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# Matrices : Theory & Applications
# Additional exercises

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## Contents



### Notation

Unless stated otherwise, a letter $k$ or $K$ denoting a set of scalars, actually denotes a (commutative) field.

### Related web pages

See the **solutions** to the exercises in the book on
`http://www.umpa.ens-lyon.fr/˜serre/exercises.pdf`,
and the **errata** about the book on
`http://www.umpa.ens-lyon.fr/˜serre/errata.pdf`.
See also a few **open problems** on
`http://www.umpa.ens-lyon.fr/˜serre/open.pdf`.

## Topics

**Calculus of variations, differential calculus:** Exercises 2, 3, 49, 55, 80, 109, 124, 184, 215, 249, 250, 292, 321, 322, 334, 371, 389, 400, 406, 408, 462, 464

**Complex analysis:** Exercises 7, 74, 82, 89, 104, 180, 245, 354, 391

**Commutator:** 81, 102, 115, 170, 203, 232, 236, 244, 256, 289, 306, 310, 315, 321, 368, 368, 369, 469, 470

**Combinatorics, complexity, P.I.D.s:** Exercises 9, 25, 28, 29, 45, 62, 68, 85, 115, 120, 134, 135, 142, 154, 188, 194, 278, 314, 315, 316, 317, 320, 332, 333, 337, 358, 381, 393, 410, 413, 425, 430, 443, 445, 457

**Algebraic identities:** Exercises 6, 11, 24, 50, 68, 80, 84, 93, 94, 95, 114, 115, 119, 120, 127, 129, 135, 144, 156, 159, 172, 173, 174, 203, 206, 242, 243, 244, 265, 288, 289, 335, 349, 379, 385, 388, 402, 415, 434, 440, 451, 461

**Inequalities:** Exercises 12, 27, 35, 40, 49, 58, 63, 69, 77, 82, 87, 101, 106, 110, 111, 125, 128, 139, 143, 155, 162, 168, 170, 182, 183, 198, 209, 210, 218, 219, 221, 222, 253, 254, 264, 272, 279, 285, 334, 336, 341, 361, 363, 364, 368, 371, 386, 419, 448, 458, 466, 471

**Bistochastic or non-negative matrices:** Exercises 9, 22, 25, 28, 76, 96, 101, 117, 138, 139, 145, 147, 148, 150, 152, 154, 155, 164, 165, 192, 193, 212, 230, 231, 251, 322, 323, 332, 333, 341, 347, 350, 365, 380, 383, 384, 389, 392, 394, 407, 421, 422, 433, 445, 468

**Determinants, minors and Pfaffian:** Exercises 8, 10,11, 24, 25, 50, 56, 84, 93, 94, 95, 112, 113, 114, 119, 120, 127, 129, 143, 144, 146, 151, 159, 163, 172, 181, 188, 190, 195, 206, 207, 208, 213, 214, 216, 221, 222, 228, 234, 246, 270, 271, 272, 273, 274, 278, 279, 285, 286, 290, 291, 292, 308, 335, 336, 349, 357, 372, 374, 383, 384, 385, 386, 387, 393, 400, 414, 424, 427, 430, 434, 446, 451, 453, 454, 457, 461, 462, 463, 467

**Hermitian or real symmetric matrices:** Exercises 12, 13, 14, 15, 16, 27, 40, 41, 51, 52, 54, 58, 63, 70, 74, 75, 77, 86, 88, 90, 92, 102, 105, 106, 110, 113, 116, 118, 126, 128, 143, 149, 151, 157, 164, 168, 170, 180, 182, 184, 192, 198, 199, 200, 201, 209, 210, 211, 215, 216, 217, 218, 219, 223, 227, 229, 230, 231, 239, 241, 248, 258, 259, 263, 264, 271, 272, 273, 274, 279, 280, 285, 291, 292, 293, 294, 301, 307, 308, 312, 313, 319, 326, 328–330, 334, 336, 347, 351, 352, 353, 355, 360, 361, 364, 371, 376, 382, 383, 386, 389, 396, 400, 404, 405, 406, 408, 409, 412, 413, 423, 424, 426, 428, 441, 446, 447, 448, 450, 461, 464, 465, 466, 467, 469, 474

**Orthogonal or unitary matrices:** Exercises 21, 64, 72, 73, 81, 82, 91, 101, 116, 158, 226, 236, 255, 257, 276, 277, 281, 297, 302, 314, 343, 344, 348, 357, 358, 366, 395, 417, 426, 437, 442, 444, 450, 471, 475, 476

**Norms, convexity:** Exercises 7, 17, 19, 20, 21, 40, 65, 69, 70, 71, 77, 81, 82, 87, 91, 96, 97, 100, 103, 104, 105, 106, 107, 108, 109, 110, 111, 117, 125, 128, 131, 136, 137, 138, 140, 141, 153, 155, 162, 170, 183, 199, 223, 227, 242, 243, 261, 297, 299, 300, 301, 302, 303, 313, 323, 361, 366, 368, 380, 408, 411, 420, 455, 460, 464, 472

**Eigenvalues, characteristic or minimal polynomial:** Exercises 22, 31, 37, 41, 47, 61, 69, 71, 83, 88, 92, 98, 99, 101, 113, 116, 121, 123, 126, 130, 132, 133, 145, 146, 147, 156, 158, 168, 175, 177, 185, 186, 187, 194, 195, 205, 252, 256, 260, 267, 269, 276, 277, 294, 295, 304, 305, 339, 340, 344, 359, 362, 367, 373, 376, 379, 384, 398, 403, 418, 438, 439, 441, 443, 449, 455, 456, 458, 474

**Diagonalization, trigonalization, similarity, Jordanization:** Exercices 4, 5, 85, 100, 187, 189, 191, 205, 212, 240, 282, 284, 287, 324, 325, 326–330, 338, 339, 340, 345, 359, 360, 378, 395, 426, 431, 432, 444, 450, 456, 469

**Singular values, polar decomposition, square root:** Exercises 30, 46, 51, 52, 69, 82, 100, 103, 108, 109, 110, 111, 139, 147, 162, 164, 185, 198, 202, 226, 240, 254, 261, 262, 281, 282, 296, 304, 354, 363, 370, 394, 395, 396, 401, 417, 450, 466, 476

**Classical groups, exponential:** Exercises 30, 32, 48, 53, 57, 66, 72, 73, 78, 79, 106, 123, 148, 153, 176, 179, 196, 201, 202, 241, 246, 268, 295, 316, 321, 436

**Numerical analysis:** Exercises 33, 42, 43, 65, 67, 140, 194, 197, 344, 353, 355, 392, 401, 409

**Matrix equations:** Exercises 16, 34, 38, 59, 60, 64, 166, 167, 237, 238, 319, 325, 331, 399, 435, 442

**Other real or complex matrices:** Exercises 26, 35, 39, 46, 80, 103, 118, 131, 166, 171, 185, 220, 221, 232, 242, 243, 245, 247, 249, 250, 253, 268, 269, 284, 290, 298, 303, 304, 306, 309, 311, 324, 331, 351, 356, 360, 369, 373, 391, 403, 418, 429, 433, 438, 452, 454, 459

**Differential equations:** Exercises 34, 44, 122, 123, 124, 145, 195, 196, 200, 236, 241, 263, 295, 365, 418

**General scalar fields:** Exercises 1, 2, 4, 5, 6, 10, 18, 23, 36, 115, 133, 160, 161, 167, 169, 173, 174, 178, 179, 181, 186, 187, 204, 206, 207, 213, 214, 224, 225, 237, 238, 239, 244, 256, 265, 266, 283, 288, 289, 310, 316, 317, 318, 320, 339, 340, 345, 349, 359, 372, 375, 377, 390, 399, 427, 430, 431, 432, 435, 444, 463, 470, 475

**Algorithms:** 11, 24, 26, 42, 43, 67, 88, 95, 112, 344, 353, 355, 392, 397

**Factorizations.** 41, 344, 349, 351, 370

**Just linear algebra:** 233, 235, 375, 382, 388

## More specific topics



## Themes of the exercises

**1.** Similarity within various fields. IX

**2.** Rank-one perturbations. III

**3.** Minors ; rank-one convexity.

**4, 5.** Diagonalizability. III

**6.** $2 \times 2$ matrices are universaly similar to their matrices of cofactors. III

**7.** Riesz–ThorinRieszThorin by bare hands. VII

**8.** Orthogonal polynomials. III

**9.** Birkhoff'sBirkhoff Theorem ; wedding lemma. VIII

**10.** Pfaffian ; generalization of Corollary 7.6.1. X

**11.** Expansion formula for the Pfaffian. Alternate adjoint. IIII

**12.** Multiplicative Weyl'sWeyl inequalities. VI

**13, 14.** Semi-simplicity of the null eigenvalue for a product of Hermitian matrices ; a contraction. VI

**15, 16.** ToeplizToepliz matrices ; the matrix equation $X + A^*X^{-1}A = H$. VI

**17.** An other proof of Proposition 3.4.2. VI

**18.** A family of symplectic matrices. X

**19, 20.** Banach–MazurBanachMazur distance. VII

**21.** Numerical range.

**22.** JacobiJacobi matrices ; sign changes in eigenvectors ; a generalization of Perron-FrobeniusPerronFrobenius for tridiagonal matrices. VIII

**23.** An other second canonical form of square matrices. IX

**24.** A recursive algorithm for the computation of the determinant. III

**25.** Self-avoiding graphs and totally positive matrices. VIII

**26.** SchurSchur parametrization of upper HessenbergHessenberg matrices. III

**27.** Weyl'sWeyl inequalities for weakly hyperbolic systems. VI

**28.** $\mathbf{SL}_2^+(\mathbb{Z})$ is a free monoid.

**29.** Sign-stable matrices. V

**30.** For a classical group $G$, $G \cap \mathbf{U}_n$ is a maximal compact subgroup of $G$. X

**31.** Nearly symmetric matrices. VI

**32.** Eigenvalues of elements of $\mathbf{O}(1, m)$. VI, X

**33.** A kind of reciprocal of the analysis of the relaxation method. XII

**34.** The matrix equation $A^*X + XA = 2\gamma X - I_n$. VI, X

**35.** A characterization of unitary matrices through an equality case. V, VI

**36.** Elementary divisors and lattices. IX

**37.** Companion matrices of polynomials having a common root. IX

**38.** Matrices $A \in \boldsymbol{M}_3(k)$ commuting with the exterior product.

**39.** Kernel and range of $I_n - P^*P$ and $2I_n - P - P^*$ when $P$ is a projector. V

**40.** Multiplicative inequalities for unitarily invariant norms. VII

**41.** Largest eigenvalue of Hermitian band-matrices. VI

**42, 43.** Preconditioned Conjugate Gradient. XII

**44.** Controllability, Kalman'sKalman criterion. X

**45.** NashNash equilibrium. The game "scissors, stone,...". V

**46.** Polar decomposition of a companion matrix. X

**47.** Invariant subspaces and characteristic polynomials. III

**48.** Eigenvalues of symplectic matrices. X

**49.** From Compensated-Compactness. V

**50.** Relations (syzygies) between minors. III

**51.** The square root map is analytic. VI

**52.** The square root map is (globally) Hölderian. VI

**53.** Lorentzian invariants of electromagnetism. X

**54.** Spectrum of blockwise Hermitian matrices. VI

**55.** Rank-one connections, again. III

**56.** The transpose of cofactors. Identities. III

**57.** A positive definite quadratic form in Lorentzian geometry. X

**58.** A convex function on $\mathbf{HPD}_n$.

**59.** When elements of $N + \mathbf{H}_n$ have a real spectrum. VI

**60.** When elements of $M + \mathbf{H}_n$ are diagonalizable. VI

**61.** A sufficient condition for a square matrix to have bounded powers. V, VII

**62.** Euclidean distance matrices. VI

**63.** A Jensen'sJensen trace inequality. VI

**64.** A characterization of normal matrices. V

**65.** Pseudo-spectrum. V

**66.** Squares in $\mathbf{GL}_n(\mathbb{R})$ are exponentials. X

**67.** The Le Verrier–Fadeevlever@Le VerrierFadeev method for computing the characteristic polynomial. III

**68.** An explicit formula for the resolvent. III

**69.** Eigenvalues *vs* singular values (**New formulation**.) IV, VI

**70.** HornHorn!Roger & Schur'sSchur theorem. VI

**71.** A theorem of P. LaxLax. VI

**72, 73.** The exchange map. III

**74.** Monotone matrix functions. VI

**75.** $S \mapsto \log\det S$ is concave, again.

**76.** An application of Perron–FrobeniusPerronFrobenius. VIII

**77.** A vector-valued form of the Hahn–BanachHahnBanach theorem, for symmetric operators. VII

**78.** Compact subgroups of $\mathbf{GL}_n(\mathbb{C})$. X

**79.** The action of $\mathbf{U}(p,q)$ over the unit ball of $\mathbf{M}_{q\times p}(\mathbb{C})$. X

**80.** Balian'sBalian formula. VI

**81.** The "projection" onto normal matrices. VI

**82.** Von Neumannvonneu@von Neumann inequality.

**83.** Spectral analysis of the matrix of cofactors. III

**84.** A determinantal identity. III

**85.** Flags.

**86.** A condition for self-adjointness. VI

**87.** Parrott'sParrott Lemma. VII

**88.** The signs of the eigenvalues of a Hermitian matrix. VI

**89.** The necessary part in Pick'sPick Theorem. VI

**90.** The borderline case for PickPick matrices. VI

**91.** Normal matrices are far from triangular. V

**92.** Tridiagonal symmetric companion matrix. III, V

**93.** An extension of the formula $\det A = (\mathrm{Pf} A)^2$. III

**94.** The Pfaffian of $A + x \wedge y$. III

**95.** A formula for the inverse of an alternate matrix. III

**96.** Isometries of $(\mathbb{R}^n; \ell^p)$ when $p \neq 2$. VII

**97.** Iteration of non-expansive matrices. VII

**98.** The eigenvalues of a $4 \times 4$ alternate matrix. III

**99.** An orthogonal companion matrix. V

**100.** The numerical range of a nilpotent matrix. V

**101.** The trace of $AB$, when $A$ and $B$ are normal. V

**102.** Compensating matrix. VI

**103.** A characterization of singular values. VII

**104.** The assumption in von Neumannvonneu@von Neumann Inequality cannot be weakened. V

**105.** An inequality for Hermitian matrices. VI

**106.** The LipschitzLipschitz norm of $A \mapsto e^{iA}$ over Hermitian matrices. X

**107.** When $(A \in H_n^+) \implies (\|A^2\| = \|A\|^2)$ VI, VII

**108.** Convexity and singular values. VII

**109.** Rank-one convexity and singular values. V

**110.** The square root in operator norm. VI, VII

**111.** And now, the cubic root. VII

**112.** A criterion for a polynomial to have only real roots. VI

**113.** Some non-negative polynomials on $\mathbf{Sym}_n(\mathbb{R})$. VI

**114.** Invariant factors of the matrix of cofactors. IX

**115.** About $\omega$-commuting matrices. V

**116.** How far from normal is a product of Hermitian matrices. VI

**117.** The extremal elements among symmetric, bistochastic matrices. VIII

**118.** "Non-negative" linear forms on $\mathbf{M}_n(\mathbb{C})$. V, VI

**119.** HilbertHilbert matrices. III

**120.** When blocks commute, the determinant is recursive. III

**121.** The characteristic polynomial of some blockwise structured matrix. III

**122.** The GreenGreen matrix for ODEs. X

**123.** Stable, unstable and neutral invariant subspaces.

**124.** The LopatinskiĭLopatinskiĭ condition in control theory. X

**125.** A trace inequality. V

**126.** Can we choose a tridiagonal matrix in Theorem 3.4.2 ? VI

**127.** $4 \times 4$ matrices. III

**128.** The Hölderhold@Hölder inequality for $A \mapsto A^\alpha$. VI

**129.** A determinantal identity. III

**130.** A disk "à la Gershgorin"Gershgorin. V

**131.** Making the diagonal constant. V

**132.** Connected components of real matrices with simple eigenvalues. V

**133.** Characteristic and minimal polynomials. IX

**134.** The converse of Proposition 8.1.3. XI

**135.** Winograd's computationWinograd.

**136.** Unitary invariant norms and adjunction. VII

**137, 138.** Convex subsets of the unit sphere of $\mathbf{M}_n(\mathbb{R})$. VIII

**139.** An other inequality from von Neumannvonneu@von Neumann. XI

**140.** Joint spectral radius. VII

**141.** Bounded semigroups of matrices. VII

**142.** Commutators and palindromes. III

**143.** Determinantal inequalities for $\mathbf{SDP}_n$ ToeplizToepliz matrices. VI

**144.** Generalized Desnanot–JacobiDesnanotJacobi formula. III

**145.** An entropy inequality for positive matrices. VIII

**146.** Exterior power of a square matrix.

**147.** The spectrum of a totally positive matrix. VIII

**148.** Totally non-negative semi-groups. VIII

**149.** More about Euclidean distance matrices. VI

**150.** Farkas' LemmaFarkas. VIII

**151.** The hyperbolic polynomial $X_0^2 - X_1^2 - \cdots - X_r^2$. V, VI

**152.** Another proof of Birkhoff's Theorem. VIII

**153.** $\mathbf{SO}_2(\mathbb{R})$ and $\mathbf{O}_2^-(\mathbb{R})$ are linked inside the unit sphere of $\mathbf{M}_2(\mathbb{R})$. X

**154.** The combinatorics of $\Delta_3$ as a polytope. VIII

**155.** Estimates of the joint spectral radius. VIII

**156.** The characteristic polynomial of a cyclic matrix. VIII, IX

**157.** Flat extension of a Hermitian matrix. VI

**158.** Spectrum of a normal matrix and of its principal submatrices. V

**159.** The characteristic polynomial of a cyclic matrix (bis). III

**160.** Schur's LemmaSchur. III

**161.** Isomorphic but not conjugated nilpotent sub-algebras of $\mathbf{M}_n(k)$. III

**162.** The unitarily invariant norms. VII

**163.** The mapping $X \mapsto A^T X A$ on the space of symmetric matrices. III

**164.** Spectral gap for tridiagonal symmetric stochastic matrices. VIII

**165.** Primitive non-negative matrices. VIII

**166.** Solvability of $AXB = C$. IX, XI

**167.** Solvability of $AX - XB = C$. III

**168.** The spectrum of $A + B$ when $A, B$ are Hermitian and $B$ is small (a case of the A. Horn'sHorn!Alfred problem). IV, VI

