation
\[ B(u) \partial_y u = f(u) - su - q, \tag{2} \]
where \( q \) is a constant of integration.

Given \((a, s, q) \in \mathcal{U} \times \mathbb{R} \times \mathbb{R}^n\), (2) admits a unique local solution \( u(x; a, s, q) \) such that \( u(0; a, s, q) = a \). It depends smoothly on all its arguments.

### 1 Periodic solutions

When \( u \) is a periodic solution of (2), we denote \( X \) the period, \( \omega := 1/X \) the frequency and \( \langle u \rangle \) the average over the period:
\[ \langle u \rangle := \frac{1}{X} \int_0^X u(y) \, dy. \]

Obviously, these quantities are unchanged when \( u \) is replaced by a shifted \( u(\cdot - h) \). For this reason, we consider the set \( P \) of periodic functions \( u \) that are solutions of (2) for some pair \((s, q)\), and then construct the quotient set \( \mathcal{P} := P/\mathbb{R} \) under the relation
\[ (u \sim v) \iff (\exists h \in \mathbb{R} ; v = u(\cdot - h)). \]

The quantities defined above are thus class functions:
\[ X = X(\hat{u}), \quad \omega = \Omega(\hat{u}), \quad s = S(\hat{u}), \quad q = Q(\hat{u}), \quad \langle u \rangle = M(\hat{u}). \]

We now specify a non-constant periodic solution \( u_0 \). Without loss of generality, we may assume that it is steady, meaning that \( S(u_0) = 0 \) (we drop the dot only for simplicity). We thus have
\[ B(u_0) \partial_y u_0 = f(u_0) - q_0, \tag{3} \]
for some constant \( q_0 = Q(u_0) \). Letting \( X_0 = X(u_0) \) and \( a_0 := u_0(0) \), the map \((x, a, s, q) \mapsto u(x; a, s, q) - a\) is smooth and well-defined in a neighbourhood of \((X_0; a_0, 0, q_0)\), and it vanishes at this special point. We shall make throughout this paper the assumption that

\[ \text{(H1) The map} \]
\[ (X; a; s, q) \xrightarrow{T} u(X; a, s, q) - a \]
\[ \mathbb{R} \times \mathcal{U} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \]

is a submersion at point \((X_0; a_0, 0, q_0)\).
Since this map vanishes exactly when the solution $u(\cdot; a, s, q)$ is $X$-periodic, the Submer-
sion Theorem ensures that the set $P$ is locally a manifold of codimension $n$, hence of
dimension $n + 2$. Since $u_0$ is non-constant, the projection
$$u \mapsto \dot{u}
$$
$$P \mapsto \mathcal{P}$$
is locally a fibration where fibers are circles, whence $\mathcal{P}$ is locally a manifold of dimension
$n + 1$.

**Remark:** Our assumption (H1) is much weaker than that of Oh & Zumbrun [13], whose
assumption (H3) roughly tells (locally) that, given a point $u_-$, there exists only one pair
$(u, s)$ with $u$ an $X_0$-periodic solution of
$$B(u)u' = f(u) - f(u_-) - s(u - u_-).$$

At the algebraic level, this reads
$$\mathbb{R}^n = \text{Span} \left\{ \left[ \frac{\partial u}{\partial s} \right], [w_2], \ldots, [w_n] \right\},$$
a much more restrictive condition, compared to that given in Lemma 1 below.

### 1.1 Variations on the theme

Our analysis makes sense even when the viscous term $\partial_x(B(u)\partial_x u)$ in (1) is replaced by
a higher order disturbance in conservation form:

$$\partial_t u + \partial_x f(u) = \partial_x (\mathcal{N}(u, \partial_x u, \ldots)).$$

(4)

Let us think for instance to KdV equation, or to the viscous-capillar fluid dynamics. For
the sake of simplicity, we write the latter in the isothermal case, with $\rho$ the mass density,
$v$ the velocity and $p = P(\rho)$ the pressure:

$$\partial_t \rho + \partial_x (\rho v) = 0,
\partial_t (\rho v) + \partial_x (\rho v^2 + p) = \nu \partial_x^2 u - \mu \partial_x^3 \rho.$$

The parameters $\nu \geq 0$ and $\mu \geq 0$ account for viscosity and capillarity respectively.

The differential system (2) must be modified accordingly:

$$\mathcal{N}(u, \partial_y u, \ldots) = f(u) - su - q.$$  

(5)

It is useful to rewrite (5) as a first-order ODE, by eliminating some coordinates if this
system contains algebraic equations, and introducing a few derivatives in the unknowns,
since $\mathcal{N}$ is of higher order. A general study of appropriate operators $\mathcal{N}$ does not seem
to exist in the literature yet, but a discussion about singular diffusion matrices $B$ can be found in Kawashima’s Thesis [11] and in [18], whereas the paper [20] describes the higher-order operators that are relevant for an Evans function analysis. See also [1] for some situations where both features occur. We shall denote by $N$ the size of the corresponding first-order ODE. In the situation described in the previous paragraph, there holds $N = n$.

For instance, in the Euler system with viscosity and capillarity, the algebraic equation $0 = \rho v - s\rho - q_1$ is used to eliminate $v$ in terms of $\rho$, and the capillarity term yields a second order derivative of $\rho$ in (2) that needs the introduction of $\partial_y \rho$ in the unknown. Hence the ODE governs the pair $(\rho, \rho')$, so that $N = 2$.

For an $N \times N$ ODE as above, the solution $U$ depends on $N + 1 + n$ parameters $(a, s, q)$; whence the map $U(X; a, s, q) - a$ depends on $N + 2 + n$ parameters, and takes values in $\mathbb{R}^N$. If it is a submersion at a given point corresponding to a non-constant periodic solution, then $P$ is a manifold of codimension $N$, thus of dimension $n + 2$ again. As above, $P$ is of dimension $n + 1$. We observe that the geometric picture does not depend much on the order of $\mathcal{N}$. In the above example of an isothermal fluid, the Euler system with viscosity and capillarity does admit non-constant periodic solutions provided the pressure is a non-monotone function $P$ of the density, as in a Van der Waals equation of state; then dim $P = 3$, as discussed in [16]. We anticipate that the analysis below is valid for a rather general perturbation $\mathcal{N}$, under the assumption that the Cauchy problem for (4) is strongly well-posed. Such an extension would be in the spirit of [1].

The submersion hypothesis ($H_1$) may be seen as generic. However, genericity depends highly on the context: the situation can be significantly different when the system (4) admits an extra conservation law. A supplementary integral has a counterpart at the level of the profile equation (5): the range of the map $U(X; a, s, q) - a$ is contained in some hypersurface of $\mathbb{R}^N$. Hence the genericity would be that its vanishing set be of codimension $N - 1$. In this situation, the dimension of $P$ will be $n + 2$ instead of $n + 1$.

Here are two examples:

- The Korteweg–de Vries equation

$$\partial_t u + 6u\partial_x u + \partial_x^3 u = 0,$$

for which the traveling waves are periodic for an open set of data $(a, s, q)$ and therefore dim $P = 3$ while $n = 1$. It is well-known that this equation has plenty of additional conservation laws.