**169.** Theorem 6.2.1 is not valid in rings that are not PID. IX

**170.** HeisenbergHeisenberg Inequality. VI

**171.** Lattices in $\mathbb{C}^m$.

**172.** Schur's Pfaffian identity.Schur III

**173.** Hua IdentityHua, Loo-Keng. III

**174.** JordanJordan!Pascual and ThedyThedy Identities for symmetric matrices. III

**175.** Diagonalization in the reals and projectors. V

**176.** Exponential in a JordanJordan!Pascual algebra. X

**177.** A characterization of the minimal polynomial. IX

**178.** The rank of SchurSchur complement. III

**179.** Group-preserving functions. X

**180.** Hyperbolic Hermitian pencils. VI

**181.** The reciprocal of Exercise 127. III

**182.** Sums of eigenvalues of a Hermitian matrices. VI

**183.** The $p$-norms of a matrix and its absolute value. VII

**184.** Inf-convolution. VI

**185.** A generalization of the fundamental Lemma of Banach algebras. VII

**186.** The characteristic polynomial and the invariant factors of submatrices. IX

**187.** The DunfordDunford decomposition.

**188.** The SmithSmith determinant. III

**189.** Similar permutation matrices. III

**190.** The determinant of $M^TM$. III

**191.** Matrices that are conjugated to their inverse. III, IX

**192.** Two kinds of positivity. V

**193.** A criterion for the spectral radius of non-negative matrices. VII

**194.** An improvement in Le Verrier'slever@Le Verrier method.

**195.** Strongly stable matrices. V

**196.** A formula for $\exp(tA)$. X

**197.** Symmetric eigenvalue problem: The order of the JacobiJacobi method when $n = 3$. XIII

**198.** The geometric mean of positive semi-definite Hermitian matrices. VI

**199.** Interpolation of Hermitian spaces. VI

**200.** Homogenization of elliptic operators. VI

**201.** Hamiltonian matrices. X

**202.** The exponential and the index. X

**203.** A polynomial identity in $\mathbf{M}_n(k)$. IV

**204.** Modular subgroups of $\mathbf{GL}_n(\mathbb{Z})$ are torsion free (with one exception).

**205.** Partial similarity and characteristic polynomial. IX

**206.** An identity in $\mathbf{M}_3(k)$.

**207.** Rank and kernel. IIII

**208.** A determinantal identity. III

**209.** $H \mapsto (\det H)^{1/n}$ is concave over $\mathbf{HPD}_n$.

**210.** A scalar inequality for vectors, which implies an inequality between Hermitian matrices. VI

**211.** An extension problem for Hermitian positive semi-definite matrices. VI

**212.** Conjugation classes in $\mathbf{SL}_2(\mathbb{Z})$.

**213.** Tensor product and determinant (I). III

**214.** Tensor product and determinant (II). III

**215.** Differentiation over $\mathbf{Sym}_n$. VI

**216.** A converse of Exercise 13 of Chapter 3. VI

**217.** Diagonalizability of pencils of real quadratic forms.VI

**218.** $H \mapsto (\det H)^{1/n}$ is concave over $\mathbf{HPD}_n$ (bis).

**219.** $H \mapsto (\det H)^{1/n}$ is concave over $\mathbf{HPD}_n$ (ter)

**220.** Symmetry group of the unit ball of $\mathbf{M}_{p\times q}(\mathbb{C})$. X

**221.** Hua's InequalityHua, Loo-Keng. V, VI

**222.** Eigenvalues *vs* singular values (II). IV, XI

**223.** A convex range for quadratic maps. V

**224.** Nilpotence. III

**225.** The dimension of nilpotent subspaces. III

**226.** The AluthgeAluthge transform. X

**227.** Computation of a convex envelop. VI

**228.** The topology of $\mathbf{Alt}_4(\mathbb{R}) \cap \mathbf{GL}_4(\mathbb{R})$. III, VI

**229.** SchurSchur complement for positive semi-definite matrices. VI

**230, 231.** Doubly stochastic $n$-tuples. VIII

**232.** A matricial equation. V

**233.** Search of an adapted orthogonal basis in $\mathbb{R}^3$.

**234.** The Pfaffian of simple alternate matrices. III

**235.** Adapted bases.

**236.** Isospectral flows. X

**237, 238.** The commutant of a square matrix. III

**239.** Sums of squares. VI

**240.** Unitarily similar matrices. V

**241.** Symmetry of a resolvant. VI, X

**242.** ZariskiZariski closure of the unit sphere of $(\mathbf{M}_2(\mathbb{R}), \|\cdot\|_2)$. VII

**243.** The algebraic nature of the unit sphere of $(\mathbf{M}_n(\mathbb{R}), \|\cdot\|_2)$. VII

**244.** A polynomial identity in $\mathbf{M}_2(k)$. III

**245.** Pencils in $\mathbf{GL}_n(\mathbb{C})$. V

**246.** A determinantal identity for orthogonal matrices. V

**247.** Small subgroups of $\mathbf{GL}_n(\mathbb{C})$ are finite. VII, X

**248.** A positive definite Hermitian matrix. VI

**249.** Differentials of power and exponential. V, X

**250.** Differentiating matrix functions. V

**251.** The strong PerronPerron–FrobeniusFrobenius property. VIII

**252.** Strange minimal polynomial ! IX

**253.** The power inequality for the numerical radius. VII

**254.** A generalized CauchyCauchy–SchwarzSchwarz inequality. V

**255.** Fuglede'sFuglede's theorem and its consequences. V

**256.** Commutators and nilpotence. III

**257.** A rational map from $\mathbf{U}_n$ to $\mathbf{U}_m$. X

**258.** The product of Hermitian matrices of which one is positive definite. VI

**259.** The *spread* of a Hermitian matrix. VI

**260.** The eigenvalues of $A^{-1}A^*$. X

**261.** Projection onto the unit ball of $\mathbf{M}_n(\mathbb{C})$. VII, X

**262.** An algorithm for the polar decomposition. X

**263.** Self-adjoint differential equations. X

**264.** A "Cauchy-Schwarz inequality" for the geometric mean. VI

**265.** Near generalized inverses. XI

**266.** Polynomial group action over nilpotent matrices. III

**267.** Matrices whose roots have integral entries. III

**268.** A characterization of finite subgroups of $\mathbf{U}_n$. X

**269.** The convex hull of the spectrum *vs* the numerical range. V

**270.** Matrices whose roots have integral entries (*bis*). III

**271.** A generalization of Hua's InequalityHua, Loo-Keng. V, X

**272.** Bellman'sBellman positive matrix. VI

**273.** Symplectic matrices and SiegelSiegel domain. X

**274.** Some $3 \times 3$ matrices, and planar triangles. V

**275.** The determinant of a sum in $\mathbf{HPD}_2$. VI

**276.** The diagonal and the spectrum of normal matrices. V

**277.** The property **L**. III

**278.** Sylvester'sSylvester Lemma. III

**279.** Oppenheim'sOppenheim (Sir A.) Inequality. VI

**280.** Positive unital linear maps. VI

**281.** Normality of $A, B, AB$ and $BA$. V

**282.** Unitary and orthogonal similarity. V, X

**283.** Matrix version of LagrangeLagrange interpolation. IX

**284.** Invariant planes for real matrices. V

**285.** An improvement of the Oppenheim'sOppenheim (Sir A.) Inequality. VI

**286.** A snail determinant. III

**287.** Matrices being diagonalizable within $\mathbf{M}_n(\mathbb{F}_p)$. III, IX

**288.** A preparation for the next one.

**289.** The Amitsur–LevitzkiAmitsurLevitzki Theorem (after S. RossetRosset!Shmuel).

**290.** The determinant and trace for positive definite matrices. V

**291.** The theorem of CraigCraig & SakamotoSakamoto. VI

**292.** The LegendreLegendre transform of $H \mapsto -\log\det H$. VI

**293.** Ando'sAndo supremum of Hermitian matrices. VI

**294.** Stability issues in HamiltonianHamilton systems. X

**295.** The exponential as a polynomial. X

**296.** The negative second moment identity. V

**297.** Von Neumann'svonneu@von Neumann proof of Fuglede'sFuglede theorem. V

**298.** Embeddings from $\mathbf{M}_m(\mathbb{C})$ into $\mathbf{M}_n(\mathbb{C})$. V

**299.** The operator norm of a nilpotent matrix of order two. VII

**300.** Contractions and ToeplizToepliz matrices. VI, VII

**301.** Positive operators that sum up to the identity. VI

**302.** A characterization of unitary matrices. V

**303.** Anticommutative systems. VII

**304.** Singular values *vs* restriction over subspaces. XI

**305.** Unit HessenbergHessenberg matrix with prescribed RitzRitz values. III

**306.** Equivalence classes under unitary conjugation in $\mathbf{M}_2(\mathbb{C})$. V

**307.** From a projection to an orthogonal projection. V, VI

**308.** Conjugation of a convex function over $\mathbf{H}_n$. VI

**309.** The resolvant and the numerical range. V

**310.** A characterization of nilpotent matrices. III

**311.** The real part of a nilpotent matrix. V

**312.** The orthogonal of a positive definite Hermitian matrix. VI

**313.** The method of alternate projections. V, VI

**314.** HadamardHadamard matrices of size $2^m$. III

**315.** Zero trace matrix is a commutator. III

**316.** The unipotent group. III

**317.** Non-isomorphic groups whose elements have the same orders. III

**318.** Elements in $\mathbf{GL}_n(\mathbb{Z}) \cap (I_n + p\mathbf{M}_n(\mathbb{Z}))$. III

**319.** Matrices $AB + BA$ with $A, B \in \mathbf{HPD}_n$. VI

**320.** Subsets with odd cardinals and even intersections. III

**321.** Formula for $e^{A+B}$ when $[A, B] = A$. X

**322.** Generators of MarkovianMarkov semi-groups. VIII

**323.** 'Log-convexity' of the Perron–FrobeniusPerronFrobenius eigenvalue. VIII

**324.** A simple algebra. III, IV, VI

**325.** The Cecioni–FrobeniusCecioniFrobenius theorem. IX

**326.** Symmetric matrices that are compatible, in the sense of linear elasticity. VI

**327–330.** HornHorn!Roger & Johnson'sJohnson!Charles R. Theorem 4.1.7: matrices similar to real ones, matrices similar to their Hermitian adjoint, products of Hermitian matrices. V, VI

**331.** Sums of squares in $\mathbf{M}_n(\mathbb{R})$. V

**332.** Sums of permutation matrices. VIII

**333.** The maximum of diagonal sums. VIII

**334.** Operator monotone functions; LoewnerLoewner's Theorem. VI

**335.** The relative gain array (after C. R. JohnsonJohnson!Charles R. & H. ShapiroShapiro). III

**336.** The relative gain array. Positive definite case. III, VI

**337.** A bound for the permanent. III

**338.** Similarity in $\mathbf{GL}_2(\mathbb{Z})$. III, IX

**339.** Diagonalizable "companion" matrix. IX

**340.** Test for diagonalizability. III

**341.** The inverse of a strictly diagonally dominant matrix. V, XII

**342.** The relative gain array. Strictly diagonally dominant case. V

**343.** The relative gain array. The case of permutation matrices. VIII

**344.** Shifted QR method (Francis'Francis algorithm.) XIII

**345.** Matrix commuting with a non-derogatory matrix. IX

**346.** Matrix commuting with a non-derogatory matrix and its transpose. V, IX

**347.** Semipositive definite symmetric matrices that have non-negative entries. VI

**348.** Parametization of $\mathbf{SO}_3(\mathbb{R})$ by unitary quaternions. X

**349.** Explicit $LU$ factorization. XI

**350.** Tridiagonal bistochastic matrices. VIII

**351.** The instability of factorization $M = \mathbf{H}_n \cdot \mathbf{H}_n$ for $M \in \mathbf{M}_n(\mathbb{R})$. V, VI, VII

**352.** Maximal positive / negative subspaces of a Hermitian matrix and its inverse. VI

**353.** Irrelevance of the choice of angle in the method of JacobiJacobi. XIII

**354.** A square root cannot be defined over $\mathbf{GL}_2(\mathbb{C})$. V, X

**355.** The Gauss–SeidelGaussSeidel algorithm for positive semi-definite matrices. XII

**356.** The numerical range of $A$ and $A^{-1}$. V

**357.** In orientable Riemannian geometry, the HodgeHodge star operator is an isometry. X, XI

**358.** Five-fold symmetry of lattices (after M. Křížek, J. Šolc and A. ŠolcováKriz@Křížeksolc@Šolcsolco@Šolco
III

**359.** The similarity of a matrix and its transpose (after O. Taussky and H. ZassenhausTausskyZassenhaus). III, IX

**360.** When a real matrix is conjugate to its transpose through a symmetric positive definite matrix. V, VI, IX

**361.** The convex cone spanned by the squares of skew-symmetric matrices. V

**362.** Computing the multiplicity of an eigenvalue from a polynomial equation $p(M) = 0$. III

**363.** Singular values *vs* eigenvalues. V, XI

**364.** $\operatorname{Tr} e^H e^K$ *vs* $\operatorname{Tr} e^{H+K}$ when $H, K$ are Hermitian. VI

**365.** Linear differential equation whose matrix has non-negative entries. X

**366.** The supporting cone of $\mathbf{SO}_n(\mathbb{R})$ at $I_n$. V

**367.** The spectrum of $(B, C) \mapsto ((BX - XC)X^T, (X^T(XC - BX))$. V

**368.** A proof of the Böttcher–Wenzelbottcher@BöttcherWenzel Inequality. VII

**369.** The numerical radius of a commutator. V

**370.** Cholesky *vs* polar factorization. X, XI

**371.** LoewnerLoewner's Theory of operator monotone functions. VI

**372.** Subspaces formed of singular matrices. III

**373.** A problem about coins and weights.

**374.** The rank of some integer matrices. III

**375.** Mixing two idempotent matrices. III

**376.** Spectrum and the equation $M = A + A^*$. V, VI

**377.** Finding $0_n$ by multiplying factors $sA + tB$, $(s, t) \neq (0, 0)$. V

**378.** Polynomials $P$ such that $P(A)$ is diagonalizable for every $A$. IX

**379.** A universal annihilator on $\mathbf{M}_n(\mathbb{F}_p)$. III

**380.** Non-negative matrices and the Hilbert distance. VIII

**381.** Left- and right- annhilators of $A \in \mathbf{M}_n(R)$ where $R$ is finite.

**382.** Linear forms over quadratic forms.

**383.** The cone of non-negative symmetric matrices with positive entries. VIII

**384.** The determinant and the permanent as eigenvalues. III

**385.** FrobeniusFrobenius determinant formula. III, V

**386.** An other proof of HadamardHadamard Inequality. XI

**387.** An integral fraction. III

**388.** Special composition of symmetries. III

**389.** A nonlinear eigenvalue problem. VI, VIII

**390.** Idempotent matrices that are sums of idempotent matrices. III

**391.** A third proof of Fuglede'sFuglede Theorem. VI, X

**392.** Iterative method for a linear system: the case of positive matrices. VIII, XII

**393.** A matrix related to Boolean algebra. III

**394.** Singular values of bistochastic matrices. VIII, XI

**395.** $T^*T$ and $TT^*$ are unitarily similar. V, XI

**396.** A decomposition lemma in $\mathbf{H}_n^+$. VI

**397.** Numerical range *vs* diagonal. V

**398.** Achieving a prescribed characteristic polynomial. III

**399.** The WaringWaring problem for matrices.

**400.** A proof that $S \mapsto \log\det S$ and $S \mapsto (\det S)^{1/n}$ are concave over $\mathbf{SDP}_n$. VI

**401.** Numerical analysis of the polar decomposition. X

**402.** An identity involving the cofactor matrix. III

**403.** Geometric multiplicity *vs* GershgorinGershgorin disks. V

**404.** HadamardHadamard product *vs* $A \mapsto AA^*$. V

**405.** HadamardHadamard factorization of positive semidefinite matrices. VI

**406.** Yosida approximation of $S \mapsto -\log\det S$. VI

**407.** $M$-matrices. V, VIII

**408.** Convex conjugate of $H \mapsto (\det H)^{1/n}$. VI

**409.** Inconditional convergence of the diagonal in the method of JacobiJacobi. XIII

**410.** Lightning. III

**411.** The $p$-norm of circulant matrices. VII

**412.** A FarkasFarkas Lemma for symmetric matrices. VI

**413.** Unit distance representation of a graph in Euclidian space. VI

**414.** A maximization problem in the unit disk. V

**415.** Polynomial identities over alternate matrices of size $n \leq 4$; after B. KostantKostant and L. H. RowenRowen. III

**416.** A variant of Cayley–HamiltonCayleyHamilton Theorem for alternate matrices. III

**417.** Singular values of blocks of LorentzLorentz transformations. X, XI

**418.** The product of two monotone matrices. V

**419.** The spread of the diagonal of a normal matrix. V

**420.** Lewis' TheoremLewis. VII

**421.** Wielandt'sWielandt Theorem for positive primitive matrices. VIII

**422.** Sharpness in Wielandt'sWielandt Theorem. VIII

**423.** Monotonicity of $S \mapsto \hat{S}$ over $\mathbf{SPD}_n$. VI

**424.** A determinantal inequality for three positive symmetric matrices. VI

**425.** FibonacciFibonacci numbers in the powers of a $0,1$-matrix. III

**426.** A convex body in $\mathbf{Sym}_3$ contained in $|\det| \leq 1$. VI

**427.** Can a vector space be the finite union of proper subspaces ?

**428.** A family of positive semi-definite matrices. VI

**429.** A parametrization of $2 \times 2$ matrices with real entries. V, VII

**430.** Large random matrix with prescribed determinant in $\mathbb{F}_q$. III

**431.** The Jordan form of the square of a Jordan block. IX

**432.** Similar permutation matrices (Brauer). VIII, IX

**433.** The positive rank of a non-negative matrix. VIII

**434.** A CapelliCapelli identity $(2 \times 2)$ case.

**435.** Matrices commuting with $A + B$ and $AB$. III

**436.** Isomorphic orthogonal groups. X

**437.** Unitary matrices *vs* numerical radius. V

**438.** Normality of $A, B, AB$ and $BA$ (bis). V

**439.** Eigenvalue on the boundary of the numerical range. V

**440.** A polynomial formula for the adjugate matrix. III

**441.** The characteristic polynomial of $S \mapsto X^T SX$. III

**442.** An equation in the unitary group. V

**443.** Computing the characteristic polynomial of real or complex matrices. III, V

**444.** Are $A$ and $A^T$ orthogonaly similar ? III, V

**445.** The orthogonal of bistochastic matrices.

**446.** Positive definite completion of a symmetric tridiagonal matrix.

**447.** Strang transform.

**448.** Comparing $P^2$ and $(P + Q)^2$ when $P$ and $Q$ are orthogonal projections.

**449.** The maximal order in $\mathbf{GL}_n(\mathbb{F}_p)$.

**450.** Real square roots of symmetric matrices.

**451.** The determinant of some complex and real matrices.

**452.** Ch. Davis'Davis proof of Toeplitz–HausdorffToeplitzHausdorff Theorem.

**453.** A Pfaffian calculation.

**454.** A maximal cone over which $\det \geq 0$.

**455.** A proof of Banach'sBanach Formula using the Cayley–HamiltonCayleyHamilton Theorem.

**456.** The persistance of eigenvalues after rank-one perturbations (after HörmanderHörmander & MelinMelin).

**457.** Elementary divisors of some integerwise matrix.

**458.** An estimate of the spread of a complex matrix.

**459.** Normal completion.

**460.** The *cut-norm*.

**461.** The Cayley–Menger determinant.

**462.** Constant-coefficient ODEs *vs* HankelHankel matrices.

**463.** A determinantal calculation.

**464.** A convex function

**465.** The case $H, K \in \mathbf{Sym}_n^+$ in the product $HK$.

**466.** Arithmetic-geometric inequality for the spectrum of a matrix product.

**467.** Two determinants of physical relevance.

**468.** The Inverse Eigenvalue Problem.

**469.** Real matrices whose square is symmetric.

**470.** Commuting matrices.

**471.** Another proof of Hadamard's Inequality.

**472.** The 3-dimensional area of the unit sphere of $\mathbf{M}_2(\mathbb{R})$.

**473.** Joint spectral radius, again.

**474.** Eigenvalues and eigenvectors of normal matrices.

**475.** The action of the group of invertible quaternions over $\mathbf{SO}_3(k)$.

**476.** The spectral radius of $QA$ when $Q$ is unitary.

**477.** Mean-by-cofactors for positive definite matrices.

## Exercises

1. Let $K$ be a field and $M, N \in \mathbf{M}_n(K)$. Let $k$ be the subfield spanned by the entries of $M$ and $N$. Assume that $M$ is similar to $N$ in $\mathbf{M}_n(K)$. Show that $M$ is similar to $N$ in $\mathbf{M}_n(k)$. Compare with Exercise 2, page 55.

2. (a) When $M, N \in \mathbf{GL}_n(k)$, show that $\mathrm{rk}(M - N) = \mathrm{rk}(M^{-1} - N^{-1})$.
   (b) If $A \in \mathbf{GL}_n(k)$ and $x, y \in k^n$ are given, let us define $B = A + xy^T$. If $B \in \mathbf{GL}_n(k)$, show that $B^{-1} = A^{-1} - B^{-1}xy^TA^{-1}$. Compute $B^{-1}x$ and deduce an explicit formula for $B^{-1}$, the ShermanSherman–MorrisonMorrison formula.
   (c) We now compute explicitly $\det B$.
      i. We begin with the case where $A = I_n$. Show that $\det(I_n + xy^T) = 1 + y^Tx$ (several solutions).
      ii. We continue with the case where $A$ is non-singular. Deduce that $\det B = (\det A)(1 + y^TA^{-1}x)$.
      iii. What is the general algebraic expression for $\det B$ ? **Hint**: Principle of algebraic identities. Such an identity is unique and can be deduced form the complex case $k = \mathbb{C}$.
      iv. Application: If $A$ is alternate and $x \in k^n$, prove that $\det(A + txx^T) \equiv \det A$.
   (d) Let $t$ vary in $k$. Check that $A + txy^T$ is invertible either for all but one values of $t$, or for all values, or for no value at all. In particular, the set $\mathbf{GL}_n^+(\mathbb{R})$ of real matrices with positive determinant is *rank-one convex* (see the next exercise), in the sense that if $A, B \in \mathbf{GL}_n^+(\mathbb{R})$ and $\mathrm{rk}(B - A) = 1$, then the interval $(A, B)$ is contained in $\mathbf{GL}_n^+(\mathbb{R})$.
   (e) We now specialize to $k = \mathbb{R}$. Check that $\det(A + xy^T)\det(A - xy^T) \le \det A^2$. Show that when the rank of $P$ is larger than one, $\det(A + P)\det(A - P)$ can be larger than $\det A^2$.

3. Given a map $f : \mathbf{GL}_n(k) \to k$, we define $f^* : \mathbf{GL}_n(k) \to k$ by $f^*(A) := f(A^{-1})\det A$.
   (a) Check that $f^{**} = f$.
   (b) If $f$ is a linear combination of minors, prove that $f^*$ is another such combination. Mind that the void minor, which is the constant function equal to 1, is allowed in these combinations. More precisely, prove that for every sets $I$ and $J$ of row and column indices, with $|I| = |J|$, the $(I, J)$-minor of $A^{-1}$ is given by the formula
   $$(\det A)A^{-1}\begin{pmatrix} I \\ J \end{pmatrix} = \epsilon(I, J)\, A\begin{pmatrix} J^c \\ I^c \end{pmatrix}$$
   for an appropriate sign $\epsilon(I, J)$.
   (c) Let $\mathbf{GL}_n^+(\mathbb{R})$ be the set of real matrices with $\det A > 0$. We say that $f : \mathbf{GL}_n^+(\mathbb{R}) \to \mathbb{R}$ is *rank-one convex* if its restrictions to segments $(A, B)$ is convex whenever $B - A$ has rank one. Show that, if $f$ is rank-one convex, then $f^*$ is rank-one convex (use the previous exercise).

(d) According to J. BallBall, we say that $f : \mathbf{GL}_n^+(\mathbb{R}) \to \mathbb{R}$ is *polyconvex* if there exists a convex function $g : \mathbb{R}^N \to \mathbb{R}$ such that $f(A) = g(m(A))$, where $m(A)$ is the list of minors of $A$. Show that, if $f$ is polyconvex, then $f^*$ is polyconvex. **Hint**: A convex function is the supremum of some family of affine functions.

4. Assume that the characteristic of the field $k$ is not equal to 2. Given $M \in \mathbf{GL}_n(k)$, show that the matrix
$$A := \begin{pmatrix} 0_n & M^{-1} \\ M & 0_n \end{pmatrix}$$
is diagonalisable. Compute its eigenvalues and eigenvectors. More generally, show that every involution ($A^2 = I$) is diagonalisable.

5. Let $P \in k[X]$ have simple roots in $k$. If $A \in \mathbf{M}_n(k)$ is such that $P(A) = 0_n$, show that $A$ is diagonalisable. Exercise 4 is a special case of this property.

6. If $n = 2$, find a matrix $P \in \mathbf{GL}_2(k)$ such that, for every matrix $A \in \mathbf{M}_2(k)$, there holds $P^{-1}AP = \hat{A}$ ($\hat{A}$ is the matrix of cofactors.)

   **Nota**: If $A \in \mathbf{SL}_2(k)$, we thus have $P^{-1}AP = A^{-T}$, meaning that the natural representation of $\mathbf{SL}_2(k)$ into $k^2$ is *self-dual*.

7. Prove the norm inequality ($1 < p < \infty$)
$$\|A\|_p \le \|A\|_1^{1/p}\|A\|_\infty^{1/p'}, \quad A \in \mathbf{M}_n(\mathbb{C}),$$
by a direct comptutation using only the Hölderhold@Hölder inequality and the explicit formulae for the norms $\|A\|_1$ and $\|A\|_\infty$. **Remark**: Exercise 20 of Chapter 4 corresponds to the case $p = 2$, where Hölder is nothing but Cauchy–SchwarzCauchySchwarz.

8. (See also Exercise 119). Let $\mu$ be a probability measure on $\mathbb{R}$, with a compact support. We assume that this support is not finite. Define its moments
$$m_k := \int_{\mathbb{R}} x^k d\mu(x), \quad k \in \mathbb{N},$$
and the determinants
$$D_n := \begin{vmatrix} m_0 & m_1 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \vdots & & & \vdots \\ m_n & m_{n+1} & \cdots & m_{2n} \end{vmatrix}, \quad D_n(x) := \begin{vmatrix} m_0 & m_1 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \vdots & & & \vdots \\ m_{n-1} & m_n & \cdots & m_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}.$$
Define at last
$$p_n(x) = \frac{1}{\sqrt{D_{n-1}D_n}} D_n(x).$$

(a) Prove that the leading order term of the polynomial $p_n$ is $c_n x^n$ for some $c_n > 0$. Then prove that

$$\int_{\mathbb{R}} p_n(x)p_m(x)d\mu(x) = \delta_n^m.$$

In other words, the family $(p_n)_{n\in\mathbb{N}}$ consists in the orthonormal polynomials relatively to $\mu$.

(b) What happens if the support of $\mu$ is finite ?

(c) Prove the formula

$$D_n = \frac{1}{(n+1)!}\int_{\mathbb{R}^{n+1}} \prod_{0\le i<j\le n}(x_i - x_j)^2 d\mu(x_0)\cdots d\mu(x_n).$$

**Hint**: Compute $D_n$, a determinant of size $n+1$, in $(n+1)!$ different ways. Then use the expression of a VandermondeVandermonde determinant.

9. Find a short proof of Birkhoff's TheoremBirkhoff (Theorem 5.5.1), using the *wedding Lemma*, also known as Hall's TheoremHall!Ph.: let $n$ be a positive integer and let $G$ (girls) and $B$ (boys) two sets of cardinality $n$. Let $\mathcal{R}$ be a binary relation on $G\times B$ ($g\mathcal{R}b$ means that the girl $g$ and the boy $b$ appreciate each other). Assume that, for every $k = 1,\dots,n$, and for every subset $B'$ of $B$, of cardinality $k$, the image $G'$ of $B'$ (that is, the set of those $g$ such that $g\mathcal{R}b$ for at least one $b\in B'$) has cardinality larger than or equal to $k$. Then there exists a bijection $f : B \to G$ such that $f(b)\mathcal{R}b$ for every $b\in B$. **Nota**: the assumption is politically correct, in the sense that it is symmetric in $B$ and $G$ (though it is not clear at a first sight). The proof of the wedding Lemma is by induction. It can be stated as a result on matrices: given an $n\times n$ matrix $M$ of zeroes and ones, assume that, for every $k$ and every set of $k$ lines, the ones in these lines correspond to at least $k$ columns. Then $M$ is larger than or equal to a permutation matrix. **Hint**: Here are the steps of the proof. Let $M$ be bistochastic. Prove that it cannot contain a null block of size $k\times l$ with $k+l>n$. Deduce, with the wedding lemma, that there exists a permutation $\sigma$ such that $m_{i\sigma(i)} > 0$ for every $i$. Show that we may restrict to the case $m_{ii} > 0$ for every $i$. Show also that we may restrict to the case where $0 < m_{ii} < 1$. In this case, show that $(1-\epsilon)M + \epsilon I_n$ is bi-stochastic for $|\epsilon|$ small enough. Conclude.

10. Let $k$ be a field. Define the *symplectic group* $\mathbf{Sp}_m(k)$ as the set of matrices $M$ in $\mathbf{M}_{2m}(k)$ that satisfy $M^T J_m M = J_m$, where

$$J_m := \begin{pmatrix} 0_m & I_m \\ -I_m & 0_m \end{pmatrix}.$$

Check that the word "group" is relevant. Using the Pfaffian, prove that every symplectic matrix (that is, $M\in\mathbf{Sp}_m(k)$) has determinant $+1$. Compare with Corollary 7.6.1.

11. Set $n = 2m$.

(a) Show the following formula for the Pfaffian, as an element of $\mathbb{Z}[x_{ij}; 1 \leq i < j \leq n]$ :

$$\operatorname{Pf}(X) = \sum (-1)^\sigma x_{i_1 i_2} \cdots x_{i_{2m-1} i_{2m}}.$$

Hereabove, the sum runs over all the possible ways the set $\{1, \dots, n\}$ can be partitionned in pairs :

$$\{1, \dots, n\} = \{i_1, i_2\} \cup \cdots \cup \{i_{2m-1} i_{2m}\}.$$

To avoid redundancy in the list of partitions, one normalized by

$$i_{2k-1} < i_{2k}, \quad 1 \leq k \leq m,$$

and $i_1 < i_3 < \cdots < i_{2m-1}$ (in particular, $i_1 = 1$ and $i_{2m} = 2m$). At last, $\sigma$ is the signature of the permutation $(i_1 i_2 \cdots i_{2m-1} i_{2m})$.

Compute the number of monomials in the Pfaffian.

(b) Deduce an "expansion formula with respect to the $i$-th row" for the Pfaffian: if $i$ is given, then

$$\operatorname{Pf}(X) = \sum_{j(\neq i)} \alpha(i,j)(-1)^{i+j+1} x_{ij} \operatorname{Pf}(X^{ij}),$$

where $X^{ij} \in \mathbf{M}_{n-2}(k)$ denotes the alternate matrix obtained from $X$ by removing the $i$-th and the $j$-th rows and columns, and $\alpha(i,j)$ is $+1$ if $i < j$ and is $-1$ if $j < i$.

(c) In particular, we have

$$\operatorname{Pf}(X) = \sum_{j=2}^{n} (-1)^j x_{1j} \operatorname{Pf}(X^{1j}).$$

**Comment**. This formula provides an *alternate adjoint* $\widehat{X}$ with the following properties:

- the formula $X\widehat{X} = \operatorname{Pf}(X) I_n$,
- the entries of $\widehat{X}$ are homogeneous polynomials of degree $m-1$ in those of $X$.

(d) Deduce $\operatorname{Pf}(A_n) = 1$, where $A_n$ denotes the alternate matrix whose upper-diagonal entries are 1s. **Hint**: Induction.

12. (a) Let $A, B$ be $n \times n$ Hermitian positive definite matrices. Denote by $\lambda_1(A) \leq \lambda_2(A) \leq \cdots$ and $\lambda_1(B) \leq \lambda_2(B) \leq \cdots$ their eigenvalues. Remarking that $AB$ is similar to $\sqrt{B} A \sqrt{B}$, show that the spectrum of $AB$ is real, positive, and satisfies

$$\lambda_i(A)\lambda_1(B) \leq \lambda_i(AB) \leq \lambda_i(A)\lambda_n(B).$$

**Hint**: Use Theorem 3.3.2.

(b) Compare with point a) of Exercise 6, Chapter 7. Show also that the conclusion still holds if $A, B$ are only positive semi-definite.

(c) More generally, prove the inequalities

$$\lambda_j(A)\lambda_k(B) \le \lambda_i(AB) \le \lambda_j(A)\lambda_l(B),$$

whenever $j + k \le i + 1$ and $j + l \ge i + n$.

(d) Set $n = 2$. Let $a_1 \le a_2$, $b_1 \le b_2$, $\mu_1 \le \mu_2$ be non-negative numbers, satisfying $\mu_1\mu_2 = a_1a_2b_1b_2$ and the inequalities

$$a_1b_1 \le \mu_1 \le \min\{a_1b_2, a_2b_1\}, \quad \max\{a_1b_2, a_2b_1\} \le \mu_2 \le a_2b_2.$$

Prove that there exist $2 \times 2$ real symmetric matrices $A$ and $B$, with spectra $\{a_1, a_2\}$ and $\{b_1, b_2\}$, such that $\{\mu_1, \mu_2\}$ is the spectrum of $AB$. **Hint**: Begin with the case where some of the inequalities are equalities. Then use the intermediate value Theorem.

(e) Set $n \ge 2$, and assume that the Hermitian matrix $B$ has eigenvalues $b_1 = 0$, $b_2 = \cdots = b_n = 1$. Show that

$$\lambda_1(AB) = 0, \quad \lambda_{i-1}(A) \le \lambda_i(AB) \le \lambda_i(A).$$

Conversely, if

$$\mu_1 = 0 \le a_1 \le \mu_2 \le a_2 \le \cdots \le \mu_{n-1} \le a_n,$$

show that there exists a Hermitian matrix $A$ with eigenvalues $a_1, \dots, a_n$, such that $AB$ has eigenvalues $\mu_1, \dots, \mu_n$.

Generalize to the case $b_0 = \cdots = b_k = 0$, $b_{k+1} = \cdots = b_n = 1$.

13. (From J. GroßGroß and G. TrenklerTrenkler.) Given $A, B$ two Hermitian positive semi-definite matrices, show that $\mathbb{C}^n = R(AB) \oplus \ker(AB)$.

14. (a) Assume that $A \in \mathbf{M}_{n\times p}(\mathbb{C})$ is injective. Prove that $H := A^*A$ is positive definite. Show that $AH^{-1}A^*$ is the orthogonal projector onto $R(A)$.

(b) Given two injective matrices $A_j \in \mathbf{M}_{n\times p_j}(\mathbb{C})$, define $H_j$ as above. Define also $F := A_1^*A_2$ and then $M := H_2^{-1}F^*H_1^{-1}F$. Using the previous exercise, show that the eigenvalues of $M$ belong to $[0, 1]$.

15. Let $A \in \mathbf{M}_n(\mathbb{C})$ and $H \in \mathbf{HPD}_n$ be given. If $\theta \in \mathbb{R}$, define $\phi(\theta) := H + e^{i\theta}A + e^{-i\theta}A^*$. Define a matrix $M_k \in \mathbf{H}_{kn}$ by

$$M_k = \begin{pmatrix} H & A & 0_n & \\ A^* & \ddots & \ddots & \ddots \\ 0_n & \ddots & \ddots & A \\ & \ddots & A^* & H \end{pmatrix}.$$

(a) Decomposing vectors $x \in \mathbb{C}^{kn}$ as $k$ blocks $x_j \in \mathbb{C}^n$, write $x^*M_kx$ as a sum of terms of the form $y^*\phi(2\ell\pi/k)y$ ($\ell = 1, \dots, k$), and $x_k^*Ax_1$.

(b) We assume that $\phi(2\ell\pi/k) > 0_n$ $(l = 1, \dots, k)$. Show that there exist two positive constants $c_k$, $d_k$ such that

$$x^* M_k x \geq c_k \|x\|^2 - d_k \|x_k\| \, \|x_1\|.$$

Deduce that there exists a $t_k > 0$, such that adding $t_k I_n$ in the bottom-right block, moves $M_k$ to a positive definite matrix.

(c) Under the same assumption as above, prove that $M_1, \dots, M_{k-1}$ are positive definite.

16. Let $A \in \mathbf{M}_n(\mathbb{C})$ and $H \in \mathbf{HPD}_n$ be given, with $A \neq 0_n$. We are interested in the equation

$$X + A^* X^{-1} A = H,$$

where $X \in \mathbf{HPD}_n$ is the unknown.

(a) Show that a necessary condition for $X$ to exist is (Property $(\mathbf{P})$)

- the Hermitian matrix $\phi(\theta) := H + e^{i\theta} A + e^{-i\theta} A^*$ is positive semi-definite for every $\theta$,
- the map $\theta \mapsto \det(\phi(\theta))$ does not identically vanish.

**Hint**: Factorize $\phi(\theta)$, and more generally $H + zA + z^{-1} A^*$.

We shall prove later on that Property $(\mathbf{P})$ is also a sufficient condition. The proof relies more or less upon infinite dimensional matrices called ToeplizToepliz matrices. For a general account of the theory, see M. RosenblumRosenblum & J. RovniakRovniak, *HardyHardy classes and operator theory*, Oxford U. Press, 1985).

(b) Check that $(\mathbf{P})$ is fulfilled in the case where $H = I_n$ and $A$ is real skew-symmetric. Then show that a solution does exist, which is not unique. **Hint**: First, solve the case $n = 2$. The nature of the solution depends on the sign of $t^2 - 1/4$, where

$$A = \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}.$$

(c) Let $A$ be invertible. Find a transformation that reduces the equation to the case $H = I_n$ (though with a different $A$). Verify that this transformation preserves the validity or the failure of Property $(\mathbf{P})$.

(d) From now on, we assume that $H = I_n$. Show that $X$ is a solution of $X + A^* X^{-1} A = I_n$, if and only if $I_n - X$ is a solution of $Y + A Y^{-1} A^* = I_n$.

(e) We temporarily assume that the equation admits at least one solution $X \in \mathbf{HPD}_n$. Here is an algorithm for the approximation of a solution (the largest one):

$$X_0 = I_n, \quad X_{n+1} := I_n - A^* X_n^{-1} A.$$

i. First, show that $X \leq I_n$ in $\mathbf{H}_n$.

ii. Prove inductively that $X_k \geq X$ for every $k$.

iii. Prove inductively that $X_k$ is non-increasing in $\mathbf{HPD}_n$. Deduce that it converges to some limit, and that this limit is a solution.

iv. Deduce that the equation admits a largest solution.

v. Show that the equation also admits a smaller solution in $\mathbf{HPD}_n$.