- The Euler system in presence of capillarity ($\mu > 0$) but in lack of viscosity ($\nu = 0$). This system turns out to conserve energy. One may check directly that dim $P = 4 = n + 2$ when $P$ is a non-monotone function of the density. See [8] for the slow modulation analysis of this case.
2 Slow modulation

An easy way to understand slow modulation is to rescale \((x,t) \mapsto (\epsilon x, \epsilon t)\), so that an \(\epsilon\) factors out in the right-hand side of (1):

\[
\partial_t u + \partial_x f(u) = \epsilon \partial_x (B(u) \partial_x u).
\]

Following Whitham [19], one looks for an homogenized system that describes the mean behaviour of solutions of (6) of the form

\[
u^\epsilon(x,t) = u^0(x,t, \frac{\phi(x,t)}{\epsilon}) + \epsilon u^1(x,t, \frac{\phi(x,t)}{\epsilon}) + \cdots
\]

Hereabove, \(y \mapsto u^0(x,t,y)\) is an unknown 1-periodic function. The local period of the oscillations of \(u^\epsilon\) is therefore \(\epsilon/\partial_x \phi\). The unknown phase \(\phi\) satisfies \textit{a priori} \(\partial_x \phi \neq 0\).

We plug the ansatz (7) into (6) and identify the powers of \(\epsilon\). The most important ones are the leading \((\epsilon^{-1})\) and the sub-leading \((\epsilon^0)\) orders. At order \(\epsilon^{-1}\), we find

\[
(\partial_x \phi)^2 \partial_y (B(u^0) \partial_y u^0) = (\partial_x \phi) \partial_y f(u^0) + (\partial_t \phi) \partial_y u^0.
\]

This is the profile equation (2), up to notations and rescaling, taking

\[
\omega := \partial_x \phi, \quad s := -\frac{\partial_t \phi}{\partial_x \phi}.
\]

In particular, we have

\[
s = S(u^0), \quad \langle u^0 \rangle = M(u^0),
\]

with a slight abuse of notations since the equation was rescaled through \(y \mapsto \omega y\) (that does not change neither \(s, q\) nor the average of the solution).

At next order \(\epsilon^0\), we obtain

\[
\partial_t u^0 + \partial_x (f(u^0) - \omega B(u^0) \partial_y u^0) = \partial_y (\cdots).
\]

Taking the average, we obtain \(n\) conservation laws, in the form

\[
\partial_t M(u^0) + \partial_x (S(u^0) M(u^0) + Q(u^0)) = 0.
\]

Under (H1), the dimension of \(P\) is \(n + 1\), and (9) is not in a closed form, since it consists of \(n\) equations only. However, the Schwarz identity

\[
\partial_t \partial_x \phi = \partial_x \partial_t \phi
\]

provides an additional conservation law\(^1\):

\[
\partial_t \Omega(u^0) + \partial_x (S(u^0) \Omega(u^0)) = 0.
\]

\(^1\)This argument has been used frequently in slow-modulation papers after Whitham’s book [19]. It is present, though hidden, in [16].
The homogenized system is (9,10). It consists in \( n+1 \) first-order conservation laws, in \( n+1 \) scalar unknowns, since the unknown \( \dot{u}^0 \) is an element of \( \mathcal{P} \). It is of evolution type, provided that the Jacobian matrix

\[
\frac{\partial(M,\Omega)}{\partial \dot{u}}
\]

be non-singular. The Cauchy problem for (9,10) is locally well-posed in \( \mathcal{C}^1 \) provided that the matrix

\[
A(\dot{u}) := \left( \frac{\partial(M,\Omega)}{\partial \dot{u}} \right)^{-1} \left( \frac{\partial(SM + Q, S\Omega)}{\partial \dot{u}} \right) = \frac{\partial(SM + Q, S\Omega)}{\partial(M,\Omega)}
\]

is uniformly diagonalizable with real eigenvalues. Rigorously speaking, it is enough that \( A(\omega, q) \) either has real eigenvalues, of constant multiplicities, or that it be uniformly symmetrizable.

2.1 Variations, again

When a more general system of conservation laws of the form (4) is taken at the beginning and a hypothesis similar to (H1) holds, the same procedure works and gives a homogenized system of \( n+1 \) first-order conservation laws, on the \( (n+1) \)-dimensional manifold \( \mathcal{P} \). This was the case for the Van der Waals gas with capillarity and viscosity, as described in [16].

The case where (4) is compatible with an additional conservation law

\[
\partial_t \eta(u, \partial_x u, \ldots) + \partial_x \psi(u, \partial_x u, \ldots) = 0
\]

is interesting too. As mentioned above, \( \mathcal{P} \) will be generically of dimension \( n + 2 \). More generally, in presence of \( m > n \) independent conservation laws, \( \mathcal{P} \) is generically of dimension \( m + 1 \). On the other hand, each conservation law, for \( l = 1, \ldots, m \), yields a zero-order equation

\[
\partial_t \eta_l(u^0, \ldots) + \partial_x \psi_l(u^0, \ldots) = \partial_y (\cdots).
\]

After averaging, this gives \( m \) first-order conservation laws on \( \mathcal{P} \). Together with the obvious conservation law (10), this makes a system of \( m + 1 \) conservation laws on the \( (m + 1) \)-dimensional manifold \( \mathcal{P} \).

Going back to the examples mentioned above, the slow modulation of periodic waves in Korteweg–de Vries equation yields a system of three equations in three unknowns, whose flux is given in terms of elliptic functions. The Euler equations for an isentropic Van der Waals gas, with capillarity but without viscosity, yields a system of four equations in four unknowns described in [8].

An other extension of the procedure consists in modulating space-time quasi-periodic solutions of (4), when non-trivial ones exist. This happens for instance in the Korteweg–de Vries equation. Typically, the leading term in the asymptotic development depends on \( l \) phases \( \psi_1(x, t), \ldots, \psi_l(x, t) \). On the one hand, assume that the set \( \mathcal{P}_l \) of such microstates, modulo translations in space and time, is a manifold whose dimension is denoted by \( p_l \). Assume on the other hand that the system is compatible with \( p \) conservation laws
(including the $n$ conservation laws of (4)). Then the homogenization procedure yields $p$ first-order conservation laws on $P_l$, which must be supplemented with $l$ compatibility conditions for the phases. If $p + l \geq p_l$, we may expect that this system be in a closed form.

Of course, it may happen that $p + l$ is greater than $p_l$. For instance, $p = \infty$ in the case of KdV equation, although each $p_l = 2l + 1$ is finite. In such a case, the homogenized system must be redundant since the unknown runs over $P_l$. In other words, the homogenized system on $P_l$ must admit $p + l - p_l$ additional entropy-flux pairs. This is the reason why the modulation in KdV yields so-called “rich systems” ; see [12, 15]. Recall that the slow modulations of KdV form an infinite hierarchy indexed by $l \in \mathbb{N}$, where $p_l = 2l + 1$. The first level ($l = 0$) concerns the constant solutions, where there is no modulation at all and the “homogenized system” is just the Burgers equation

$$\partial_t u + 3\partial_x (u^2) = 0.$$ 

The next one is the level of non-constant periodic solutions described above.

### 2.2 Link with the stability of periodic waves

In the light of the computations of [16] for a Van der Waals fluid, and of the general analysis of [13], it is expected that the local well-posedness of the Cauchy problem for (9,10), is a necessary (though not sufficient) condition for the spectral stability of the periodic waves of (1).

The case where all the periodic waves are stationary, that is $S \equiv 0$ is particularly interesting since it happens for the Van der Waals fluid in Lagrangian coordinates. Then the homogenized system semi-decouples into

$$\partial_t M + \partial_x Q = 0, \quad \partial_t \Omega = 0,$$

and its hyperbolicity is a property of

$$(13) \quad A_0(\dot{u}) := \partial Q/\partial M$$

alone: it should have a real spectrum and an eigenbasis. This is precisely one of the necessary conditions for spectral stability obtained in [13]. It is not satisfied by the Euler system of a Van der Waals fluid, hence all periodic waves are unstable in this case. But it holds true in other examples.

The situation where $S$ is not identically zero is a little bit wilder. It is of great importance in applications. For instance, the periodic traveling waves in KdV do not share the same velocity. As explained in [14], the presence of a Jordan block associated to the null eigenvalue of $L$ is a serious obstacle on the road of linear stability\footnote{However, we should keep in mind that the property that $S$ be or not a constant, is not intrinsic, since a transformation of the system does not necessarily preserve it. For instance, the situation for a Van der Waals gas depends on whether we work in Lagrangian, or Eulerian coordinates.}, at least against
localized perturbations. There is some hope that quasi-periodic disturbances would lead to better stability results. For instance, homogenization of periodic solutions of KdV equations yields a system that is hyperbolic, as shown in [12]. We might expect that quasi-periodic data could be handled through the inverse scattering machinery.

It should be quoted that in the general case where $S$ is non-constant, the well-posedness for the homogenized system is a property of the matrix $A(\dot{u})$, rather than of $A_0(\dot{u})$. Therefore, one expects that the spectrum of $A(\dot{u})$ is relevant in the study of the weak stability of periodic waves, rather than that of $A_0(\dot{u})$. In other words, the ambiguous “with $s$ held fixed at the base value under consideration, without loss of generality $s = 0$” in [13], pp 129–130, might have been the source of a lack of generality. The main purpose of the present text is to explain in a rigorous way the role that the matrix $A(\dot{u})$ plays in the spectral stability problem. For this purpose, we shall have to unify and develop further the calculations done in [13] and in [16]. In particular, we shall get rid of the assumption made in [13] that the profile equation (2) admit a first integral.

2.3 Open questions and comments

• Which structure is needed in (1) in order that the homogenized system (9,10) admits an entropy-flux pair? Such a structure could guarantee the local well-posedness of the Cauchy problem if it was possible to prove the convexity of the entropy (see [3] and [17] for instance.) Could one also prove the stability of periodic waves in such a framework?

We remind that we found such a pair in the system obtained from the Van der Waals fluid in [16]. This example suggests the following simple mechanism. Assume that (4) is compatible with a dissipation identity, of the form

$$\partial_t \eta(u, \partial_x u, \ldots) + \partial_x \psi(u, \partial_x u, \ldots) + Q(u, \partial_x u, \ldots) = 0,$$

(14)

where $Q$ is a non-negative quantity. Then the profile ODE implies

$$Q(u, u', \ldots) + (\psi - s \eta)' = 0,$$

(15)

which by integration over a period yields $Q \equiv 0$ for periodic solutions, meaning that $u, u', \ldots$ belong to some non-trivial submanifold. Of course, one needs that $Q$ controls only a small part of the derivatives of $u$ in order that non-constant periodic solutions exist. Then, in the homogenization procedure, we face the identity

$$\partial_t \eta(u', \ldots) + \partial_x \psi(u', \ldots) + \frac{1}{\epsilon} Q(u', \ldots) = \partial_y (\cdots),$$

where it happens that the dissipation term $\epsilon^{-1} Q'$ is of order $\epsilon$ (think that $Q$ is quadratic and its argument is small, thus of order $\epsilon$). Averaging and then passing to the limit as $\epsilon \to 0$, we obtain a new homogenized equation that plays the role of an entropy-flux law:

$$\partial_t \langle \eta \rangle + \partial_x \langle \psi \rangle = 0.$$

(16)
The law (14) in the isothermal Van der Waals fluid is provided by the energy balance, where \( Q \) is the Newtonian dissipation for an isentropic fluid. Equation (15) implies that the fluid velocity remains constant. The extra homogenized equation (16) plays the role of energy conservation, since the dissipation vanishes in the limit. This reflects the fact that, at the homogenized level, the fluid behaves as if it was non-isothermal.

We warn the reader about an obstacle in the mechanism described above. Denote \( G := \psi - s\eta \). For the sake of simplicity, we write \( G = G(u, s, q) \), having in mind a parabolic system (1), and where we eliminate \( u_x \) by means of the profile equation. Then we have \( G(u(X; a, s, q)) - G(0) \leq 0 \) for every \( X > 0 \). Using the fact that this expression vanishes for every periodic solution and therefore achieves a maximum, the map \( T \) in \((H1)\) satisfies

\[
\frac{\partial G}{\partial u} \xi^T = 0
\]

along periodic solutions. In particular, \((H1)\) is violated whenever \( \partial G/\partial u \neq 0 \). Thus one needs a very special structure in order to have the correct dimensions for \( P \) and \( \mathcal{P} \). Fluid dynamics with capillarity and viscosity has such a structure, as \( \partial G/\partial u \neq 0 \), but \( \partial G/\partial u \) vanishes along periodic solutions.

- It is hard to say whether discontinuous solutions of the homogenized system make sense in the slow modulation analysis. If so, that would suggest the existence of special solutions of the original system, which behave like two periodic traveling waves \( u^\pm \) as \( x - ct \) tends to \( \pm \infty \) (\( c \) the speed of the discontinuity). In such a case, the term \( \epsilon^{-1}Q^\epsilon \) might tend to a measure, supported by the discontinuity, and (16) would have to be converted into an inequality

\[
\partial_t \langle \eta \rangle + \partial_x \langle \psi \rangle \leq 0.
\]

This speculative comment suggests that the author was wrong in [16], when he considered the average of specific energy as a conserved quantity, and the frequency \( \Omega \) (denoted \( Y \) in that paper) as an entropy. This thought was influenced by the nature of the Euler equations for full gas dynamics.

3 The Evans function

From now on, we fix a space-periodic solution \( u_0 \) of (1) that is steady: \( S(u_0) = 0 \). There is no loss of generality since we always may reduce to this case through the choice of a moving frame. However, we must keep track of this specialization since it could cancel some important terms in the general result. In particular, we have in mind the fact that the term \( S(\dot{u})I_{n+1} \) in the matrix \( A(\dot{u}) \) disappears under our assumption. The correct treatment consists therefore in starting from an arbitrary periodic solution, then choosing a moving frame in which this solution is steady, making the analysis below and then going back to the original reference frame. We leave the reader completing this task and we
limit ourselves to describing the role of \( A(\dot{u}) \) in the case \( S(u_0) = 0 \). We mind however that we do not assume \( S \equiv 0 \); in general, \( S \) is a non-constant function on \( \mathcal{P} \).

Since \( u_0 \) is a steady solution of (1), its spectral stability is encoded in the linearized equation about \( u_0 \):

\[
\partial_t w = Lw. \tag{17}
\]

Because of conservativeness, \( L \) has the form \( \partial_x \ell w \), where \( \ell \) is the first-order differential operator in a single variable, defined by

\[
\ell w = B(u_0)w' + (dB(u_0)w)u'_0 - df(u_0)w = : B_0 w' - A_0 w.
\]

Here, the prime stands for derivation. We warn that \( A_0, B_0 \) are non-constant matrices that are periodic of period \( X_0 = X(u_0) \). An important fact, related to translation invariance is that

\[
\ell u'_0 = 0. \tag{18}
\]

**Definition 1** The solution \( u_0 \) is said to be spectrally unstable if the spectrum of \( L \) contains an element of positive real part. It is weakly spectrally stable otherwise.