(f) We now turn to the existence of a solution.

i. Define a block-tridiagonal matrix $M_k \in \mathbf{H}_{kn}$ by

$$M_k = \begin{pmatrix} I_n & A & 0_n & \\ A^* & \ddots & \ddots & \ddots \\ 0_n & \ddots & \ddots & A \\ & \ddots & A^* & I_n \end{pmatrix}.$$

If $\phi(0) > 0_n$ and (**P**) holds, show that $M_k \in \mathbf{HPD}_{kn}$ for every $k \geq 1$. **Hint**: use Exercise 15.

ii. Then show that there exists a unique blockwise lower-bidiagonal matrix $L_k$, with diagonal blocks in $\mathbf{HPD}_n$, such that $L_k L_k^* = M_k$.

iii. Then prove that there exist matrices $B_j \in \mathbf{HPD}_n$ and $C_j \in \mathbf{M}_n(\mathbb{C})$, such that, for every $k \geq 1$, there holds

$$L_k = \begin{pmatrix} B_1 & 0_n & & \\ C_1 & \ddots & \ddots & \\ 0_n & \ddots & \ddots & 0_n \\ & \ddots & C_{k-1} & B_k \end{pmatrix}.$$

iv. Write the recursion satisfied by $(B_j, C_j)$, and check that $X_k := B_k^2$ satisfies the algorithm above. Then, show that $X_k$ converges as $k \to +\infty$, and that its limit is a solution of our equation (therefore the greatest one).

v. Assuming (**P**) only, show that we may assume $\phi(0) > 0_n$ (consider the matrix $e^{i\alpha}A$ instead of $A$, with $\alpha$ suitably chosen). Conclude.

17. (a) Let $a \in \mathbb{R}^n$ have positive entries. Recall (Exercise 20.a of Chapter 5) that the extremal points of the convex set defined by the "inequality" $b \succ a$ are obtained from $a$ by permutation of its entries.

Show

$$(b \succ a) \Longrightarrow (\prod_j b_j \geq \prod_i a_i). \tag{1}$$

(b) Deduce an other proof of Proposition 3.4.2, with the help of Theorem 3.4.1. (One may either deduce (1) from Proposition 3.4.2 and Theorem 3.4.2. These are rather long proofs for easy results !)

18. Let $A \in \mathbf{M}_n(k)$ be invertible and define $M \in \mathbf{M}_{2n}(k)$ by

$$M := \begin{pmatrix} 0_n & A^{-1} \\ -A^T & A^{-1} \end{pmatrix}.$$

Show that $M$ is symplectic: $M^T J M = J$, with

$$J := \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}.$$

19. (The Banach–MazurBanachMazur distance.)


**Stefan Banach.**
Banach, *Stefan* (Poland)

(a) Let $N$ and $N'$ be two norms on $k^n$ ($k = \mathbb{R}$ or $\mathbb{C}$). If $A \in \mathbf{M}_n(k)$, we may define norms

$$\|A\|_{\rightarrow} := \sup_{x \neq 0} \frac{N'(Ax)}{N(x)}, \quad \|A^{-1}\|_{\leftarrow} := \sup_{x \neq 0} \frac{N(A^{-1}x)}{N'(x)}.$$

Show that $\|A\|_{\rightarrow} \|A^{-1}\|_{\leftarrow}$ achieves its upper bound. We shall denote by $\delta(N, N')$ the minimum value. Verify

$$0 \leq \log \delta(N, N'') \leq \log \delta(N, N') + \log \delta(N', N'').$$

When $N = \|\cdot\|_p$, we shall write $\ell^p$ instead. If in addition $N' = \|\cdot\|_q$, we write $\|\cdot\|_{p,q}$ for $\|\cdot\|_{\rightarrow}$.

(b) In the set $\mathcal{N}$ of norms on $k^n$, let us consider the following equivalence relation: $N \sim N'$ if and only if there exists an $A \in \mathbf{GL}_n(k)$ such that $N' = N \circ A$. Show that $\log \delta$ induces a metric $d$ on the quotient set $\mathbf{Norm} := \mathcal{N}/\sim$. This metric is called the *Banach–MazurBanachMazur distance.*How many classes of Hermitian norms are there ?

(c) Compute $\|I_n\|_{p,q}$ for $1 \leq p, q \leq n$ (there are two cases, depending on the sign of $q - p$). Deduce that

$$\delta(\ell^p, \ell^q) \leq n^{\left|\frac{1}{p} - \frac{1}{q}\right|}.$$

(d) Show that $\delta(\ell^p, \ell^q) = \delta(\ell^{p'}, \ell^{q'})$, where $p', q'$ are the conjugate exponents.

(e) i. Given $H \in \mathbf{H}_n^+$, find that the average of $x^* H x$, as $x$ runs over the set defined by $|x_j| = 1$ for all $j$'s, is Tr $H$ (the measure is the product of $n$ copies of the normalized Lebesgue measure on the unit disk). Deduce that

$$\sqrt{\text{Tr } M^* M} \leq \|M\|_{\infty,2}$$

for every $M \in \mathbf{M}_n(k)$.

ii. On the other hand, prove that

$$\|A\|_{p,\infty} = \max_{1\le i\le n} \|A^{(i)}\|_{p'},$$

where $A^{(i)}$ denotes the $i$-th row vector of $A$.

iii. Deduce that $\delta(\ell^2, \ell^\infty) = \sqrt{n}$.

iv. Using the triangle inequality for $\log\delta$, deduce that

$$\delta(\ell^p, \ell^q) = n^{\left|\frac{1}{p}-\frac{1}{q}\right|}$$

whenever $p, q \ge 2$, and then for every $p, q$ such that $(p-2)(q-2) \ge 0$. **Nota**: The exact value of $\delta(\ell^p, \ell^q)$ is not known when $(p-2)(q-2) < 0$.

v. Remark that the "curves" $\{\ell^p\,;\, 2 \le p \le \infty\}$ and $\{\ell^p\,;\, 1 \le p \le 2\}$ are geodesic, in the sense that the restrictions of the Banach–Mazur distance to these curves satisfy the triangular *equality*.

(f) When $n = 2$, prove that $\delta(\ell^1, \ell^\infty) = 1$. On the other hand, if $n \ge 3$, then $\delta(\ell^1, \ell^\infty) > 1$.

(g) A Theorem proved by F. JohnJohn states that the diameter of $(\mathbf{Norm}, d)$ is precisely $\frac{1}{2}\log n$. Show that this metric space is compact. **Nota**: One may consider the norm whose unit ball is an $m$-agon in $\mathbb{R}^2$, with $m$ even. Denote its class by $N_m$. It seems that $d(\ell^1, N_m) = \frac{1}{2}\log 2$ when $8|m$.

20. (Continuation of Exercise 19.) We study here classes of norms (in **Norm**) that contain a pair $\{\|\cdot\|, \|\cdot\|_*\}$. We recall that the dual norm of $\|\cdot\|$ is defined by

$$\|x\|_* := \inf_{x\ne 0} \frac{\Re(y^*x)}{\|y\|}.$$

As shown in the previous exercise, $\|\cdot\|$ and $\|\cdot\|_*$ are in the same class if and only if there exists an $A \in \mathbf{GL}_n(\mathbb{C})$ such that

$$\|Ax\|_* = \|x\|, \quad \forall x \in \mathbb{C}^n. \tag{2}$$

(a) Let $A$, $\|\cdot\|$ and $\|\cdot\|_*$ satisfy (2). Show that $A^{-*}A$ is an isometry of $(\mathbb{C}^n, \|\cdot\|)$, where $A^{-*} := (A^*)^{-1}$. Deduce that $A^{-*}A$ is diagonalizable, with eigenvalues of modulus one. **Nota**: The whole exercise is valid with the field $\mathbb{R}$ instead of $\mathbb{C}$, but the latter result is a bit more difficult to establish.

(b) Let $P \in \mathbf{GL}_n(\mathbb{C})$ be such that $D := P^{-1}A^{-*}AP$ is diagonal. Define a norm $N \sim \|\cdot\|$ by $N(x) := \|Px\|$. Show that $D$ is an isometry of $(\mathbb{C}^n, N)$, and that

$$N_*(Bx) = N(x), \quad \forall x \in \mathbb{C}^n,$$

where $B := P^*AP$.

(c) Using Exercise 7 of Chapter 3 (page 56), prove that the class of $\|\cdot\|$ contains a norm $\mathcal{N}$ such that

$$\mathcal{N}_*(\Delta x) = \mathcal{N}(x), \quad \forall x \in \mathbb{C}^n$$

for some diagonal matrix $\Delta$. Show also that $\mathcal{N}(\Delta^{-*}\Delta x) \equiv \mathcal{N}(x)$. Show that one may choose $\Delta$ unitary.

(d) Find more than one example of such classes of norms on $\mathbb{C}^2$.

21. (Numerical range.)

Given $A \in \mathbf{M}_n(\mathbb{C})$, define $r_A(x) = (Ax, x) = x^*Ax$. The numerical range of $A$ is

$$\mathcal{H}(A) = \{r_A(x) \,;\, \|x\|_2 = 1\}.$$

(a) We show that if $n = 2$, then $\mathcal{H}(A)$ is an ellipse whose foci are the eigenvalues of $A$.

i. First check that it suffices to consider the cases of matrices

$$\begin{pmatrix} 0 & 2a \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2a \\ 0 & -1 \end{pmatrix}, \qquad a \in \mathbb{R}^+.$$

ii. Treat the first case above.

iii. From now on, we treat the second case. First prove that $\mathcal{H}(A)$ is the union of circles with center $p \in [-1, 1]$ and radius $r(p) = a\sqrt{1-p^2}$.

iv. We define the (full) ellipse $\mathcal{E} \in \mathbb{C} \sim \mathbb{R}^2$ by the inequality

$$\frac{x^2}{1+a^2} + \frac{y^2}{a^2} \leq 1.$$

Show that $\mathcal{H}(A) \subset \mathcal{E}$.

v. Define $p \mapsto g(p) := y^2 + (x-p)^2 - r(p)^2$ over $[-1, 1]$. If $(x, y) \in \mathcal{E}$, prove that $\min g \leq 0$ ; deduce that $g$ vanishes, and thus that $(x, y) \in \mathcal{H}(A)$. Conclude.

vi. Show that for a general $2 \times 2$ matrix $A$, the area of $\mathcal{A}$ equals

$$\frac{\pi}{4}\, |\det[A^*, A]|^{1/2}.$$

(b) Deduce that for every $n$, $\mathcal{H}(A)$ is convex (Toeplitz–HausdorffToeplizHausdorff Theorem). **See** an application in Exercise 131.

(c) When $A$ is normal, show that $\mathcal{H}(A)$ is the convex hull of the spectrum of $A$.

(d) Let us define the *numerical radius* $w(A)$ by

$$w(A) := \sup\{|z| \,;\, z \in \mathcal{H}(A)\}.$$

Prove that

$$w(A) \leq \|A\|_2 \leq 2w(A).$$

**Hint**: Use the polarization principle to prove the second inequality.

Deduce that $A \mapsto w(A)$ is a norm (there are counter-examples showing that it is not submultiplicative.)

(e) Let us assume that there exists a complex number $\lambda \in \mathcal{H}(A)$ such that $|\lambda| = \|A\|_2$, say that $\lambda = r_A(x)$ for some unit vector $x$. Prove that $\lambda$ is an eigenvalue of $A$, with $x$ an eigenvector. Thus $\rho(A) = \|A\|_2 = w(A)$. Prove also that $x$ is an eigenvector of $A^*$, associated with $\bar{\lambda}$. **Hint**: Show at first that $\lambda A^* x + \bar{\lambda} A x = 2|\lambda|^2 x$.

(f) If $A^2 = 0_n$, show that $w(A) = \frac{1}{2}\|A\|_2$ (see also Exercise 100.)

22. (JacobiJacobi matrices.) Let $A \in \mathbf{M}_n(\mathbb{R})$ be tridiagonal, with $a_{i,i+1}a_{i+1,i} > 0$ when $i = 1, \dots, n-1$.

(a) Show that $A$ is similar to a tridiagonal symmetric matrix $S$ with $s_{i,i+1} < 0$.

(b) Deduce that $A$ has $n$ real and simple eigenvalues. We denote them by $\lambda_1 < \cdots < \lambda_n$.

(c) For $j = 1, \dots, n$, let $A^{(j)}$ be the principal submatrix, obtained from $A$ by keeping the first $j$ rows and columns. Without loss of generality, we may assume that the off-diagonal entries of $A$ are non-positive, and denote $b_j := -a_{j,,j+1} > 0$. If $\lambda$ is an eigenvalue, show that

$$x := (b_1^{-1} \cdots b_{j-1}^{-1} D_{j-1}(\lambda))_{1 \le j \le n}, \quad D_j(X) := \det(A^{(j)} - XI_j)$$

is an eigenvector associated with $\lambda$.

(d) Deduce that the sequence of coordinates of the eigenvector associated with $\lambda_j$ has exactly $j-1$ sign changes (one says that this eigenvector has $j-1$ *nodes*). **Hint**: Use the fact that $(D_1, \dots, D_n)$ is a SturmSturm sequence.

How could this be proven rapidly when $j = 1$ ?

23. Let $k$ be a field with zero characteristic. Let $A \in \mathbf{M}_n(k)$ and $l \ge 1$ be given. We form the block-triangular matrix

$$M := \begin{pmatrix} A & I_n & 0_n & \dots & 0_n \\ 0_n & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0_n \\ \vdots & & \ddots & \ddots & I_n \\ 0_n & \dots & \dots & 0_n & A \end{pmatrix} \in \mathbf{M}_{nl}(k).$$

(a) Let $P \in k[X]$ be a polynomial. Show that $P(M)$ is block-triangular, with the $(i, i+r)$-block equal to $\frac{1}{r!}P^{(r)}(A)$.

(b) If an irreducible polynomial $q$ divides a polynomial $Q$, together with its derivatives $Q^{(r)}$ up to $Q^{(s)}$, prove that $q^{s+1}$ divides $Q$.

(c) Assume that the minimal polynomial of $A$ is irreducible. Compute the minimal polynomial of $M$. Compute the lists of its invariant polynomials.

(d) Deduce an alternate and somehow simpler second canonical form of square matrices.

24. Given $M \in \mathbf{M}_n(k)$, denote $|M_j^i|$ the minor obtained by removing the $i$-th row and the $j$-column. We define similarly $|M_{j,k}^{i,l}|$.

   (a) Prove Desnanot–JacobiDesnanotJacobi formula, also called DodgsonDodgson (see Lewis C.)Lewis Caroll[1] condensation formula,

   $$|M_{1,n}^{1,n}| \det M = |M_1^1|\,|M_n^n| - |M_n^1|\,|M_1^n|.$$

   See Exercise 144 for a generalization.


Left: **Charles L. Dodgson.**

*(Strangely enough, this is the only stamp with his portrait.)*

   (b) Deduce a recursive algorithm for the computation of $\det M$, in about $4n^3/3$ operations.

   (c) Does this algorithm always produce a result?

   (d) (GantmacherGantmacher & KreinKrein, KotelyanskiĭKotelyanskiĭ.) If $S \in \mathbf{Sym}_n$, one defines $|S(I)|$ as the minor of $S$ corresponding to lines and columns of indices $i, j \in I$.
   If $S$ is positive semi-definite, show that $|S(I \cap J)| \cdot |S(I \cup J)| \leq |S(I)| \cdot |S(J)|$.

   **Hint**: One may always assume that $I \cup J = [\![1, n]\!]$. By a density argument, one may assume that $S$ is positive definite. Argue by induction upon the cardinal of the symmetric difference $I \Delta J$. If $I \setminus J$ and $J \setminus I$ are singletons, apply Desnanot–Jacobi.

25. Consider the oriented graph whose vertices are the points of $\mathbb{Z}^2$ and edges are the horizontal (left to right) and vertical (downwards) segments. Given two vertices $A$ and $B$, denote by $n(A, B)$ the number of paths from $A$ to $B$. Thus $n(A, B) \geq 1$ iff $A_1 \leq B_1$ and $A_2 \geq B_2$.

   Given $2m$ points $A^1, \dots, A^m$ and $B^1, \dots, B^m$, we assume that $A^{i+1}$ (respectively $B^{i+1}$) is strictly upper right with respect to $A^i$ (resp. $B^i$) and that $B^m$ is lower right with respect to $A^1$.

[1] Charles Lutdwidge Dodgson, English mathematician. Educated people, as well as children, prefer to call him *Lewis Caroll.*

(a) Consider $m$-tuples of paths $\gamma_j$, each one joining $A^j$ to $B^j$. Prove that the number of such $m$-tuples, for which the $\gamma_j$'s are pairwise disjoint, equals the determinant of

$$N := \left(n(A^i, B^j)\right)_{1\le i,j\le m}.$$

(b) Prove that the matrix $N$ is totally positive.

26. Let $U \in \mathbf{U}_n$ be upper HessenbergHessenberg. Up to a multiplication by a unitary diagonal matrix, we may assume that the entries $\beta_k := u_{k+1,k}$ are real non-negative. Prove that there exist numbers $\alpha_k \in \mathbb{C}$ such that $|\alpha_k|^2 + \beta_k^2 = 1$ for $k = 1, \dots, n-1$, $|\alpha_n| = 1$ and $U = G_1(\alpha_1)\cdots G_n(\alpha_n)$, where

$$G_k(\alpha_k) := \operatorname{diag}\left(I_{k-1}, \begin{bmatrix} -\alpha_k & \beta_k \\ \beta_k & \bar{\alpha}_k \end{bmatrix}, I_{n-k-1}\right), \quad k = 1, \dots, n-1,$$

and

$$G_n(\alpha_n) := \operatorname{diag}(1, \dots, 1, -\alpha_n).$$

This is the *SchurSchur parametrization*. Notice that the matrices $G_k(\alpha_k)$ are unitary.

27. (P. D. LaxLax, H. F. WeinbergerWeinberger (1958).) Let $V$ be a linear subspace of $\mathbf{M}_n(\mathbb{R})$, with the property that the spectrum of every matrix $M \in V$ is real. We shall denote $\lambda_1(M) \le \cdots \le \lambda_n(M)$ the eigenvalues of $M \in V$, repeated with multiplicities.

(a) Prove that the functions $M \mapsto \lambda_k(M)$ are continous over $V$.

(b) Let $A, B \in V$, with $\lambda_1(B) > 0$ (one says that $B$ is *positive*.) Given $\lambda \in \mathbb{R}$, show that the polynomial $x \mapsto \det(\lambda I_n - A - xB)$ has $n$ real roots (counting with multiplicities.) **Nota**: This is really a difficult question, but just assume that the functions $\lambda_k$ are infinitely differentiable away from the origin, a fact that is true when the eigenvalues are simple for every non-zero $M \in V$.

(c) With $A, B$ as above, prove that $\mu \mapsto \lambda_k(A + \mu B)$ is strictly increasing. Deduce that $\mu \mapsto \lambda_k(A + \mu B) - \mu\lambda_1(B)$ is non-decreasing, and that $\mu \mapsto \lambda_k(A + \mu B) - \mu\lambda_n(B)$ is non-increasing.

(d) Given $X, Y \in V$, show that

$$\lambda_k(X) + \lambda_1(Y) \le \lambda_k(X + Y) \le \lambda_k(X) + \lambda_n(Y).$$

Deduce that $\lambda_1$ and $\lambda_n$ are respectively a concave and a convex functions.

(e) Prove that the subset of positive matrices is a convex cone in $V$. Deduce that the relation $A \prec B$, defined on $V$ by

$$\lambda_k(A) \le \lambda_k(B), \quad \forall k = 1, \dots, n,$$

is an order relation.

(f) Give an example of such a subspace $V$, of dimension $n(n+1)/2$. Did we already know all the results above in this case ?

28. Denote by $\mathbf{SL}_2^+(\mathbb{Z})$ the set of non-negative matrices with entries in $\mathbb{Z}$ and determinant $+1$. Denote by $E, F$ the "elementary" matrices:
$$E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$
Show that $\mathbf{SL}_2^+(\mathbb{Z})$ is the disjoint union of
$$\{I_2\}, \quad E \cdot \mathbf{SL}_2^+(\mathbb{Z}) \text{ and } F \cdot \mathbf{SL}_2^+(\mathbb{Z}).$$
Deduce that $\mathbf{SL}_2^+(\mathbb{Z})$ is the free monoid spanned by $E$ and $F$, meaning that for every $A \in \mathbf{SL}_2^+(\mathbb{Z})$, there exists a unique word $m$ in two letters, such that $A = m(E, F)$. Notice that $I_2$ corresponds to the void word.

Show that
$$A_m := \begin{pmatrix} 1 & m & m \\ m & 1+m^2 & 0 \\ m & 0 & 1+m^2+m^4 \end{pmatrix}, \quad m \in \mathbb{N}$$
is an element of $\mathbf{SL}_3^+(\mathbb{Z})$, which is irreducible, in the sense that $A_m = MN$ and $M, N \in \mathbf{SL}_3^+(\mathbb{Z})$ imply that $M$ or $N$ is a permutation matrix (the only invertible elements in $\mathbf{SL}_3^+(\mathbb{Z})$ are the matrices of even permutations.) Deduce that $\mathbf{SL}_3^+(\mathbb{Z})$ cannot be generated by a finite number of elements.

29. (R. M. MayMay, C. JeffriesJeffries, D. LogofetLogofet, UlianovUlianov.) We distinguish three signs $-, 0, +$ for real numbers, which exclude each other. In mathematical terms, $- = (-\infty, 0)$, $0 = \{0\}$ and $+ = (0, +\infty)$. The product of signs is well-defined.

Two given matrices $A, B \in \mathbf{M}_n(\mathbb{R})$ are said *sign-equivalent* if the entries $a_{ij}$ and $b_{ij}$ have same sign, for every pair $(i, j)$. Sign-equivalence is obviously an equivalence relation. An equivalence class is written as a matrix $S$, whose entries are signs. Here are three examples of sign-classes:
$$S_1 = \operatorname{diag}(-, \cdots, -), \quad S_2 = \begin{pmatrix} 0 & - \\ + & - \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & - & 0 \\ + & 0 & 0 \\ + & + & - \end{pmatrix}.$$
In some applications of dynamical system theory, we are concerned with the asymptotic stability of the origin in the system $\dot{x} = Ax$. For some reason, we do know the signs of the entries of $A$, but the magnitude of non-zero entries is unknown. This arises for instance in ecology or in the study of chemical reactions. Hence we ask whether the sign structure of $A$ (its sign-class) ensures that the whole spectrum lies in the half-space $\Sigma := \{z \in \mathbb{C} \,;\, \Re z < 0\}$, or not. If it does, then we say that this class (or this matrix) is *sign-stable*.

Given a sign-class $S$, we denote $\mathcal{G}(S)$ the oriented graph whose vertices are the indices $j = 1, \ldots, n$, and arrows correspond to the non-zero entries $s_{ij}$ with $i \neq j$.

(a) Show that the classes $S_1$ and $S_2$ above are sign-stable, but that $S_3$ is not.

(b) Actually (**Hint**), $S_3$ is reducible. Show that the determination of the stable sign-classes reduces to that of the stable irreducible sign-classes.

From now on, we restrict to irreducible sign-classes.

(c) Given a class $S$, we denote by $\bar{S}$ its closure, where $- = (-\infty, 0)$ is replaced by $(-\infty, 0]$, and similarly for $+$.

If a class $S$ is sign-stable, show that $\bar{S}$ is *weakly* sign-stable, meaning that its elements $A$ have their spectra in the closed half-space $\bar{\Sigma}$. Deduce the following facts:

i. $s_{ii} \leq 0$ for every $i = 1, \ldots, n$,

ii. $\mathcal{G}(S)$ does not contain cycles of length $p \geq 3$.

(d) We restrict to sign-classes that satisfy the two necessary conditions found above.

i. Considering the trace of a matrix in $S$, show that sign-stability requires that there exists a $k$ such that $s_{kk} = -$.

ii. Deduce that, for a class to be stable, we must have, for every pair $i \neq j$, either $s_{ij} = s_{ji} = 0$ or $s_{ij}s_{ji} < 0$. In ecology, one says that the matrix is a *predation* matrix. **Hint**: Use the irreducibility and the absence of cycle of length $p > 2$.

iii. Under the two additional restrictions just found, show that every monomial ($\sigma$ a permutation)

$$\epsilon(\sigma)s_{1\sigma(1)} \cdots s_{n\sigma(n)}$$

is either 0 or $(-)^n$. Deduce that the sign of the determinant is not ambiguous: Either every element of $S$ satisfies $(-)^n \det A > 0$, or every element of $S$ satisfies $\det A = 0$. In the latter case, every monomial in $\det S$ must vanish.

iv. Check that sign-stability requires that the sign of the determinant be $(-)^n$.

(e) Check that the following class satisfies all the necessary conditions found above, but that it is not sign-stable because it contains an element $A$ with eigenvalues $\pm i$:

$$S_5 = \begin{pmatrix} 0 & - & 0 & 0 & 0 \\ + & 0 & - & 0 & 0 \\ 0 & + & - & - & 0 \\ 0 & 0 & + & 0 & - \\ 0 & 0 & 0 & + & 0 \end{pmatrix}.$$

(f) Show that the following class satisfies all the necessary conditions found above, but one ($\det S_7 = 0$):

$$S_7 = \begin{pmatrix} 0 & - & 0 & 0 & 0 & 0 & 0 \\ + & - & - & 0 & 0 & 0 & 0 \\ 0 & + & - & - & - & 0 & 0 \\ 0 & 0 & + & 0 & 0 & 0 & 0 \\ 0 & 0 & + & 0 & 0 & - & 0 \\ 0 & 0 & 0 & 0 & + & - & - \\ 0 & 0 & 0 & 0 & 0 & + & 0 \end{pmatrix}.$$

Actually, show that every element $A$ of $S_7$ satisfies $Ax = 0$ for some non-zero vector $x$ with $x_1 = x_2 = x_3 = x_6 = 0$.

30. Let $G$ be a classical group of real or complex $n \times n$ matrices. We only need that it satisfies Proposition 7.3.1. Let $G_1$ be a compact subgroup of $G$, containing $G \cap \mathbf{U}_n$.

    (a) Let $M$ be an element of $G_1$, with polar decomposition $QH$. Verify that $H$ belongs to $G_1 \cap \mathbf{HPD}_n$.

    (b) Using the fact that $H^m \in G_1$ for every $m \in \mathbb{Z}$, prove that $H = I_n$.

    (c) Deduce that $G_1 = G \cap U_n$. Hence, $G \cap \mathbf{U}_n$ is a maximal compact subgroup of $G$.

31. Following C. R. JohnsonJohnson!Charles R. and C. J. HillarHillar (SIAM J. Matrix Anal. Appl., **23**, pp 916-928), we say that a word with an alphabet of two letters is *nearly symmetric* if it is the product of two palindromes (a palindrome can be read in both senses; for instance the French city LAVAL is a palindrome). Thus $ABABAB = (ABABA)B$ is nearly symmetric. Check that every word in two letters of length $\ell \leq 5$ is nearly symmetric. Show that if a word $m(A, B)$ is nearly symmetric, then the matrix $m(S_1, S_2)$ is diagonalizable with positive real eigenvalues, for every symmetric, positive definite matrices $S_1, S_2$ (see Exercise 258).

32. In $\mathbb{R}^{1+m}$ we denote the generic point by $(t, x)^T$, with $t \in \mathbb{R}$ and $x \in \mathbb{R}^m$. Let $\mathcal{C}^+$ be the cone defined by $t > \|x\|$. Recall that those matrices of $\mathbf{O}(1, m)$ that preserve $\mathcal{C}^+$ form the subgroup $G_{+\pm}$. The quadratic form $(t, x) \mapsto \|x\|^2 - t^2$ is denoted by $q$.

    Let $M$ belong to $G_{+\pm}$.

    (a) Given a point $x$ in the unit closed ball $B$ of $\mathbb{R}^m$, let $(t, y)^T$ be the image of $(1, x)^T$ under $M$. Define $f(x) := y/t$. Prove that $f$ is a continous map from $B$ into itself. Deduce that it has a fixed point. Deduce that $M$ has at least one real positive eigenvalue, associated with an eigenvector in the closure of $\mathcal{C}^+$. **Nota**: If $m$ is odd, one can prove that this eigenvector can be taken in the light cone $t = \|x\|$.

    (b) If $Mv = \lambda v$ and $q(v) \neq 0$, show that $|\lambda| = 1$.

    (c) Let $v = (t, x)$ and $w = (s, y)$ be *light* vectors (that is $q(v) = q(w) = 0$), linearly independent. Show that $v^* J w \neq 0$.

    (d) Assume that $M$ admits an eigenvalue $\lambda$ of modulus different from 1, $v$ being an eigenvector. Show that $1/\lambda$ is also an eigenvector. Denote by $w$ a corresponding eigenvector. Let $< v, w >^\circ$ be the orthogonal of $v$ and $w$ with respect to $q$. Using the previous question, show that the restriction $q_1$ of $q$ to $< v, w >^\circ$ is positive definite. Show that $< v, w >^\circ$ is invariant under $M$ and deduce that the remaining eigenvalues have unit modulus.

    (e) Show that, for every $M \in G_{+\pm}$, $\rho(M)$ is an eigenvalue of $M$.

33. Assume that $A \in \mathbf{M}_n(\mathbb{C})$ is tridiagonal, with an invertible diagonal part $D$. Assume that the relaxation method converges for every parameter $\omega$ in the disc $|\omega - 1| < 1$.

(a) Show that, for every $\omega$ in the circle $|\omega - 1| = 1$, the spectrum of $\mathcal{L}_\omega$ is included in the unit circle.

(b) Deduce that the spectrum of the iteration matrix $J$ of the JacobiJacobi method is included in the interval $(-1, 1)$. Compare with Theorem 9.4.1 and Exercise 7 of the book.

34. Let $A \in \mathbf{M}_n(\mathbb{C})$ be given, and $U(t) := \exp(tA)$.

(a) Show that $\|U(t)\| \leq \exp(t\|A\|)$ for $t \geq 0$ and any matrix norm. Deduce that the integral

$$\int_0^{+\infty} e^{-2\gamma t} U(t)^* U(t)\, dt$$

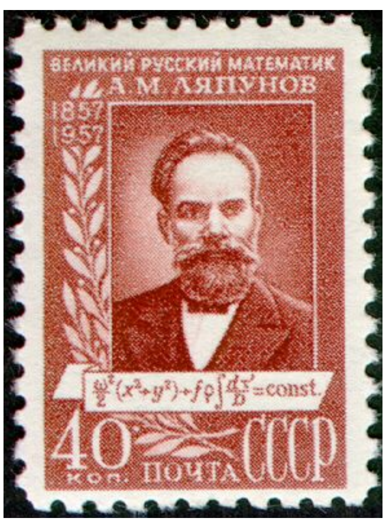
converges for every $\gamma > \|A\|$.

(b) Denote $H_\gamma$ the value of this integral, when it is defined. Computing the derivative at $h = 0$ of $h \mapsto U(h)^* H_\gamma U(h)$, by two different methods, deduce that $H_\gamma$ is a solution of

$$A^* X + XA = 2\gamma X - I_n, \quad X \in \mathbf{HPD}_n. \tag{3}$$

(c) Let $\gamma$ be larger than the supremum of the real parts of eigenvalues of $A$. Show that Equation (3) admits a unique solution in $\mathbf{HPD}_n$, and that the above integral converges.

(d)




**Alexandr M. Lyapunov.**
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In particular, if the spectrum of $M$ has positive real part, and if $K \in \mathbf{HPD}_n$ is given, then the LyapunovLyapunov equation

$$M^* H + HM = K, \qquad H \in \mathbf{HPD}_n$$

admits a unique solution.
Let $x(t)$ be a solution of the differential equation $\dot{x} + Mx = 0$, show that $t \mapsto x^* H x$ decays, and strictly if $x \neq 0$.

35. Show that if $M \in \mathbf{M}_n(\mathbb{C})$ and if $\operatorname{Tr} M^* M \leq n$, then $|\det M| \leq 1$, with equality if, and only if, $M$ is unitary.

36. Let $k$ be a field of characteristic zero, meaning that 1 spans an additive subgroup, isomorphic to $\mathbb{Z}$. By a slight abuse of notation, this subgroup is denoted by $\mathbb{Z}$. We call $\Lambda$ a *lattice of rank* $n$ if there exists a basis $\{x^1, \dots, x^n\}$ of $k^n$ such that

$$\Lambda = \mathbb{Z}x^1 \oplus \cdots \oplus \mathbb{Z}x^n.$$

Such a basis of $k^n$ is called *a basis of* $\Lambda$.

Let $\Lambda$ and $\Lambda'$ be two lattices of rank $n$, with $\Lambda' \subset \Lambda$. Prove that there exist a basis $\{y^1, \dots, y^n\}$ of $\Lambda$, together with integers $d_1, \dots, d_n$, such that $d_1|d_2$, $d_2|d_3$, $\dots$, and $\{d_1 y^1, \dots, d_n y^n\}$ is a basis of $\Lambda'$. Show that $d_1, \dots, d_n$ are uniquely defined by this property, up to their signs. Finally, prove that the product $d_1 \cdots d_n$ equals the order $[\Lambda : \Lambda']$ of the quotient group $\Lambda/\Lambda'$.

37. (From E. S. KeyKey.) Given the companion matrix of a polynomial $X^n - a_1 X^{n-1} - \cdots - a_n$, in the form

$$\begin{pmatrix} 0 & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ a_n & \cdots & \cdots & a_1 \end{pmatrix},$$

and given a root $x$ of $P$, compute an eigenvector associated with $x$. Deduce that, if $P_1, \dots, P_k$ have a common root $z$, then $z^k$ is an eigenvalue of the product of their companion matrices.

38. Define the wedge product in $k^3$ in the same way as in $\mathbb{R}^3$. Given a non-zero vector $a$ in $k^3$, find all matrices $A \in \mathbf{M}_3(k)$ with the property that $(Aa) \wedge x = A(a \wedge x) = a \wedge (Ax)$ for every $x \in k^3$. **Hint**: The result depends on whether $a \cdot a = a_1^2 + a_2^2 + a_3^2$ vanishes or not.

39. (From Y. TianTian, Yongge.) Let $P$ be a projector (that is $P^2 = P$) on a real or complex finite dimensional space.

    (a) Prove that $I_n - P^*P$ is positive semi-definite if and only if $P$ is an orthogonal projector, that is $R(P) \perp \ker(P)$.

    (b) In general, prove the equalities

    $$\ker(I_n - P^*P) = R(P) \cap R(P^*) = \ker(2I_n - P - P^*).$$

    and deduce that

    $$R(I_n - P^*P) = \ker P + \ker P^* = R(2I_n - P - P^*).$$

40. Let $\|\cdot\|$ be a unitary invariant norm on $\mathbf{M}_n(\mathbb{C})$, and let $A \in \mathbf{HPD}_n$ and $B \in \mathbf{M}_n(\mathbb{C})$ be given. Recall that $A$ admits a unique logarithm $\log A$ in $\mathbf{H}_n$, a matrix such that $\exp(\log A) = A$. For complex numbers $z$, we thus define $A^z := \exp(z \log A)$.

(a) Define $F(z) := A^z B A^{1-z}$. Show that $\|F(z)\| = \|F(\Re z)\|$, then that $F$ is bounded on the strip $0 \le \Re z \le 1$.

(b) If $\|AB\| \le 1$ and $\|BA\| \le 1$, deduce that $\|A^{1/2}BA^{1/2}\| \le 1$. More generally,

$$\|A^{1/2}BA^{1/2}\|^2 \le \|AB\| \cdot \|BA\|.$$

Compare with Exercise 21, page 79.

(c) Replacing $B$ by $B' := (1 + c\|B\|)^{-1}B$ and $A$ by $A' := A + cI_n$ with $c > 0$, show that the same result holds true if we suppose only that $A \in \mathbf{H}_n$ is positive semi-definite.

(d) More generally, if $H, K$ are Hermitian positive semi-definite, prove that

$$\|HBK\|^2 \le \|H^2 B\| \cdot \|BK^2\|.$$

41. Let $H \in \mathbf{H}_n$ be positive semidefinite and have bandwidth $2b - 1$, meaning that $|j - i| \ge b$ implies $m_{ij} = 0$.

(a) Prove that the CholeskyCholesky factorization $H = LL^*$ inherits the band-property of $M$.

(b) Deduce that the largest eigenvalue of $H$ is lower than or equal to the maximum among the sums of $b$ consecutive diagonal entries. Compare to Exercise 20, page 59 of the book.

42. (Preconditionned Conjugate Gradient Method.)

Let $Ax = b$ be a linear system whose matrix $A$ is symmetric positive definite (all entries are real.) Recall that the convergence ratio of the Conjugate gradient method is the number

$$\tau_{GC} = -\log \frac{\sqrt{K(A)} - 1}{\sqrt{K(A)} + 1},$$

that behaves like $2/\sqrt{K(A)}$ when $K(A)$ is large, as it uses to be in the real life. The number $K(A) := \lambda_{max}(A)/\lambda_{min}(A)$ is the condition number of $A$.

Preconditioning is a technique that reduces the condition number, hence increases the convergence ratio, through a change of variables. Say that a new unknown is $y := B^T x$, so that the system is equivalent to $\tilde{A}y = \tilde{b}$, where

$$\tilde{A} := B^{-1}AB^{-T}, \quad b = B\tilde{b}.$$

For a given preconditioning, we associate the matrices $C := BB^T$ and $T := I_n - C^{-1}A$. Notice that preconditioning with $C$ or $\alpha^{-1}C$ is essentially the same trick if $\alpha > 0$, although $T = T(\alpha)$ differs significantly. Thus we mere associate to $C$ the whole family

$$\{T(\alpha) = I_n - \alpha C^{-1}A\,;\, \alpha > 0\}.$$

(a) Show that $\tilde{A}$ is similar to $C^{-1}A$.

(b) Consider the decomposition $A = M - N$ with $M = \alpha^{-1}C$ and $N = \alpha^{-1}C - A$. This yields an iterative method

$$C(x^{k+1} - x^k) = b - \alpha A x^k,$$

whose iteration matrix is $T(\alpha)$. Show that there exist values of $\alpha$ for which the method is convergent. Show that the optimal parameter (the one that maximizes the convergence ratio) is

$$\alpha_{opt} = \frac{2}{\lambda_{min}(\tilde{A}) + \lambda_{max}(\tilde{A})},$$

with the convergence ratio

$$\tau_{opt} = -\log \frac{K(\tilde{A}) - 1}{K(\tilde{A}) + 1}.$$

(c) When $K(\tilde{A})$ is large, show that

$$\frac{\tau_{GCP}}{\tau_{opt}} \sim \sqrt{K(\tilde{A})},$$

where $\tau_{GCP}$ stands for the preconditioned conjugate gradient, that is the conjugate gradient applied to $\tilde{A}$.