Of course, the notion of spectral stability depends on the choice of the functional space where the Cauchy problem is well-posed. Since \( L \) has periodic coefficients, there are two interesting cases, whether the space consists only on functions of the same period \( X_0 \), or it consists on arbitrary functions of given smoothness and/or integrability on \( \mathbb{R} \). The latter notion of stability is stronger than the former. Both cases were considered in [13]. In particular, in the case of spaces of \( X_0 \)-periodic functions, Oh and Zumbrun gave a sign condition that happens to be necessary for spectral stability, in the spirit of [7] and [2]. In the more general setting of a non-periodic functional space, they gave a supplementary necessary condition that appears to be correct only if \( S \equiv 0 \) (or more generally if \( dS(\dot{u}_0) = 0 \)).

The spectral analysis (in \( L^p(\mathbb{R}) \) where \( 1 \leq p \leq \infty \)) of differential operators with periodic coefficients is known as Floquet’s theory. It is described in terms of the \((2n) \times (2n)\) monodromy matrix \( M(X_0; \lambda) \) of the operator \( L - \lambda \), where \( \lambda \) is an arbitrary complex number. This number belongs to the spectrum of \( L \) if and only if

\[
\det(M(X_0; \lambda) - e^{i\theta}I_{2n}) = 0 \tag{19}
\]

for some real number \( \theta \). Such an equality means that the eigenvalue equation

\[
Lw = \lambda w \tag{20}
\]

admits a non-trivial solution with the property that

\[w(x + kX_0) = e^{ik\theta}w(x), \quad k \in \mathbb{Z}.\]

This is equivalent to saying that (20) admits a non-trivial bounded solution.

Following Gardner [4, 5, 6], a convenient way to rewrite (19) is to choose a basis \( \{W_1(\cdot; \lambda), \ldots, W_{2n}(\cdot; \lambda)\} \) of the kernel of \( L - \lambda \), which is holomorphic in terms of \( \lambda \) and
real when $\lambda$ is real. This can be done, using an argument of Kato [10]. We define then the *Evans function* of the problem:

$$D(\lambda, \theta) := \left\| W_j(X_0; \lambda) - e^{i\theta} W_j(0; \lambda) \right\|_{1 \leq j \leq 2n}.$$ 

Then (19) is equivalent to $D(\lambda, \theta) = 0$.

The main result of this article gives an equivalent of $D$ at the origin.

**Theorem 1** Let $u_0$ be a steady solution of (1) that is $X_0$-periodic in space. Denote $\omega_0 := 1/X_0$, assume (H1). Then there exists a number $\Gamma_0 \neq 0$ such that

$$D(\lambda, \theta) = \Gamma_0 \Delta(\lambda, \theta) + O(\lambda^{n+2} + |\theta|^{n+2}),$$

where $\Delta$, a homogeneous polynomial of degree $n + 1$, is given by

$$\Delta(\lambda, \theta) := \det \left( \lambda \frac{\partial (M, \Omega)}{\partial \dot{u}} (\dot{u}_0) + i\omega_0 \theta \frac{\partial (SM + Q, S\Omega)}{\partial \dot{u}} (\dot{u}_0) \right).$$

Since $D$ is a holomorphic function of $(\lambda, \theta)$, we deduce as in [20] the following criterion:

**Theorem 2** Under the assumptions of Theorem 1, assume moreover that

$$\det \left( \frac{\partial (M, \Omega)}{\partial \dot{u}} (\dot{u}_0) \right) \neq 0.$$

Then a necessary condition for spectral stability of $u_0$ is that the spectrum of $A(\dot{u}_0)$ be real, where $A(\dot{u})$ is defined in (13).

As a matter of fact, if $A(\dot{u}_0)$ had a non real eigenvalue, then $\Delta(\cdot, 1)$ would have a root $\lambda^*$ with a non-zero real part. Consider the continuous family of holomorphic functions

$$g_\theta(z) := \theta^{-n-1} D(z\theta, \theta), \quad g_0(z) = \Gamma_0 \Delta(z, 1).$$

By Rouché’s Theorem, $g_\theta$ admits a zero $z(\theta)$ close to $\lambda^*$. Hence $D$ vanishes along a curve $(\lambda(\theta), \theta)$, where $\lambda \sim \theta \lambda^*$ when $\theta \to 0$. There is a sign of $\theta$ for which $\lambda(\theta)$ has a positive real part. Hence $u_0$ is spectrally unstable.

The case where $A(\dot{u}_0)$ has a non-semi-simple real eigenvalue $\alpha$ is interesting too. Assume for the sake of simplicity that the algebraic multiplicity of $\alpha$ is two. Then $g_\theta$ has a double root at some purely imaginary $z^*$. This root splits as two distinct roots $z_{\pm}(\theta)$ of $g_\theta$, of the form $\sigma(\theta) + \sqrt{\delta(\theta)}$, where both $\sigma$ and $\delta$ are analytic, and $\delta(0) = 0$. Generically, one has

$$\frac{d \delta}{d \theta}(0) \neq 0.$$ 

This property implies that for small enough $\theta \in \mathbb{R}$, possibly under a sign condition (if $\delta'(0)$ is real), $g_\theta$ admits two roots whose real parts are of opposite signs. Hence $u_0$ itself is unstable. Whence the improved necessary condition:

**Theorem 3** Under the assumptions of Theorem 2, periodic solutions $u_0$ of which the matrix $A(\dot{u}_0)$ is not diagonalizable are (generically) spectrally unstable.
Remarks.

1. We have proved that a (generically) necessary condition for spectral stability is that the homogenized system be hyperbolic at $\dot{u}_0$. This is very similar to the necessary condition for the stability of shock profiles, that the Cauchy problem for the underlying first-order system

$$\partial_t u + \partial_x f(u) = 0,$$

(24)

given by the right-hand side of (1), be well-posed near the step shock associated to the profile, see [7] in one space dimension and [20] for several space dimensions. As a matter of fact, such shock profiles are micro-states in (24), which thus appears as the limit of (6). A meta-theorem is that the spectral stability of a micro-state needs the well-posedness of the “limit system” at the associated macro-state, here the step shock. The limit system is understood as the one that describes the evolution of slowly modulated waves. In the homogenization procedure described above, the periodic solutions are the micro-states, whereas the homogenized system is the limit system of (6). Since the macro-states are constants, it is not surprising that the stability of the periodic solutions (micro-states) requires the well-posedness of the homogenized system.

2. Theorem 1 might let the reader think that the spectral stability of the micro-state depends only on the properties of the system governing the macro-states. This is far from being true, as noticed in [16]. As far as the Euler equations of a Van der Waals fluid with capillarity and viscosity is concerned, the periodic solutions and the homogenized system do not depend on the specific viscosity term\(^3\), while the linearized operator $L$ does. Depending on the viscous term, a periodic solution may or may not be stable, although the nature of the homogenized system does not change.

3. With the dependency of $D$ on two variables $(\lambda, \theta)$, our one-dimensional periodic case has some feature in common with the case of multi-dimensional shock profiles, as studied in [20]; this was pointed out by Oh & Zumbrun [13]. However, the Taylor expansion in Theorem 1 is of a different nature, as its leading term is a polynomial. We remind that the leading term obtained in [20] was the Lopatinski determinant of the underlying inviscid shock, a non-smooth homogeneous function. Also, the Evans function could not be defined everywhere, due to the presence of an essential spectrum. As in [13, 14], we point out that the tangency (in our present context) to the imaginary axis of the spectrum of $L$ is a pretty bad feature, which forbids high-quality bounds on the Green function, contrary to the case of shock profiles when the inviscid shock is strongly stable.