Conclusion ?

43. (Continuation.) We now start from a decomposition $A = M - N$ and wish to construct a preconditioning.

Assume that $M + N^T$, obviously a symmetric matrix, is positive definite. We already know that $\|M^{-1}N\|_A < 1$, where $\|\cdot\|_A$ is the Euclidean norm associated with $A$ (Lemma 9.3.1.)

(a) Define $T := (I_n - M^{-T}A)(I_n - M^{-1}A)$. Prove that $\|T\|_A < 1$. Deduce that the "symmetric" method

$$Mx^{k+1/2} = Nx^k + b, \quad M^T x^{k+1} = N^T x^{k+1/2} + b$$

is convergent (remark that $A = M^T - N^T$.)

This method is called *symmetric S.O.R.*, or *S.S.O.R.* when $M$ is as in the relaxation method.

(b) From the identity $T = I_n - M^{-T}(M + N^T)M^{-1}A$, we define $C = M(M + N^T)^{-1}M^T$. Express the corresponding preconditioning $C(\omega)$ when $M$ and $N$ come from the S.O.R. method:

$$M = \frac{1}{\omega}D - E, \quad \omega \in (0, 2).$$

This is the *S.S.O.R.* preconditioning.

(c) Show that $\lambda_{max}(C(\omega)^{-1}A) \le 1$, with equality when $\omega = 1$.

(d) Compute $\rho(T)$ and $K(\tilde{A})$ when $A$ is tridiagonal with $a_{ii} = 2$, $a_{i,i\pm 1} = -1$ and $a_{ij} = 0$ otherwise. Compare the S.S.O.R. method and the S.S.O.R. preconditioned conjugate gradient method.

44. In control theory of linear systems, we face differential equations

$$\dot{x} = Ax + Bu,$$

where $A \in \mathbf{M}_n(\mathbb{R})$ and $B \in \mathbf{M}_{n\times m}(\mathbb{R})$. We call $x(t)$ the *state* and $u(t)$ the *control*. Controllability is the property that, given a time $T > 0$, an initial state $x_0$ and a final state $x_f$, it is possible to find a control $t \mapsto u(t)$ such that $x(0) = x_0$ and $x(T) = x_f$.

(a) Assume first that $x(0) = 0$. Express $x(T)$ in terms of $Bu$ and $e^{tA}$ $(0 \le t \le T)$ in a closed form. Deduce that controllability is equivalent to $x_f \in H$, where

$$H := \mathop{+}_{k=0}^{n-1} R(A^k B).$$

This is Kalman'sKalman criterion.

(b) Reversing the time, show that controllability from a general $x(0)$ to $x_f = 0$ is equivalent to Kalman'sKalman criterion. Conclude that controllability for general initial and final states is equivalent to Kalman's criterion.

(c) Prove the following forms of Kalman's criterion:

i. $\ker B^T$ does not contain any eigenvector of $A^T$.

ii. For every complex number $z$, the matrix

$$\begin{pmatrix} A^T - zI_n \\ B^T \end{pmatrix}$$

has rank $n$.

(d) Assume $m = 1$: The control is scalar. We shall denote $b$ instead of $B$, since it is a vector. Furthermore, assume controllability. Show that there exists a vector $c \in \mathbb{R}^n$ such that $c^T(A+I_n)^{-1}b = -\delta_1^k$ for $k = 1, \dots, n$. Deduce that the spectrum of $A+bc^T$ reduces to $\{1\}$. Hence the *feedback* $u(t) = c(t) \cdot x(t)$ yields *stabilization*, since then $x(t)$ decays exponentially for every initial data.

45. Consider a zero-sum repeated game between two players A and B. Each player chooses one object among a list $O_1, \dots, O_n$. When A chooses $O_i$ and B chooses $O_j$, the payoff is $m_{ij} \in \mathbb{R}$, which is positive if A wins, negative if B wins. Obviously, the matrix $M$ is skew-symmetric.

Players play a large number of games. One may represent their strategies by vectors $x^A$, $x^B$, where $x_i^A$ is the probability that A chooses the $i$-th object. Hence $x^A \ge 0$ and $\sum_i x_i^A = 1$, and the same for $x^B$. Given a pair of strategies, the expectation has the form

$\phi(x^A, x^B)$ where $\phi(x,y) := x^T My$. Player A tries to maximize, while B tries to minimize the expectation.

A *NashNash equilibrium* $(\bar{x}, \bar{y})$ is a pair of strategies that is optimal for both players, in the sense that, for all strategies $x$ and $y$:

$$\phi(x, \bar{y}) \leq \phi(\bar{x}, \bar{y}) \leq \phi(\bar{x}, y).$$

(a) Prove that a NashNash equilibrium always exists, and that the set of NashNash equilibria is a product $C \times D$ of convex subsets.

(b) Deduce that, for every skew-symmetric matrix with real entries, there exists a non-negative vector $x \neq 0$, such that $Mx$ is non-negative, and $x_j(Mx)_j = 0$ for each $j = 1, \dots, n$.

(c) **Example**: The list of objects consists in scissors, a stone, a hole and a sheet of paper. The payoff is $\pm 1$, according to the following natural rules. Scissors win against paper, but looses against the hole and the stone. Paper wins against the hole and the stone. The hole wins again the stone.

Find the (unique and rather counter-intuitive) NashNash equilibrium.

**Remark**: In many countries, this game is more symmetric, with only scissors, stone and the sheet of paper. It was illustrated during WWII, when the leaders of Great Britain, USSR and Germany each had their own choice. Ultimately, the cissors and the stone defeated the sheet of paper. During the Cold War, there remained the cissors and the stone, untill Staline'sStaline death in 1953 and Churchill'sChurchill loss of 1955 elections. One had to wait untill 1991 to see the cissors defeating the stone, unlike in the usual game.

46. (P. Van den Driesschevanden@van den Driessche, H. K. WimmerWimmer.)

(a) Characterize the complex numbers $a, b, c$ and the vectors $X \in \mathbb{C}^{n-1}$, such that the following matrix is unitary

$$U := \begin{pmatrix} aX^* & c \\ I_{n-1} - bXX^* & X \end{pmatrix}.$$

(b) Let $C$ be a companion matrix, given in the form

$$C = \begin{pmatrix} 0^* & m \\ -I_{n-1} & V \end{pmatrix}, \quad m \in \mathbb{C},\, V \in \mathbb{C}^{n-1}.$$

Find the polar decomposition $C = QH$. **Hint**: $Q$ equals $-U$, where $U$ is as in the previous question.

47. Let $E$ be an invariant subspace of a matrix $M \in \mathbf{M}_n(\mathbb{R})$.

(a) Show that $E^\perp$ is invariant under $M^T$.

(b) Prove the following identity between characteristic polynomials:

$$P_M(X) = P_{M|E}(X) P_{M^T|E^\perp}(X). \tag{4}$$

48. (See also YakubovichYakubovich & StarzhinskiiStarzhinskii, *Linear differential equations with periodic coefficients*. Wiley & Sons, 1975.)

Let $M$ belong to $\mathbf{Sp}_n(\mathbb{R})$. We recall the notations of Chapter 7: $M^T JM = J$ and $J^2 = -I_{2n}$ as well as $J^T = -J$.

(a) Show that the characteristic polynomial is reciprocal:

$$P_M(X) = X^{2n} P_M\left(\frac{1}{X}\right).$$

Deduce a classification of the eigenvalues of $M$.

(b) Define the quadratic form

$$q(x) := 2x^T JMx.$$

Verify that $M$ is a $q$-isometry.

(c) Let $(e^{-i\theta}, e^{i\theta})$ be a pair of *simple* eigenvalues of $M$ on the unit circle. Let $\Pi$ be the corresponding invariant subspace:

$$\Pi := \ker(M^2 - 2(\cos\theta)M + I_{2n}).$$

i. Show that $J\Pi^\perp$ is invariant under $M$.

ii. Using the formula (4) above, show that $e^{\pm i\theta}$ are not eigenvalues of $M|_{J\Pi^\perp}$.

iii. Deduce that $\mathbb{R}^{2n} = \Pi \oplus J\Pi^\perp$.

(d) (Continued.)

i. Show that $q$ does not vanish on $\Pi \setminus \{0\}$. Hence $q$ defines a Euclidian structure on $\Pi$.

ii. Check that $M|_\Pi$ is direct (its determinant is positive.)

iii. Show that $M|_\Pi$ is a rotation with respect to the Euclidian structure defined by $q$, whose angle is either $\theta$ or $-\theta$.

(e) More generally, assume that a plane $\Pi$ is invariant under a symplectic matrix $M$, with corresponding eigenvalues $e^{\pm i\theta}$, and that $\Pi$ is not Lagrangian: $(x, y) \mapsto y^T Jx$ is not identically zero on $\Pi$. Show that $M|_\Pi$ acts as rotation of angle $\pm\theta$. In particular, if $M = J$, show that $\theta = +\pi/2$.

(f) Let $H$ be an invariant subspace of $M$, on which the form $q$ is either positive or negative definite. Prove that the spectrum of $M|_H$ lies in the unit circle and that $M|_H$ is semisimple (the JordanJordan!Camille form is diagonal).

(g) Equivalently, let $\lambda$ be an eigenvalue of $M$ (say a simple one) with $\lambda \not\in \mathbb{R}$ and $|\lambda| \neq 1$. Let $H$ be the invariant subspace associated with the eigenvalues $(\lambda, \bar{\lambda}, 1/\lambda, 1/\bar{\lambda})$. Show that the restriction of the form $q$ to $H$ is neither positive nor negative definite. Show that the invariant subspace $K$ associated with the eigenvalues $\lambda$ and $\bar{\lambda}$ is $q$-isotropic. Thus, if $q|_H$ is non-degenerate, its signature is $(2, 2)$.

49. (From L. TartarTartar.) In $\mathbf{M}_n(\mathbb{R})$, prove the inequality

$$(\operatorname{Tr} M)^2 \leq (\operatorname{rk} M) \operatorname{Tr}(M^T M).$$

**Hint**: Apply Schur'sSchur trigonalization Theorem.

Use the latter to built as many as possible non-trivial quadratic forms on $\mathbf{M}_n(\mathbb{R})$, non-negative on the cone of singular matrices.

50. The minors of general matrices are not independent on each other. For instance, each entry is a minor (of order one) and the determinant (an other minor) is defined in terms of entries. An other instance is given by the row- or column-expansion of the determinant. See also Exercise 24 above. Here is another relation.

Denote $P_{2m}$ the set of partitions $I \cup J$ of $\{1, \dots, 2m\}$ into two sets $I$ and $J$ of equal lengths $m$. If $(I, J) \in P_m$, let $\sigma(I, J)$ be the signature of the permutation $(i_1, \dots, i_m, j_1, \dots, j_m)$, where

$$I = \{i_1 \leq \cdots \leq i_m\}, \quad J = \{j_1 \leq \cdots \leq j_m\}.$$

Prove that, for every matrix $A \in \mathbf{M}_{2m \times m}(k)$, there holds

$$\sum_{(I,J) \in P_m} \sigma(I, J) A(I) A(J) = 0,$$

with

$$A(I) := A \begin{pmatrix} 1 & \cdots & m \\ i_1 & \cdots & i_m \end{pmatrix}.$$

Find other algebraic relations (syzygies) between minors.

51. (a) Let $\Sigma$ belong to $\mathbf{SPD}_n$. Prove that the linear map $\sigma \mapsto \sigma\Sigma + \Sigma\sigma$ is an automorphism of $\mathbf{Sym}_n(\mathbb{R})$. **Hint:** Consider the spectrum of $\sigma^2\Sigma$. Show that it is real non-negative on one hand, non-positive on the other hand. Then conclude.

(b) Let $\Sigma$ belong to $\mathbf{SPD}_n$. Compute the differential of the map $\Sigma \mapsto \Sigma^2$.

(c) Deduce that the square root map $S \mapsto \sqrt{S}$ is analytic on $\mathbf{SPD}_n$. **Remark:** The same result holds true on $\mathbf{HPD}_n$, same proof.

52. We consider real symmetric $n \times n$ matrices. We use the Schur–FrobeniusSchurFrobenius norm $\| \cdot \|_F$. The result would be the same for complex Hermitian matrices.

(a) Given two matrices $A = \mathrm{diag}(a_1, \dots, a_n)$ and $B = P\mathrm{diag}(b_1, \dots, b_n)P^T$, with $P$ an orthogonal matrix, verify that

$$\|B - A\|_F^2 = \sum_{i,l} p_{il}^2 (a_i - b_l)^2.$$

(b) Assume that both $A$ and $B$ are positive definite and as above. Prove that

$$\|\sqrt{B} - \sqrt{A}\|_F^4 \leq n\|B - A\|_F^2.$$

**Hint:** Use $|\sqrt{b} - \sqrt{a}|^2 \leq |b - a|$ for positive real numbers, together with Cauchy–Schwarz inequality.

(c) Deduce that the square root map, defined on $\mathbf{SPD}_n$, is Hölderian with exponent $1/2$. Verify that the supremum of

$$\|\sqrt{B} - \sqrt{A}\|_F^2 \|B - A\|_F^{-1},$$

taken either on $\mathbf{SPD}_n$ or on the subset of diagonal matrices, takes the same value. **Remark**: See the improvement of this result in Exercise 110.

53. The electromagnetic field $(E, B)$ must be understood as an alternate 2-form:

$$\omega = dt \wedge (E \cdot dx) + B_1 dx_2 \wedge dx_3 + B_2 dx_3 \wedge dx_1 + B_3 dx_1 \wedge dx_2.$$

In coordinates $(t, x_1, x_2, x_3)$, it is thus represented by the alternate matrix

$$A = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} \in \mathbf{M}_4(\mathbb{R}).$$

In another choice of LorentzianLorentz coordinates, $\omega$ is thus represented by the new matrix $A' := M^T A M$. Matrix $M \in \mathbf{O}(1,3)$ is that of change of variables. With $A'$ are associated $(E', B')$. Namely, the decomposition of the electro-magnetic field into an electric part and a magnetic one depends on the choice of coordinates. The purpose of this exercise is to find Lorentzian invariants, that is quantities associated with $\omega$ that do not depend on the choice of coordinates.

(a) If $M = \mathrm{diag}(\pm 1, Q)$ with $Q \in \mathbf{O}_3(\mathbb{R})$, express $E', B'$ in terms of $E, B$ and $Q$.

(b) Verify that $M$ belongs to $\mathbf{O}(1,3) \cap \mathbf{SPD}_4$ if and only if there exists a unit vector $q$ and numbers $c, s$ with $c^2 - s^2 = 1$, such that

$$M = \begin{pmatrix} c & sq^T \\ sq & I_3 + (c-1)q\,q^T \end{pmatrix}.$$

Find how $M$ transforms $E$ and $B$.

(c) Verify that $|B|^2 - |E|^2$ and $|E \cdot B|$ are Lorentzian invariants.

(d) We now show that $|B|^2 - |E|^2$ and $|E \cdot B|$ are the only Lorentzian invariants. Thus let $(E_i, B_i)$ and $(E_f, B_f)$ be two pairs of vectors in $\mathbb{R}^3$, defining two 2-forms $\omega_i$ and $\omega_f$. We assume that

$$|B_f|^2 - |E_f|^2 = |B_i|^2 - |E_i|^2 =: \epsilon, \quad |E_f \cdot B_f| = |E_i \cdot B_i| =: \delta.$$

i. Choose $q \in \mathbf{S}^2$ that is orthogonal to both $E_i$ and $B_i$. If $\theta \in \mathbb{R}$ is given, define

$$f(\theta) := |(\cosh\theta)E + (\sinh\theta)B \wedge q|^2.$$

Show that the range of $f$ covers exactly the interval $[\frac{1}{2}(\epsilon + \sqrt{\epsilon^2 + 4\delta}), +\infty)$.

ii. Deduce that there exist a pair $(E_m, B_m)$ of vectors, both belonging to the plane spanned by $E_i$ and $B_i$ and with

$$|B_m|^2 = |B_f|^2, \quad |E_m|^2 = |E_f|^2, \quad |E_m \cdot B_m| = \delta,$$

and a symmetric positive definite Lorentzian matrix $M_i$ such that

$$\omega_i(M_i x, M_i y) \equiv \omega_m(x, y).$$

iii. Show also that there exists an orthogonal Lorentzian matrix $R_f$ such that

$$\omega_m(R_f x, R_f y) \equiv \omega_f(x, y).$$

Conclude.

54. Let $A \in \mathbf{H}_n$ be given blockwise in the form

$$A = \begin{pmatrix} B & C \\ C^* & D \end{pmatrix}, \quad B \in \mathbf{H}_m,\ D \in \mathbf{H}_{n-m}.$$

Assume that the least eigenvalue $\lambda$ of $B$ is greater than the highest eigenvalue $\mu$ of $D$. Prove that the spectrum of $A$ is included into $(-\infty, \mu] \cup [\lambda, +\infty)$. Prove also that $[\lambda, +\infty)$ contains exactly $m$ eigenvalues, counting with multiplicities.

55. Let $k$ be a field and $A, B, C$ be non-colinear matrices in $\mathbf{M}_n(k)$, such that $A - B$, $B - C$ and $C - A$ have rank one. Show that every distinct matrices $P, Q$ in the affine plane spanned by $A, B, C$ differ from a rank-one matrix.

On the contrary, find in $M_2(k)$ four matrices $A, B, C, D$ such that $A - B$, $B - C$, $C - D$, $D - A$ have rank one, but for instance $A - C$ has rank two.

56. Given an abelian ring $A$, recall that for a square matrix $M \in \mathbf{M}_n(A)$, adj$M$ denotes the transpose of the cofactors of $M$, so that the following identity holds

$$M(\mathrm{adjM}) = (\mathrm{adj}M)M = (\det M)I_n.$$

(a) Prove that the rank of adj$M$ equals either $n$, 1 or 0. Characterize the three cases by the rank of $M$.

(b) Describe in more details adj$M$ when rk$M = n-1$ and $A$ is a field. Show that if 0 is a simple eigenvalue, then the right and left eigenvectors ($\ell$ and $r$), normalized by $\ell r = 1$, are related by

$$\ell_i r_i = \frac{\det M^{(i)}}{\sum_j \det M^{(j)}},$$

where $M^{(j)}$ is the matrix obtained from $M$ by removing its $j$-th row and column. Justify that the denominator is non-zero.

(c) Given $n \geq 2$, show that $\det(\mathrm{adj}M) = (\det M)^{n-1}$ and $\mathrm{adj}(\mathrm{adj}M) = (\det M)^{n-2} M$. In the special case $n = 2$, the latter means $\mathrm{adj}(\mathrm{adj}M) = M$. In particular, if $n \geq 3$ (but not when $n = 2$), we have

$$(\det M = 0) \Longrightarrow (\mathrm{adj}(\mathrm{adj}M) = 0_n).$$

**Hint**: First prove the formulæ when $A$ is a field and $\det M \neq 0$. Then, considering that the required identities are polynomial ones with integral coefficients, deduce that they hold true in $\mathbb{Z}[X_{11}, \ldots, X_{nn}]$ by choosing $A = \mathbb{Q}$, and conclude.