However, as noticed by the anonymous referee, the tangency of its vanishing set reflects the smoothness of $D$ at the origin and it has a positive counterpart. The

\(^3\)Although it depends dramatically on the fact that viscosity is present.
Evans function is used to construct and to estimate the Green function of the linearized operator. In this “tangent” case, the Green function decays slower than in the uniformly stable case. However, the corresponding estimates are obtained more easily, and with better accuracy, thanks to the smoothness of $D$. Such observations were made by Hoff and Zumbrun [9], in their multidimensional stability analysis of scalar viscous shock fronts. This example is a typical situation where the Lopatinski determinant (there linear) is smooth and vanishes along the imaginary axis.

4. The fact that the leading term in the Taylor expansion of $D$ be of degree $n + 1$ is due to the algebraic multiplicity $n + 1$ of the null eigenvalue of $L$. More generally, this degree always equals that multiplicity. Hence we expect that the homogenized system governing the slow modulations of periodic waves be a system of $N$ conservation laws, with $N$ this common number. See Section 5 for an example where $N = n$ instead!

5. When $\theta$ vanishes, formulæ (21) and (22) imply that

$$D(\lambda, 0) \sim \lambda^{n+1} \Gamma_0 \det \left( \frac{\partial(M, \Omega)}{\partial \dot{u}} (\dot{u}_0) \right)$$

near the origin. This suggests that the stability index, defined as

$$\text{ind}_{u_0} := \text{sign}(D(\lambda, 0)D(\Lambda, 0))$$

for real arguments $0 < \lambda << 1 << \Lambda < +\infty$, has the form

$$\text{ind}_{u_0} = \text{sign} \left\{ \gamma \det \left( \frac{\partial(M, \Omega)}{\partial \dot{u}} (\dot{u}_0) \right) \right\}, \quad (25)$$

where $\gamma$ is some transversality number. This stability index was computed in [13], where Formula (5.48) of Theorem 5.9 can be rewritten as

$$\text{ind}_{u_0} = \text{sign} \left\{ \gamma_0 \det \left( \frac{\partial M}{\partial \dot{u}} \bigg|_{\Omega \equiv \omega_0} (\dot{u}_0) \right) \right\}. \quad (26)$$

The apparent discrepancy between (25) and (26) is an illusion. As a matter of fact, assumption (H3) of [13] implies that $d\Omega(\dot{u}_0) \neq 0$. Then the $(n + 1) \times (n + 1)$ determinant in (25) vanishes if, and only if, the $n \times n$ determinant in (26) does. We emphasize that the computation of the stability index in [13] was done without any restrictive assumption\(^4\). Hence there is no need to revisit this calculus here.

Let $L_0$ be the restriction of $L$ to $X_0$-periodic functions. We recall that when $\text{ind}_{u_0}$ is non-zero, the number of unstable (i.e. positive real part) eigenvalues of $L_0$ is even if, and only if, this index is positive. In particular, a negative stability index implies a spectral instability.

\(^4\)Contrary to the Taylor expansion at the origin.
6. The stability index is non-zero if, and only if, System (9,10) is of evolution type. As mentioned in [13], this property means that, given a small initial disturbance \( \delta u_0 \), periodic with the same period \( X_0 \), so that the solution \( u \) remains \( X_0 \)-periodic, there exists a unique (up to a shift) \( X_0 \)-periodic traveling wave \( \tilde{u} \), close to \( u_0 \), such that \( M(\tilde{u}) = \langle u_0 + \delta u_0 \rangle \). In case of asymptotic stability, \( \tilde{u} \) must be the (orbital) limit of \( u(t) \), because of the conservation of mass.

4 Proof of Theorem 1

There is a lot of arbitrariness in the choice of the basis \( \{W_1(\cdot; \lambda), \ldots, W_{2n}(\cdot; \lambda)\} \). We do not need specifying in more details these functions, except that we fix, for practical purposes,

\[
W_1(\cdot; 0) = u'_0.
\]

This is allowed, thanks to (18). Next, we denote by \( w_j(x) \), \( z_j(x) \) and \( y_j(x) \) the first terms in the Taylor expansion of \( W(x; \lambda) \) at the origin:

\[
W_j(\cdot; \lambda) = w_j + \lambda z_j + \lambda^2 y_j + \cdots
\]

The functions \( w_j \) form a basis of the kernel of \( L \). In particular, these are independent solutions of \( \ell w = \text{cst} \). We immediately see that the functions

\[
\tilde{w}_j = \begin{cases} \frac{\partial}{\partial u_j}(u_0(0), 0, Q(u_0)), & 1 \leq j \leq n, \\
\frac{\partial}{\partial q_j-n}(u_0(0), 0, Q(u_0)), & n + 1 \leq j \leq 2n,
\end{cases}
\]

gives such a basis, except that it does not satisfy the specialization

\[
w_1 = u'_0
\]
in general.

We warn the reader that the functions \( w_j, z_j, \ldots \) do not need to be periodic, except for \( w_1 \). They satisfy the following ODEs

\[
(\ell w_j)' = 0, \quad (\ell z_j)' = w_j, \quad (\ell y_j)' = z_j.
\]

In particular, denoting by \( [h] \) the quantity \( h(X_0) - h(0) \) for a general function \( h \), we have the formulæ

\[
[\ell w_j] = 0, \quad [\ell z_j] = \int_0^{X_0} w_j \, dx, \quad [\ell y_j] = \int_0^{X_0} z_j \, dx,
\]

from which we infer

\[
[\ell z_1] = 0.
\]

These formulæ allow us to give the likely leading terms in an appropriate form of \( D(\lambda, \theta) \). We first rewrite

\[
D(\lambda, \theta) = \left\| \begin{bmatrix} [W_j(\lambda)] + (1 - e^{i\theta})W_j(0; \lambda) \\
[W_j'(\lambda)] + (1 - e^{i\theta})W_j'(0; \lambda) \end{bmatrix} \right\|_{1 \leq j \leq 2n}.
\]
We multiply the second row in (31) by \( B_0(0) \) and then subtract \( A_0(0) \) times the first one. We obtain

\[
(\det B_0(0)) D(\lambda, \theta) = \left\| \left[ W_j(\lambda) \right] + (1 - e^{i\theta}) W_j(0; \lambda) \right\|_{1 \leq j \leq 2n}.
\]

The order of the leading term in each entry depends on the line and the row under consideration:

- For \( j \geq 2 \), the leading term in \([W_j(\lambda)] + (1 - e^{i\theta}) W_j(0; \lambda)\) is \([w_j]\).
- However, for \( j = 1 \), the leading term in \([W_1(\lambda)] + (1 - e^{i\theta}) W_1(0; \lambda)\) is \(\lambda[z_1] - i\theta w_1(0) = \lambda[z_1] - i\theta u_0'(0)\).
- For \( j \geq 2 \), the leading term in \([\ell W_j(\lambda)] + (1 - e^{i\theta}) (\ell W_j)(0; \lambda)\) is, thanks to (29),
  \[\lambda[\ell z_j] - i\theta \ell w_j = \lambda \int_0^{X_0} w_j \, dx - i\theta \ell w_j.\]

Hereabove, we do not need to specify the argument of \( \ell w_j \), since this function is constant.