57. (A. MajdaMajda, ThomannThomann) Let $A := \mathrm{diag}\{1, -1, \ldots, -1\}$ be the matrix of the standard scalar product $\langle \cdot, \cdot \rangle$ in Lorentzian geometry. The vectors belong to $\mathbb{R}^{1+n}$, and read $x = (x_0, \ldots, x_n)^T$. The forward cone $K^+$ is defined by the inequalities

$$\langle x, x \rangle > 0, \quad x_0 > 0.$$

The words *non-degenerate* and *orthogonal* are used with respect to the Lorentzian scalar product.

(a) Let $q, r \in K^+$ be given. Show the "reverse Cauchy–SchwarzCauchySchwarz inequality"

$$\langle q, q \rangle \langle r, r \rangle \leq \langle q, r \rangle^2, \tag{5}$$

with equality if, and only if, $q$ and $r$ are colinear. This is a special case of a deep result by L. GårdingGaa@Gårding.

(b) More generally, if $q \in K^+$ and $r \in \mathbb{R}^{1+n}$, prove (5).

(c) Given $\mu \in \mathbb{R}$ and $q, r \in K^+$, define the quadratic form

$$H_\mu(x) := \mu \langle q, x \rangle \langle r, x \rangle - \langle q, r \rangle \langle x, x \rangle.$$

i.

Lorentz, *Hendrik* (Holland)

**Hendrik Lorentz.**

In the case $q = r = e_0 := (1, 0, \dots, 0)^T$, check that $H_\mu$ is positive definite whenever $\mu > 1$.
Deduce that if $\mu > 1$ and if $q, r$ are colinear, then $H_\mu$ is positive definite.
**Hint**: Use a LorentzLorentz transformation to drive $q$ to $e_0$.

ii. We now assume that $q$ and $r$ are not colinear. Check that the plane $P$ spanned by $q$ and $r$ is non-degenerate and that the LorentzLorentz "norm" is definite on $P^\perp$. Deduce that $H_\mu$ is positive definite if, and only if, its restriction to $P$ has this property.

iii. Show that $H_\mu$ is not positive definite for $\mu \leq 1$, as well as for large positive $\mu$'s. On the other hand, show that it is positive definite for every $\mu$ in a neighbourhood of the interval

$$\left[2, \frac{2\langle q, r\rangle^2}{\langle q, r\rangle^2 - \langle q, q\rangle \langle r, r\rangle}\right].$$

58. (a) Given a matrix $M \in \mathbf{M}_n(\mathbb{C})$ that have a positive real spectrum, show that

$$\det(I_n + M) \geq 2^n \sqrt{\det M}.$$

(b) Deduce that, for every positive definite Hermitian matrices $H, K$, there holds

$$(\det(H + K))^2 \geq 4^n (\det H)(\det K).$$

**Nota**: This inequality can be improved into

$$(\det(H + K))^{1/n} \geq (\det H)^{1/n} + (\det K)^{1/n},$$

the proof of which being slightly more involved. This can be viewed as a consequence of an inequality of L. GårdingGaa@Gårding about hyperbolic polynomials. See Exercise 218 for a proof without polynomials. See Exercise 219 for an improvement of the inequality above.

(c) Show that the map

$$H \mapsto -\log\det H$$

is strictly convex upon $\mathbf{HPD}_n$. Notice that Gårding'sGaa@Gårding inequality tells us the better result that $H \mapsto (\det H)^{1/n}$ is concave. This is optimal since this map is homogeneous of degree one, and therefore is linear on rays $\mathbb{R}^+ H$. However, it depends on $n$, while $H \mapsto \log\det H$ does not.

(d) Deduce that the set of positive definite Hermitian matrices such that $\det H \geq 1$ (or greater or equal to some other constant) is convex.

59. Let $N \in \mathbf{M}_n(\mathbb{C})$ be given, such that every matrix of the form $N + H$, $H$ Hermitian, has a real spectrum.

    (a) Prove that there exists a matrix $M$ in $N + \mathbf{H}_n$ that has only simple eigenvalues.

    (b) Because of simplicity, the eigenvalues $M + H \mapsto \lambda_j$ are $\mathcal{C}^\infty$-functions for $H$ small. Compute their differentials:
    $$d\lambda_j(M) \cdot K = \frac{Y_i^* K X_i}{Y_i^* X_i},$$
    where $X_i$, $Y_i$ are eigenvectors of $M$ and $M^*$, respectively.

    (c) Show that $Y_i$ and $X_i$ are colinear. **Hint**: Take $K$ of the form $xx^*$ with $x \in \mathbb{C}^n$.

    (d) Deduce that $N$ is Hermitian.

60. One wishes to prove the following statement: A matrix $M \in \mathbf{M}_n(\mathbb{C})$, such that $M + H$ is diagonalizable for every $H \in \mathbf{H}_n$, has the form $iaI_n + K$ where $a \in \mathbb{R}$ and $K$ is Hermitian.

    (a) Prove the statement for $n = 2$.

    (b) Let $M$ satisfy the assumption. Show that there exists a Hermitian matrix $K$ such that $M + K$ is upper triangular with pure imaginary diagonal entries. Then conclude by an induction.

61. (P. D. LaxLax) Let $A \in \mathbf{M}_n(k)$ be given, with $k = \mathbb{R}$ or $\mathbb{C}$. Assume that for every $x \in k^n$, the following bound holds true:
    $$|\langle Ax, x\rangle| \leq \|x\|^2.$$
    Deduce that the sequence of powers $(A^m)_{m\in\mathbb{N}}$ is bounded. **Hint**: Prove that the unitary eigenvalues are semi-simple. Then use Exercise 10 of Chapter 4.

62. Let $e \in \mathbb{R}^n$ denote the vector $(1, \dots, 1)^T$. A square matrix $M \in \mathbf{M}_n(\mathbb{R})$ is called a "Euclidean distance matrix" (EDM) if there exist vectors $p_1, \dots, p_n$ in a Euclidean vector space $E$, such that $m_{ij} = \|p_i - p_j\|^2$ for every pair $(i, j)$.


Euclid (Vatican state)

    (a) Show that every Euclidean distance matrix, besides being non-negative and symmetric with a diagonal of zeroes (obvious conditions), also defines a non-positive quadratic form on the hyperplane $e^\perp$.

(b) Given a symmetric matrix $M$ with a diagonal of zeroes, assume that it defines a non-positive quadratic form on the hyperplane $e^\perp$. Check that $M \geq 0$ (in the sense that the entries are nonnegative.) Prove that there exists a vector $v \in \mathbb{R}^n$ such that the quadratic form

$$q(x) := (v \cdot x)(e \cdot x) - \frac{1}{2}x^T Mx$$

is non-negative. Let $S$ be the symmetric matrix associated with $q$, and let $P$ be a symmetric square root of $S$. Prove that $M$ is an EDM, associated with the column vectors $p_1, \dots, p_n$ of $P$.

(c) Prove that the minimal dimension $r$ of the Euclidean space $E$, equals the rank of $JMJ$, where $J$ is the orthogonal projection onto $e^\perp$:

$$J := I_n - \frac{1}{n}ee^T.$$

63. (F. HansenHansen, G. PedersenPedersen) Hereafter, we denote by $(x, y) = \sum_j x_j \bar{y}_j$ the usual Hermitian product in $\mathbb{C}^n$. Given a numerical function $f : I \to \mathbb{R}$ defined on an interval, and given a Hermitian $n \times n$ matrix $H$, with $\mathrm{Sp}(H) \subset I$, we define $f(H)$ in the following natural way: Let $H = U^*DU$ be a diagonalization of $H$ in a unitary basis, $D = \mathrm{diag}\{d_1, \dots, d_n\}$, then $f(H) := U^*f(D)U$, where

$$f(D) = \mathrm{diag}\{f(d_1), \dots, f(d_n)\}.$$

(a) Find a polynomial $P \in \mathbb{R}[X]$, that depends only on $f$ and on the spectrum of $H$, so that $f(H) = P(H)$. Deduce that the definition above is not ambiguous, namely that it does not depend on the choice of the unitary eigenbasis.

(b) Let $m$ be any positive integer and $H_1, \dots, H_m$ be Hermitian. We also give $m$ matrices $A_1, \dots, A_m$ in $\mathbf{M}_n(\mathbb{C})$, with the property that

$$A_1^*A_1 + \cdots + A_m^*A_m = I_n.$$

Finally, we define

$$H := A_1^*H_1A_1 + \cdots + A_m^*H_mA_m.$$

i. Let $I$ be an interval of $\mathbb{R}$ that contains all the spectra of $H_1, \dots, H_m$. Show that $H$ is Hermitian and that $I$ contains $\mathrm{Sp}(H)$.

ii. For each $\lambda \in I$, we denote by $E_k(\lambda)$ the orthogonal projector on $\ker(H_k - \lambda)$. If $\xi$ is a unit vector, we define the (atomic) measure $\mu_\xi$ by

$$\mu_\xi(S) = \sum_{k=1}^m \sum_{\lambda \in S} (E_k(\lambda)A_k\xi, A_k\xi).$$

Show that $\mu_\xi$ is a probability. Also, show that if $\xi$ is an eigenvector of $H$, then

$$(H\xi, \xi) = \int \lambda \, d\mu_\xi(\lambda).$$

iii. Under the same assumptions, show that

$$(f(H)\xi,\xi) = f\left(\int \lambda \, d\mu_\xi(\lambda)\right).$$

iv. If $f$ is convex on $I$, deduce that

$$\mathrm{Tr} f(H) \le \mathrm{Tr}\left(\sum_{k=1}^{m} A_k^* f(H_k) A_k\right).$$

**Hint:** Use Jensen'sJensen inequality, plus the fact that

$$\mathrm{Tr} M = \sum_{l=1}^{n} (M\xi_l, \xi_l),$$

for every unitary basis $\{\xi_1, \dots, \xi_n\}$, for instance an eigenbasis if $M$ is Hermitian.

64. We deal with complex $n \times n$ matrices. We denote by $\sigma(A)$ the spectrum of $A$ and by $\rho(A)$ its complement, the resolvant set of $A$. We use only the canonical Hermitian norm on $\mathbb{C}^n$ and write $\|A\|$ for the induced norm on $\mathbf{M}_n(\mathbb{C})$ (we wrote $\|A\|_2$ in the book). We denote $\mathrm{dist}(z; F)$ the distance from a complex number $z$ to a closed subset $F$ in $\mathbb{C}$.

(a) Prove that for every matrix $A \in \mathbf{M}_n(\mathbb{C})$ and complex number $z \in \rho(A)$, there holds

$$\|(z - A)^{-1}\| \le \frac{1}{\mathrm{dist}(z; \sigma(A))}. \tag{6}$$

(b) When $A$ is normal, prove that the equality holds in (6).

(c) Conversely, we consider a matrix $A$ such that the equality holds in (6).

i. Show that we may assume, up to a unitary conjugation, that $A$ be block-triangular

$$A = \begin{pmatrix} \lambda & X^* \\ 0 & B \end{pmatrix},$$

with $X \in \mathbb{C}^{n-1}$. **Hint:** Apply Theorem 3.1.3 (SchurSchur).

ii. When $A$ is block-triangular as above, compute the inverse of $z - A$ blockwise, when $z$ is close to (but distinct from) $\lambda$. Establish the following inequality

$$2\Re\left(\bar{\alpha} X^* (z - B)^{-1} v\right) + |z - \lambda|^2 \|(z - B)^{-1} v\|^2 \le \|v\|^2,$$

for every complex number $\alpha$ and $v \in \mathbb{C}^{n-m}$. Deduce that $X = 0$.

iii. Conclude, by an induction, that $A$ is diagonal. Finally, show that a matrix satisfies for every $z$ the equality in (6), if and only if it is normal.

65. We use the notations of the previous exercise. In addition, if $\epsilon > 0$ we define the $\epsilon$-pseudospectrum as

$$\sigma_\epsilon(A) := \sigma(A) \cup \left\{ z \in \rho(A) \,;\, \|(z-A)^{-1}\| \geq \frac{1}{\epsilon} \right\}.$$

We recall (Exercise 21 in this list) that the numerical range

$$\mathcal{H}(A) := \{ r_A(x) \,;\, \|x\|_2 = 1 \}$$

is a convex compact subset.

(a) Prove that

$$\sigma_\epsilon(A) = \bigcup_{\|B\| \leq \epsilon} \sigma(A+B).$$

(b) Prove also that

$$\sigma_\epsilon(A) \subset \{ z \in \mathbb{C} \,;\, \mathrm{dist}(z; \mathcal{H}(A)) \leq \epsilon \}.$$

66. (From notes by M. CosteCoste.) This exercise shows that a matrix $M \in \mathbf{GL}_n(\mathbb{R})$ is the exponential of a real matrix if, and only if, it is the square of another real matrix.

(a) Show that, in $\mathbf{M}_n(\mathbb{R})$, every exponential is a square.

(b) Given a matrix $A \in \mathbf{M}_n(\mathbb{C})$, we denote $\mathcal{A}$ the $\mathbb{C}$-algebra spanned by $A$, that is the set of matrices $P(A)$ as $P$ runs over $\mathbb{C}[X]$.

i. Check that $\mathcal{A}$ is commutative, and that the exponential map is a homomorphism from $(\mathcal{A}, +)$ to $(\mathcal{A}^*, \times)$, where $\mathcal{A}^*$ denotes the subset of invertible matrices (a multiplicative group.)

ii. Show that $\mathcal{A}^*$ is an open and connected subset of $\mathcal{A}$.

iii. Let $E$ denote $\exp(\mathcal{A})$, so that $E$ is a subgroup of $\mathcal{A}^*$. Show that $E$ is a neighbourhood of the identity. **Hint**: Use the Implicit Function Theorem.

iv. Deduce that $E$ is closed in $\mathcal{A}^*$; **Hint**: See Exercise 21, page 135. Conclude that $E = \mathcal{A}^*$.

v. Finally, show that every matrix $B \in \mathbf{GL}_n(\mathbb{C})$ reads $B = \exp(P(B))$ for some polynomial $P$.

(c) Let $B \in \mathbf{GL}_n(\mathbb{R})$ and $P \in \mathbb{C}[X]$ be as above. Show that

$$B^2 = \exp(P(B) + \bar{P}(B)).$$

Conclusion ?

67. (The Le Verrier–Faddeevlever@Le VerrierFaddeev method.) Given $A \in \mathbf{M}_n(k)$, we define inductively a sequence $(A_j, a_j, B_j)_{1 \leq j \leq n}$ by

$$A_j = AB_{j-1} \quad (\text{or } A_1 = A), \quad a_j = -\frac{1}{j}\,\mathrm{Tr}\, A_j, \quad B_j = A_j + a_j I_n.$$

Hamilton, *Rowan* (Eire)

Show that the characteristic polynomial of $A$ is

$$X^n + a_1 X^{n-1} + \cdots + a_n.$$

Apply Cayley–Hamilton'sCayleyHamilton theorem and compare with Exercise 25, page 37.

**Rowan Hamilton.**

68. Given $A \in \mathbf{M}_n(k)$, with its characteristic polynomial

$$P_A(X) = X^n + a_1 X^{n-1} + \cdots + a_n,$$

we form a sequence of polynomials by the *Horner'sHorner rule*:

$$p_0(X) := 1, \quad p_1(X) := X + a_1, \quad p_j(X) = X p_{j-1}(X) + a_j, \ldots$$

Prove that

$$(XI_n - A)^{-1} = \frac{1}{P_A(X)} \sum_{j=0}^{n-1} p_j(A) X^{n-j-1}.$$

69. Let $A \in \mathbf{M}_n(\mathbb{C})$ be given, with eigenvalues $\lambda_j$ and singular values $\sigma_j$, $1 \leq j \leq n$. We choose the decreasing orders:

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|, \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n.$$

Recall that the $\sigma_j$'s are the square roots of the eigenvalues of $A^*A$.

We wish to prove the inequality

$$\prod_{j=1}^{k} |\lambda_j| \leq \prod_{j=1}^{k} \sigma_j, \quad 1 \leq k \leq n.$$

(a) Prove directly the case $k = 1$. Show the equality in the case $k = n$.

(b) Working within the exterior algebra, we define $A^{\wedge p} \in \mathbf{End}(\Lambda^p(\mathbb{C}^n))$ by

$$A^{\wedge p}(x_1 \wedge \cdots \wedge x_p) := (Ax_1) \wedge \cdots \wedge (Ax_p), \qquad \forall x_1, \ldots, x_p \in \mathbb{C}^n.$$

Prove that the eigenvalues of $A^{\wedge p}$ are the products of $p$ terms $\lambda_j$ with pairwise distinct indices. Deduce the value of the spectral radius.

(c) We endow $\Lambda^p(\mathbb{C}^n)$ with the natural Hermitian norm in which the canonical basis made of $\mathbf{e}^{i_1} \wedge \cdots \wedge \mathbf{e}^{i_p}$ with $i_1 < \cdots < i_p$, is orthonormal. We denote by $\langle \cdot, \cdot \rangle$ the scalar product in $\Lambda^p(\mathbb{C}^n)$.

i. If $x_1, \dots, x_p, y_1, \dots, y_p \in \mathbb{C}^n$, prove that

$$\langle x_1 \wedge \cdots \wedge x_p, y_1 \wedge \cdots \wedge y_p \rangle = \det (x_i^* y_j)_{1 \le i,j \le p} .$$

ii. For $M \in \mathbf{M}_n(\mathbb{C})$, show that the Hermitian adjoint of $M^{\wedge p}$ is $(M^*)^{\wedge p}$.

iii. If $U \in \mathbf{U}_n$, show that $U^{\wedge p}$ is unitary.

iv. Deduce that the norm of $A^{\wedge p}$ equals $\sigma_1 \cdots \sigma_p$.

(d) Conclude.

70. Use Exercise 20.a of Chapter 5 to prove the theorem of R. HornHorn!Roger & I. SchurSchur : The set of diagonals $(h_{11}, \dots, h_{nn})$ of Hermitian matrices with given spectrum $(\lambda_1, \dots, \lambda_n)$ is the convex hull of the points $(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(n)})$ as $\sigma$ runs over the permutations of $\{1, \dots, n\}$.

71. (A theorem by P. D. LaxLax.)

Assume that a subspace $V$ of $\mathbf{M}_n(\mathbb{R})$ has dimension 3, and that its non-zero elements have a real spectrum, with pairwise distinct eigenvalues. When $M \in V$ and $M \neq 0_n$, denote

$$\lambda_1(M) < \cdots < \lambda_n(M)$$

its eigenvalues. We equip $\mathbb{R}^n$ with the standard Euclidean norm.

(a) Verify that

$$\lambda_j(-M) = -\lambda_{n-j+1}(M).$$

(b) Prove that there exists a continuous map

$$M \mapsto \mathcal{B}(M) = \{r_1(M), \dots, r_n(M)\},$$

defined on $V \setminus \{0_n\}$, such that $\mathcal{B}(M)$ is a unitary basis of $M$. **Hint**: The domain $V \setminus \{0_n\}$ is simply connected.

(c) Let choose $M_0$ a non-zero element of $V$. We orient $\mathbb{R}^n$ in such a way that $\mathcal{B}(M_0)$ be a direct basis. Show that $\mathcal{B}(M)$ is always direct.