- However, the latter expression vanishes for \( j = 1 \), because of (18) and (30). Hence the leading term in \([\ell W_1(\lambda)] + (1 - e^{i\theta}) (\ell W_1)(0; \lambda)\) is, using again (18) and (30),
  \[\lambda^2[\ell y_1] - i\theta \lambda (\ell z_1)(0) + \frac{\theta^2}{2}(\ell w_1)(0) = \lambda^2 \int_0^{X_0} z_1 \, dx - i\theta \lambda (\ell z_1)(0).\]

Denote by \( \Delta_1(\lambda, \theta) \) the determinant of leading terms. It has the form

\[
\Delta_1(\lambda, \theta) = \left\| \begin{array}{cccc}
P_1 & P_0 & \cdots & P_0 \\
P_2 & P_1 & \cdots & P_1 \\
\end{array} \right\|,
\]

where \( P_d \) denotes various homogenous polynomials of degree \( d \), with \( n \) components. From this, we infer that \( \Delta_1 \) is a homogeneous polynomial of degree \( n + 1 \), and that

\[
(\det B_0(0)) D(\lambda, \theta) = \Delta_1(\lambda, \theta) + O(|\lambda|^{n+2} + |\theta|^{n+2}).
\]

It thus remains to compute the polynomial \( \Delta_1 \).

### 4.1 The computation of \( \Delta_1(\lambda, \theta) \)

We first remark that \( z_1 \) can be expressed in terms of \( w_1, \ldots, w_{2n} \) and \( \partial u/\partial s(u_0(0), 0, Q_0) \), which we denote simply \( \partial u/\partial s \). As a matter of fact,

\[
Lz_1 = w_1 = u_0' = -L \frac{\partial u}{\partial s}
\]
gives $L(z_1 + \partial u/\partial s) = 0$. Therefore there exists a vector $\alpha \in \mathbb{R}^{2n}$ such that

\begin{equation}
z_1 = -\frac{\partial u}{\partial s} + \sum_{1}^{2n} \alpha_j w_j.
\end{equation}

Writing $\Delta_1$ explicitly as

\begin{equation}
\begin{vmatrix}
\lambda[z_1] - i\theta u'_0(0) \\
\lambda^2 \int_0^{X_0} z_1 \, dx - i\theta \lambda(\ell z_1)(0) \\
\lambda \int_0^{X_0} w_2 \, dx - i\theta \ell w_2 \\
\vdots \\
\lambda \int_0^{X_0} w_{2n} \, dx - i\theta \ell w_{2n}
\end{vmatrix},
\end{equation}

we thus obtain

\begin{equation}
\Delta_1(\lambda, \theta) \equiv \begin{vmatrix}
-\lambda[\partial u/\partial s] - i\theta u'_0(0) \\
\lambda^2 \int_0^{X_0} \partial u/\partial s \, dx + i\theta \lambda(\ell \partial u/\partial s)(0) \\
\lambda \int_0^{X_0} w_2 \, dx - i\theta \ell w_2 \\
\vdots \\
\lambda \int_0^{X_0} w_{2n} \, dx - i\theta \ell w_{2n}
\end{vmatrix},
\end{equation}

noticing that the quantities $[w_1], \int_0^{X_0} w_1 \, dx, \ell w_1$ vanish.

**Lemma 1** Assumption (H1) is equivalent to

$$\mathbb{R}^n = \text{Span} \left\{ \left[ \frac{\partial u}{\partial s} \right], [w_2], \ldots, [w_{2n}], u'_0(0) \right\}.$$ 

**Proof.**

The vectors $[w_1], \ldots, [w_{2n}]$ span the same space as the vectors $\partial u/\partial a_j$ and $\partial T/\partial q_j$ do. Besides, there holds

$$\frac{\partial T}{\partial X} = u'_0(X_0) = u'_0(0)$$

and

$$\frac{\partial T}{\partial s} = [\partial u/\partial s].$$

At last, $[w_1]$ may be omitted in the list since it vanishes.

The tangent space $T_0P$ to $P$ at $u_0$ is made of linear combinations

$$w = \beta_0 \frac{\partial u}{\partial s} + \sum_{1}^{2n} \beta_j w_j,$$

such that $[w]$ is parallel to $u'_0(0)$. In other words, it is obtained by projecting, along the $\beta$-component, the kernel of

\begin{equation}
Z(\beta, \gamma): = \beta_0 \left[ \frac{\partial u}{\partial s} \right] + \sum_{1}^{2n} \beta_j [w_j] + \gamma u'_0(0).
\end{equation}
However, the tangent space $TP$ to $P$ at $\dot{u}_0$ is isomorphic to the quotient of $T_0P$ by the line spanned by $u'_0 = w_1$. Hence it may be identified as the $\beta$-projection of the kernel of

$$Z(\beta_0, \beta_2, \ldots, \beta_{2n}, \gamma) := \beta_0 \left[ \frac{\partial u}{\partial s} \right] + \sum_2^{2n} \beta_j[w_j] + \gamma u'_0(0).$$

We have used $u'_0(0) \neq 0$, as a non-constant solution of a first-order autonomous ODE may not have a critical point.

From now on, we shall never use $w_1$ and therefore we shall feel free to relabel $\beta_1$ instead of $\beta_0$ in the above formula:

$$Z(\beta, \gamma) := \beta_1 \left[ \frac{\partial u}{\partial s} \right] + \sum_2^{2n} \beta_j[w_j] + \gamma u'_0(0), \quad \beta \in \mathcal{C}^{2n}, \quad \gamma \in \mathcal{C}. \quad (37)$$

When computing the differentials that are involved in the right-hand side of (22), we easily recognize

$$dX \cdot (\beta, \gamma) = \gamma, \quad dS \cdot (\beta, \gamma) = \beta_1, \quad dQ \cdot (\beta, \gamma) = -\sum_2^{2n} \beta_j \ell w_j, \quad (38)$$

and thus

$$d\Omega \cdot (\beta, \gamma) = -\omega_0^2 \gamma. \quad (39)$$

On the other hand, differentiation of

$$XM = \int_0^X u(x) \, dx$$

gives

$$d(XM) \cdot (\beta, \gamma) = \gamma u_0(0) + \beta_1 \int_0^X \frac{\partial u}{\partial s} \, dx + \sum_2^{2n} \beta_j \int_0^X w_j. \, dx. \quad (40)$$

Hereabove, we have made use of (38). We point out that the right-hand sides in (38), (39) and (40) are well-defined linear maps on $\mathcal{C}^{n+1}$, although the left-hand sides make sense only on the kernel of $Z$. These right-hand sides are denoted by $\delta X, \ldots, \delta(XM)$. Of course, they define additional linear maps, as $\delta M$, through the standard rule:

$$X_0 \delta M := \delta(XM) - M_0 \delta X.$$

We now compute explicitly the expression $\Delta_2(\lambda, \theta)$ in the right-hand side of (22). It is the determinant of the restriction to ker $Z$ of the linear map

$$T_{\lambda, \theta}(\beta, \gamma) := \left( \begin{array}{c} \lambda \delta \Omega + i \omega_0^2 \theta \delta S \\ \lambda \delta M + i \omega_0 \theta(M_0 \delta S + \delta Q) \end{array} \right).$$

We notice that the notion of determinant is not intrinsic since our maps are not endomorphisms. Thus our determinants are defined up to non-zero constants. What is important
in the sequel is that the constants depend only on the choice of bases in the spaces, but not on the parameters \((\lambda, \theta)\).

In particular, an appropriate choice of the bases yields the formula

\[
\Delta_2(\lambda, \theta) = \det \left( T_{\lambda,\theta}|_{\text{ker} Z} \right) = \det \left( \frac{T_{\lambda,\theta}}{Z} \right).
\]

Therefore, we are led to the computation of the determinant of

\[
(\beta, \gamma) \mapsto \left( \frac{\lambda \gamma - i\theta \beta_1}{\lambda \delta M + i\omega_0 \theta (M_0 \delta S + \delta Q)} \right).
\]

The last line of \(T_{\lambda,\theta}(\beta, \gamma)\) takes the form

\[
\lambda \omega_0 \left( -\gamma (\ell \frac{\partial u}{\partial s})(0) + \beta_1 \int_0^{\lambda_0} \frac{\partial u}{\partial s} \, dx + \sum_{\beta_2}^{2n} \beta_j \int_0^{\lambda_0} w_j \, dx \right) - \lambda \omega_0 M_0 \gamma + i\omega_0 \theta (M_0 \beta_1 - \sum_{\beta_2}^{2n} \beta_j \ell w_j).
\]

Expansion with respect to the fist line amounts to specializing the other lines of \(T_{\lambda,\theta}(\beta, \gamma)\) to the subspace defined by \(\lambda \gamma - i\theta \beta_1 = 0\). Parametrizing this hyperplane by \(\beta_2, \ldots, \beta_{2n}\) and \(\beta_1 = \lambda \rho, \gamma = i\theta \rho\), we obtain

\[
\Delta_2(\lambda, \theta) = \Gamma \det (Y_{\lambda,\theta}),
\]

where \(Y_{\lambda,\theta}\) maps \((\rho, \beta_2, \ldots, \beta_{2n})\) to

\[
\left( \begin{array}{c}
\lambda \rho [\partial u/\partial s] + \sum_{\beta_2}^{2n} \beta_j [w_j] + i\theta \rho u'_0(0) \\
\lambda \omega_0 \{ -i\theta \rho (\ell \partial u/\partial s)(0) + \lambda \rho \int \partial u/\partial s + \sum_{\beta_2}^{2n} \beta_j \int w_j \} - i\omega_0 \theta \sum_{\beta_2}^{2n} \beta_j \ell w_j
\end{array} \right),
\]

and \(\Gamma\) denotes a fixed constant. Therefore,

\[
\Delta_2(\lambda, \theta) = \omega_0^n \Gamma \left| \begin{array}{cc}
\lambda [\partial u/\partial s] + i\theta u'_0(0) & [w_2] \\
\lambda^2 \int \partial u/\partial s - i\theta \lambda (\ell \partial u/\partial s)(0) & \lambda \int w_2 - i\theta \ell w_2
\end{array} \right|.
\]

This exactly give \(\Delta_2(\lambda, \theta) = -\omega_0^n \Gamma \Delta_1(\lambda, \theta)\), as expected. This completes the proof of Theorem 1.

5 Stability of discrete periodic waves

We consider in this section the approximation of a system of conservation laws by finite difference schemes. Typically, such a scheme reads

\[
u_j^{m+1} = u_j^m + \sigma (F_{j-1/2}^m - F_{j+1/2}^m),
\]

where \(\sigma = \Delta t/\Delta x\), the mesh ratio, is usually kept fixed. The numerical flux is given by a formula

\[
F_{j+1/2}^m := F(\sigma; u_{j-p+1}^m, \ldots, u_{j+q}^m).
\]
Since $u_{j}^{n+1}$ is given in terms of $u_{j-p}^{n}, \ldots, u_{j+q}^{n}$ (assuming that $p, q \geq 0$), we speak of a $(p + q + 1)$-points scheme. Although our analysis below can be done in a fairly general context, we shall assume $p = q = 1$ (three-points scheme) for the sake of simplicity. Since $\sigma$ will not vary, we drop this argument in $F$ and write $a$ and $b$ for the two other arguments: $F = F(a, b)$ is defined and smooth enough on some open domain of $\mathbb{R}^{2n}$.

We are interested in the sequel in the spectral stability of space-periodic, stationary solutions of (43). As above, we do not discuss the existence of such a solution; we only assume that there exists at least one, denoted by $\bar{u}$, and make a generic assumption that turns out to ensure the existence of a manifold $P$ of such solutions.

A steady periodic solution $\bar{u}$ is a discrete sequence $(\bar{u}_{j})_{j \in \mathbb{Z}}$, with the property that $u_{j+N} = u_{j}$ ($N$ the period), which satisfies the profile equation

$$F(\bar{u}_{j}, \bar{u}_{j+1}) = \bar{q},$$

with $\bar{q} \in \mathbb{R}^{n}$ a constant. Our first assumption is that (44) is locally solvable in terms of either $u_{j}$ or $u_{j+1}$:

**(H2)** There holds

$$d_{a}F(\bar{u}_{j}, \bar{u}_{j+1}), d_{b}F(\bar{u}_{j}, \bar{u}_{j+1}) \in \text{GL}_{n}(\mathbb{R}), \quad 0 \leq j \leq N - 1.$$

Remark that Godunov-like schemes do not satisfy (H2), while Lax–Friedrichs-like schemes do.

With (H2) in our hand, we may rewrite the general profile equation in an explicit form

$$u_{j+1} = S(u_{j}; q)$$

for profiles that remain close to $\bar{u}$. The map $S(\cdot; q)$ being a local diffeomorphism, our next assumption is generic:

**(H3)** The periodic orbit $\bar{u}$ of $S(\cdot; \bar{q})$ is transversal. In other words, the spectrum of the differential of $S(\cdot; \bar{q})^{N}$ at $\bar{u}_{0}$ does not contain $\lambda = 1$.

This ensures that, in a neighbourhood of $(\bar{u}; \bar{q})$, the set of pairs $(u, q)$, where $q \in \mathbb{R}^{n}$ and $u$ is an $N$-periodic solution of

$$F(u_{j}, u_{j+1}) = q,$$

is smoothly parametrized by $q$. We denote by $u^{q}$ the unique periodic solution associated to $q$, which is close to $\bar{u}$.

The linearized operator $L$ for (43) about $\bar{u}$ is given by

$$(Lv)_{j} = v_{j} + \sigma((Lv)_{j-1/2} - (Lv)_{j+1/2}),$$

where

$$(Lv)_{j+1/2} = d_{a}F(\bar{u}_{j}, \bar{u}_{j+1})v_{j} + d_{b}F(\bar{u}_{j}, \bar{u}_{j+1})v_{j+1}.$$
Notice that $L - 1$ is conservative. In particular, given an $N$-periodic datum $w$, the equation $(L - 1)v = w$ admits a bounded solution if, and only if, the average
\[ \langle w \rangle := \sum_{j=0}^{N-1} w_j \]
vanishes.

Our goal is the analysis of the spectrum of $L$ over $\ell^p$ spaces, in a neighbourhood of the unity. We notice that, as in the “continuous” case, the manifold of steady periodic waves provides a geometric multiplicity $n$, because of the identity
\[ (L - 1) \frac{\partial u^q}{\partial q_l}(\bar{q}) = 0, \quad 1 \leq l \leq n. \] (47)
Actually, the kernel of $L - 1$ is exactly of dimension $n$, because of (H3). However, a significant difference from the continuous case is that, because of the lack of translational invariance, we lose the Jordan part. As a matter of fact, if $(L - 1)v = du^q \cdot \delta q$ holds for some bounded $v$, then (see above) there holds
\[ \langle du^q \cdot \delta q \rangle = 0. \] (48)
Generically, (48) implies that $\delta q = 0$, and therefore $v \in \ker(L - 1)$. We formalize this important observation by making the generic assumption

(H4) The map
\[ q \mapsto U(q) := \langle u^q \rangle \]
is a local diffeomorphism.

Under (H4), the algebraic multiplicity of the unity in the spectrum of $L$ is precisely $n$. Hence we expect that the leading term in the Taylor expansion of the Evans function $D(\lambda, \theta)$ at $(1, 0)$ be of degree $n$, instead of $n + 1$ in the continuous case. We confirm this belief herebelow. For the moment, we define the Evans function as follows. Let $\phi^1(\lambda), \ldots, \phi^{2n}(\lambda)$ be $2n$ linearly independent solutions of
\[ (L - \lambda)\phi = 0, \] (49)
which depend analytically on $\lambda$. We denote\(^5\)
\[ w^l = \phi^l(1), \quad z^l = \frac{\partial \phi^l}{\partial \lambda}(1), \quad 1 \leq l \leq 2n. \] (50)
From discrete Floquet’s Theory, we know that $\lambda$ is a spectral value of $L$ if, and only if, there exists a non-trivial linear combination of the $\phi^l(\lambda)$’s, which is periodic up to a phase rotation $e^{i\theta}$. This amounts to saying that the following determinant vanishes
\[ D(\lambda, \theta) := \begin{vmatrix} \phi^1_N(\lambda) - e^{i\theta} \phi^1_{N+1}(\lambda) & \phi^2_N(\lambda) - e^{i\theta} \phi^2_{N+1}(\lambda) & \cdots & \phi^{2n}_N(\lambda) - e^{i\theta} \phi^{2n}_{N+1}(\lambda) \\ \phi^1_{N+1}(\lambda) - e^{i\theta} \phi^1_N(\lambda) & \phi^2_{N+1}(\lambda) - e^{i\theta} \phi^2_N(\lambda) & \cdots & \phi^{2n}_{N+1}(\lambda) - e^{i\theta} \phi^{2n}_N(\lambda) \end{vmatrix}_{1 \leq l \leq 2n}. \]
The main result of this section gives the leading term in the Taylor expansion of the Evans function:

\(^5\)Contrary to the continuous case, we do not need a second order derivative.
Theorem 4 Assume (H2,H3,H4). For \((\lambda, \theta)\) close to \((1,0)\), there holds
\[
D(\lambda, \theta) \sim \gamma \Delta(\lambda - 1, \theta),
\]
where \(\Delta\) is a homogeneous polynomial:
\[
\Delta(\mu, \theta) = \det \left( \mu \frac{\partial U}{\partial q}(\bar{u}) - i\theta I_n \right),
\]
and the constant \(\gamma\) does not vanish.

From (51), we obtain the local branches of the spectrum, as
\[
\lambda_k(\theta) = 1 + \frac{i\theta}{\alpha_k} + O(\theta^2),
\]
\(\alpha_1, \ldots, \alpha_n\) being the spectrum of \(\partial U/\partial q(\bar{u})\) (remind (H4) that this matrix is non-singular.) Since the spectral stability of \(\bar{u}\) is the property that the spectrum of \(L\) be contained in the unit disc, we deduce:

Corollary 1 Assume (H2,H3,H4). Then a necessary condition for \(\bar{u}\) being spectrally stable is that the spectrum of \(\partial U/\partial q(\bar{u})\) be real.

As in the continuous case, a more careful examination would lead generically to the necessary condition that \(\partial U/\partial q(\bar{u})\) be semi-simple, meaning that the system of conservation laws
\[
\partial_t U(q) + \partial_x q = 0
\]
be hyperbolic. As a matter of fact, System (52) governs the slow modulations of periodic waves in (43). It is obtained by letting \(\Delta x, \Delta t\) going to zero, while keeping the ratio \(\sigma\) fixed, with the ansatz
\[
u^m_j = u^{q(x,t)}_j + l.o.t., \quad x = j\Delta x, t = m\Delta t.
\]
Let us point out that (52) is \(n \times n\), instead of \((n + 1) \times (n + 1)\) in the continuous case. As mentioned above, this lower complexity is due to a lack of translational invariance at the discrete level.

5.1 Proof of Theorem 4

The proof of the theorem follows the same lines as in the continuous case. Of course, we encounter slight differences, as the final result bears a different form. We discuss later on the nature of these differences.

We begin with a linear combinations of rows:
\[
\left( \det d_b F_{1/2} \right) D(\lambda, \theta) := \left\| \phi_N^0(\lambda) - e^{i\theta} \phi_0(\lambda) \right\|_{1 \leq i \leq 2n}.
\]
We rewrite (53) in the following form, with obvious notations:

\[
\left( \det d_b F_{1/2} \right) D(\lambda, \theta) := \left\| \left[ \phi'(\lambda) + (1 - e^{i\theta})\phi_0'(\lambda) \right] \right\|_{1 \leq l \leq 2n}.
\]

Retaining only the leading term in each entry, we have

\[
\left( \det d_b F_{1/2} \right) D(\lambda, \theta) := \Delta_1(\lambda - 1, i\theta) + O(|\lambda - 1|^{n+1} + |\theta|^{n+1}),
\]

where \(\Delta_1\), a homogeneous polynomial of degree \(n\), is defined by

\[
\Delta_1(\mu, \beta) := \left\| \mu[\ell w] - \beta\ell w \right\|_{1 \leq l \leq 2n}.
\]

On the one hand, we remind that \(\ell w\) is a constant for every \(l\). On the other hand, we have from (50) that

\[
[\ell z] = \langle w \rangle.
\]

Finally, there comes

\[
\Delta_1(\mu, \beta) := \left\| \mu\langle w \rangle - \beta\ell w \right\|_{1 \leq l \leq 2n}.
\]

Assumption (H3) exactly tells that the linear map

\[
\ker(L - 1) \rightarrow \mathbb{R}^n,
\]

\[
w \overset{Y}{\rightarrow} [w]
\]

is onto, we see as above that \(\Delta_1\) is, up to some non-zero constant depending on the choice of bases, the determinant of the linear map

\[
\ker Y \rightarrow \mathbb{R}^n,
\]

\[
w \overset{T_{\mu,\beta}}{\mapsto} \mu\langle w \rangle - \beta w.
\]

We conclude with the following observations:

- \(\ker Y\) is precisely the tangent space at \(\bar{u}\) to the manifold \(P\) of \(N\)-periodic, stationary solutions of the scheme,

- \(w \mapsto \langle w \rangle\) is the differential of the moment map

\[
P \rightarrow \mathbb{R}^n,
\]

\[
u \mapsto \langle u \rangle,
\]

- \(w \mapsto \ell w\) is the differential at \(\bar{u}\) of the \(q\)-map

\[
P \rightarrow \mathbb{R}^n,
\]

\[
u \mapsto q := F(u_j, u_{j+1}),
\]

where this definition does not depend on the grid point \(j\). The latter differential is non-singular, from (H4).

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Continuous vs discrete cases. As noticed above, the Evans function does not behave the same way in the continuous case and in the discrete one. As a matter of fact, their Taylor expansion for large wavelength disturbances are related to the well-posedness of a system that governs the slow modulation of periodic waves, and it happens that the system differs dramatically between the two cases. It might be strange, at a first sight, that the same calculation as we performed in the discrete case, may formally be done in the continuous case! The reason why we do not reach a contradiction is that, in the continuous case, the corresponding \( n \times n \) determinant vanishes identically. This is due to the fact that, as long as we focus on \( X \)-periodic waves only, the translational invariance provides a direction on the manifold, along which neither the average, nor the integration constant \( q \), vary. Since the factor \( \Delta \) appearing in Theorem 4 is

\[
\det(\mu \partial U - i\theta \partial q),
\]

where \( U \) denotes the average and \( \partial \) stands for the Jacobian with respect to a suitable parametrization, we find \( \Delta \equiv 0 \).

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References