(d) Show also that for every $j$, there exists a constant $\rho_j = \pm 1$ such that, for every non-zero $M$, there holds

$$r_j(-M) = \rho_j r_{n-j+1}(M).$$

(e) From the former questions, show that if $n \equiv 2, 3 \pmod 4$, then

$$\prod_{j=1}^{n} \rho_j = -1.$$

(f) On another hand, show that there always holds

$$\rho_j \rho_{n-j+1} = 1.$$

Deduce that $n \not\equiv 2 \pmod 4$.

72. (The exchange, gyration or sweep operator.) Given a matrix $M \in \mathbf{M}_n(k)$ in block form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A \in \mathbf{GL}_p(k),$$

we define a matrix ($q := n - p$)

$$\operatorname{exc}(M) := \begin{pmatrix} I_p & 0_{p\times q} \\ C & D \end{pmatrix} \times \begin{pmatrix} A & B \\ 0_{q\times p} & I_q \end{pmatrix}^{-1} = \begin{pmatrix} A & -A^{-1}B \\ CA^{-1} & D - CA^{-1}B \end{pmatrix}.$$

(a) Show that the *exchange map* is an involution:

$$\operatorname{exc}(\operatorname{exc}(M)) = M.$$

(b) If $D$ is non-singular, prove that $\operatorname{exc}(M)$ is non-singular, with

$$\operatorname{exc}(M^{-1}) = \operatorname{exc}(M)^{-1}.$$

(c) Let $J := \operatorname{diag}\{I_p, -I_q\}$. Show that

$$\operatorname{exc}(JMJ)^T = \operatorname{exc}(M^T).$$

(d) We restrict to $k = \mathbb{R}$. Recall that $\mathbf{O}(p,q)$ is the orthogonal group associated with $J$. Show that the exchange map is well defined on $\mathbf{O}(p,q)$. With the previous formulæ, prove that it maps $\mathbf{O}(p,q)$ on a subset of $\mathbf{O}_n(\mathbb{R})$.

(e) Show that the image of the exchange map is a dense open subset of $\mathbf{O}_n(\mathbb{R})$.

73. Using the quadratic forms of $\mathbb{R}^n$ that are preserved by elements of the groups $\mathbf{O}(p,q)$ and $\mathbf{O}_n(\mathbb{R})$, find a simpler proof of the fact that the exchange map maps the former into the latter.

74. Given a function $f : (0, +\infty) \to \mathbb{R}$, we may define a map

$$\begin{array}{rcl} M & \mapsto & f(M), \\ \mathbf{SPD}_n & \to & \mathbf{Sym}_n(\mathbb{R}) \end{array}$$

in the same way as we defined the square root. The uniqueness is proved with the same argument (see for instance Exercise 63.a). We say that $f$ is a *monotone matrix function* if, whenever $0_n < M < N$ in the sense of quadratic forms, there holds $f(M) < f(N)$.

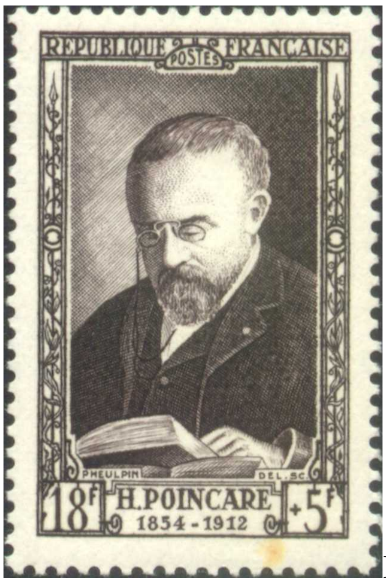
(a) Prove that $f(s) := -1/s$ is a monotone matrix function.

(b) Verify that the set of monotone matrix functions is a convex cone. Deduce that, for every nonnegative, non-zero measure $m$,

$$f(s) := -\int_0^{+\infty} \frac{dm(t)}{s+t}$$

is a monotone matrix function.

(c)



Poincaré, *Henri* (France)

**Henri Poincaré.**

Prove that, given numbers $a \geq 0$, $b \in \mathbb{R}$ and a non-negative bounded measure $m$, such that $(a, m) \neq (0, 0)$,

$$(7) \qquad f(s) := as + b - \int_0^{+\infty} \frac{1-st}{s+t} dm(t)$$

is a monotone matrix function (Loewner'sLoewner theorem asserts that every monotone matrix function is of this form ; such functions have a holomorphic extension to the domain $\mathbb{C} \setminus \mathbb{R}^-$, and send the PoincaréPoincaré half-space $\Im z > 0$ into itself. The formula above is the NevanlinnaNevanlinna representation of such functions.)

(d) Prove that, given a non-negative measure $\nu \neq 0$,

$$s \mapsto \left( \int_0^{+\infty} \frac{d\nu(t)}{s+t} \right)^{-1}$$

is a monotone matrix function (this function is the inverse of the CauchyCauchy transform of the measure $\nu$.)

(e) Compute

$$\int_0^{\infty} \frac{dt}{t^\alpha (s+t)}.$$

Deduce that $f_\alpha(s) = s^\alpha$ is a monotone matrix function for every $\alpha \in (0, 1)$.

(f) Prove that $0_n < M \leq N$ implies $\log M \leq \log N$. **Hint**: Consider the map $\alpha^{-1}(f_\alpha - 1)$.

(g) Find two matrices $M, P \in \mathbf{SPD}_2(\mathbb{R})$ such that $MP + PM \notin \mathbf{SPD}_2(\mathbb{R})$. Deduce that $f_2(s) = s^2$ is not a monotone matrix function.

75. Given $S \in \mathbf{SPD}_n(\mathbb{R})$, prove the formula

$$\frac{\pi^{n/2}}{\sqrt{\det S}} = \int_{\mathbb{R}^n} e^{-x^T S x} dx.$$

Deduce that

$$S \mapsto \log \det S$$

is concave on $\mathbf{SPD}_n(\mathbb{R})$. **Nota**: an other proof was given in Exercise 58.

76. Let $M \in \mathbf{M}_n(\mathbb{R})$ be non-negative and let us choose numbers $s_1, \dots, s_n$ in $(1, +\infty)$. For $\lambda \in \mathbb{R}$, define

$$S^\lambda := \operatorname{diag}\{s_1^\lambda, \dots, s_n^\lambda\}.$$

Assume that $M - S^\lambda$ is non-singular for every $\lambda \geq 0$. prove that $\rho(M) < 1$.

77. (From T. BarbotBarbot.) Let $S \in \mathbf{M}_m(\mathbb{R})$ and $R \in \mathbf{M}_{p\times m}(\mathbb{R})$ be given, with $S$ symmetric. Our goal is to prove that there exists a symmetric $\Sigma \in \mathbf{M}_p(\mathbb{R})$ such that

$$\left\| \begin{pmatrix} S & R^T \\ R & \Sigma \end{pmatrix} \right\|_2 \leq \left\| \begin{pmatrix} S \\ R \end{pmatrix} \right\|_2 =: \rho. \tag{8}$$

Indeed, since the reverse inequality is always true, we shall have an equality. **Nota**: This is a particular case of Parrott'sParrott lemma. See Exercise number 87.

This property may be stated as a vector-valued version of the Hahn–BanachHahnBanach theorem for symmetric operators: Let $u : F \to E$, defined on a subspace of $E$, a Euclidian space, with the symmetric property that $\langle u(x), y\rangle = \langle x, u(y)\rangle$ for every $x, y \in F$, then there exists a symmetric extension $U \in \mathcal{L}(E)$, such that $\|U\| \leq \|u\|$ (and actually $\|U\| = \|u\|$) in operator norm.

By homogeneity, we are free to assume $\rho = 1$ from now on. In the sequel, matrix inequalities hold in the sense of quadratic forms.

(a) Show that $S^2 + R^T R \leq I_m$.

(b) For $|\mu| > 1$, show that

$$H_\mu := \mu I_p - R(\mu I_m - S)^{-1} R^T$$

is well-defined, and that the map $\mu \mapsto H_\mu$ is monotone increasing on each of the intervals $(-\infty, -1)$ and $(1, +\infty)$.

(c) We begin with the case where $\rho(S) < 1$. Then $(I_m - S^2)^{-1}$ is well-defined and symmetric, positive definite. Prove that $\|R(I_m - S^2)^{-1/2}\|_2 \leq 1$. Deduce that $\|(I_m - S^2)^{-1/2} R^T\|_2 \leq 1$. Conclude that $H_{-1} \leq H_1$.

(d) Let $\Sigma$ be symmetric with $H_{-1} \leq \Sigma \leq H_1$. For instance, $\Sigma = H_{\pm 1}$ is convenient. Prove the inequality (8). **Hint**: Consider an eigenvector $(x, y)^T$ for an eigenvalue $\mu > 1$. Compute $y^T \Sigma y$ and reach a contradiction. Proceed similarly if $\mu < -1$.

(e) In the general case, we have by assumption $\rho(S) = \|S\|_2 \leq 1$. Replace $S$ by $tS$ with $0 < t < 1$ and apply the previous result. Then use a compactness argument as $t \to 1^-$.

78. Let $K$ be a compact subgroup of $\mathbf{GL}_n(\mathbb{R})$. We admit the existence of a *HaarHaar measure*, that is a probability $\mu$ on $K$, with the left-invariance property:

$$\int_K \phi(gh)\, d\mu(h) = \int_K \phi(h)\, d\mu(h), \quad \forall \phi \in \mathcal{C}(K), \forall g \in K.$$

(a) Let $|\cdot|$ denote the canonical Euclidian norm, and $(\cdot,\cdot)$ its scalar product. For $x, y \in \mathbb{R}^n$, define

$$\langle x, y\rangle := \int_K (hx, hy)\, d\mu(h), \quad \|x\| := \langle x, x\rangle^{1/2}.$$

Show that $\langle\cdot,\cdot\rangle$ is a scalar product, for which every element of $K$ is an isometry.

(b) Deduce that $K$ is conjugated to a subgroup of $\mathbf{O}_n(\mathbb{R})$. Similarly, prove that every compact subgroup of $\mathbf{GL}_n(\mathbb{C})$ is conjugated to a subgroup of $\mathbf{U}_n$.

79. (Thanks to P. de la Harpedela@de la Harpe and E. GhysGhys.) Let $p, q \geq 1$ be integers. We endow $\mathbb{C}^p$ and $\mathbb{C}^q$ with the canonical Hermitian scalar products $\langle y, z\rangle := y_1\bar{z}_1 + \cdots$. On $\mathbb{C}^{p+q}$, we consider their difference. The corresponding Hermitian form is

$$Q(z) = |z_1|^2 + \cdots - |z_{p+1}|^2 - \cdots.$$

Denote by $\mathcal{X}$ the set of linear subspaces of $\mathbb{C}^{p+q}$ on which the restriction of $Q$ is positive definite, and which are maximal for this property.

(a) Show that $E \in \mathcal{X}$ if, and only if, it is the graph

$$\{(x, Mx) \mid x \in \mathbb{C}^p\}$$

of a matrix $M \in \mathbf{M}_{q\times p}(\mathbb{C})$ with $\|M\| < 1$ ; this norm is taken with respect to the Hermitian norms of $\mathbb{C}^p$ and $\mathbb{C}^q$, in particular, $\|M\|^2 = \rho(M^*M) = \rho(MM^*)$.

(b) Let $Z \in \mathbf{U}(p,q)$, written blockwise as

$$Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Verify that, if $E \in \mathcal{X}$, then $ZE \in \mathcal{X}$. Deduce that if $M$ lies in the unit ball $\mathcal{B}$ of $\mathbf{M}_{q\times p}(\mathbb{C})$, then so does the matrix

$$\sigma_Z(M) := (AM + B)(CM + D)^{-1}.$$

(c) Show that $\sigma : \mathbf{U}(p,q) \to \mathbf{Bij}(\mathcal{B})$ is a homomorphism. In other words, it is a *group action* of $\mathbf{U}(p,q)$ over $\mathcal{B}$.

(d) Let $M \in \mathcal{B}$ be given. Prove that there exists a unique element $Z$ of $\mathbf{U}(p,q) \cap \mathbf{HPD}_n$ such that $M = \sigma_Z(0_{q\times p})$. This shows that the group action is *transitive*.

(e) Find the *stabilizer* of $0_{q\times p}$, that is the set of $Z$'s such that $\sigma_Z(0_{q\times p}) = 0_{q\times p}$.

(f) Deduce that $\mathcal{B}$ is diffeormorphic to the *homogeneous space*

$$\mathbf{U}(p,q)/(\mathbf{U}_p \times \mathbf{U}_q).$$

80. Given a function $\phi : \mathbb{R} \to \mathbb{R}$, we admit that there is a unique way to define a $\Phi : \mathbf{Sym}_n(\mathbb{R}) \to \mathbf{Sym}_n(\mathbb{R})$ such that $\Phi(O^TSO) = O^TSO$ if $O$ is orthogonal, and $\Phi(D) = \operatorname{diag}\{\phi(d_1), \dots, \phi(d_n)\}$ in the diagonal case (see for instance Exercise 63.a). It is clear that the spectrum of $\Phi(S)$ is the image of that of $S$ under $\phi$.

Let $B \in \mathbf{M}_n(\mathbb{R})$ be symmetric and $A \in \mathbf{M}_n(\mathbb{R})$ be diagonal, with diagonal entries $a_1, \dots, a_n$. If $\phi$ is of class $\mathcal{C}^2$, prove Balian'sBalian formula:

$$\lim_{t\to 0} t^{-2} \operatorname{Tr}(\Phi(A+tB) + \Phi(A-tB) - 2\Phi(A)) = \sum_i b_{ii}^2 \phi''(a_i) + \sum_{i,j\,(j\neq i)} b_{ij}^2 \frac{\phi'(a_j) - \phi'(a_i)}{a_j - a_i}.$$

**Hint**: Prove the formula first in the case where $\phi(t) = t^m$ for some integer $m$. Then pass to general polynomials, then to $\mathcal{C}^2$ functions.

81. We denote by $\|\cdot\|_F$ the FrobeniusFrobenius norm: $\|A\|_F^2 = \operatorname{Tr}(A^*A)$.

(a) Show that the set $\mathcal{N}_n$ of $n \times n$ normal matrices is closed. Deduce that if $A \in \mathbf{M}_n(\mathbb{C})$, there exists an $N$ in $\mathcal{N}_n$ for which $\|A - N\|_F$ is minimum.

(b) Given $h \in \mathbf{H}_n$ and $t \in \mathbb{R}$, $\exp(ith)$ is unitary. Therefore we have

$$\|A - N\|_F \le \|A - e^{ith} N e^{-ith}\|_F.$$

By letting $t \to 0$, deduce that

$$(A - N)N^* - N^*(A - N) \in \mathbf{H}_n. \tag{9}$$

(c) Using a unitary conjugation, show that we may assume that $N$ is diagonal. In that case, write $N = \operatorname{diag}\{d_1, \dots, d_n\}$. Then:

i. Show that $d_j = a_{jj}$. **Hint**: Compare with other diagonal matrices.

ii. Suppose that $d_j = d_k$ $(=: d)$ for some pair $(j,k)$ $(j \neq k)$. Verify that

$$\|B - dI_2\|_F \le \|B - n\|_F, \quad \forall n \in \mathcal{N}_2,$$

where

$$B := \begin{pmatrix} a_{jj} & a_{jk} \\ a_{kj} & a_{kk} \end{pmatrix}.$$

Deduce that $a_{jk} = a_{kj} = 0$ (in other words, $B = dI_2$).

iii. From (9) and the previous question, deduce that one can define a Hermitian matrix $H$, such that

$$h_{jk}(d_k - d_j) = a_{jk}, \quad \forall j, k\, (j \neq k).$$

(d) In conclusion, prove that for every $A$ in $\mathbf{M}_n(\mathbb{C})$, there exist a normal matrix $N$ (the one defined above) and a Hermitian one $H$, such that

$$A = N + [H, N].$$

82. (von Neumannvonneu@von Neumann inequality.)


Neumann, *John*!(USA)

Let $M \in \mathbf{M}_n(\mathbb{C})$ be a contraction, meaning that $\|M\|_2 \leq 1$. In other words, there holds $M^*M \leq I_n$ in the sense of Hermitian matrices. We recall that $\|M^*\|_2 = \|M\|_2$, so that we also have $MM^* \leq I_n$. We denote $S = \sqrt{I_n - M^*M}$ and $T := \sqrt{I_n - MM^*}$. Such positive square roots do vonneu@von exist, from unitary diagonalisation ; they turn out to be unique, but we do not use this fact.

**John von Neumann.**

Given an integer $k \geq 1$ and $K = 2k + 1$, we define a matrix $V_k \in \mathbf{M}_{Kn}(\mathbb{C})$ blockwise:

$$V_k = \begin{pmatrix} & \ddots & & & & & & & \\ & & I_n & & & & & & \\ & & & I_n & & & & & \\ & & & & S & -M^* & & & \\ & & & & M & T & & & \\ & & & & & & I_n & & \\ & & & & & & & I_n & \\ & & & & & & & & \ddots \\ I_n & & & & & & & & \end{pmatrix},$$

where the dots represent blocks $I_n$, while missing entries are blocks $0_n$. The column and row indices range from $-k$ to $k$. In particular, the central block indexed by $(0, 0)$ is $M$. All the other diagonal blocks are null.

(a) We begin with the easy case, where $M$ is normal. Prove that

$$\|p(M)\|_2 = \max\{|p(\lambda)|\,;\, \lambda \in \mathrm{Sp}(M)\}.$$

(b) We turn to the general case. Check that $MS = TM$ and $SM^* = M^*T$. Deduce that $V_k$ is unitary.

(c) Show that, whenever $q \leq 2k$, the central block of the $q$-th power $V_k^q$ equals $M^q$. Deduce that if $p \in \mathbb{C}[X]$ has degree at most $2k$, then the central block of $p(V_k)$ equals $p(M)$.

(d) Then, still assuming $d^o p \leq 2k$, show that $\|p(M)\|_2 \leq \|p(V_k)\|_2$.

(e) Deduce von Neumannvonneu@von Neumann inequality:

$$\|p(M)\|_2 \leq \max\{|p(\lambda)| \, ; \, \lambda \in \mathcal{S}^1\},$$

where $\mathcal{S}^1$ is the unit circle.

83. Let $k$ be field and $A \in \mathbf{M}_n(k)$ be given. We denote $B := \mathrm{adj}A = \hat{A}^T$ the transpose of the cofactors matrix. We recall $BA = AB = (\det A)I_n$. Denote also the respective characteristic polynomials

$$p_A(X) = X^n - a_1 X^{n-1} + \cdots + (-1)^n a_n, \quad p_B(X) = X^n - b_1 X^{n-1} + \cdots + (-1)^n b_n.$$

(a) Prove the identity
$$b_n p_A(X) = (-1)^n X^n p_B\left(\frac{a_n}{X}\right).$$

(b) Deduce that, if $\det A \neq 0$, there holds $b_j = a_{n-j} a_n^{j-1}$ for $j = 1, \ldots, n$.

(c) Extend these formulas to the general case. **Hint**: Apply the previous question when $k$ is replaced by the ring $A[a_{11}, \ldots, a_{nn}]$, where the indeterminates $a_{ij}$ are the entries of a general matrix $A$. See for instance the proof of Theorem 2.1.1.

(d) Conclude that the spectrum of $B$ is given, counting with multiplicities, by

$$\lambda_1 \cdots \lambda_{j-1} \lambda_{j+1} \cdots \lambda_n, \quad j = 1, \ldots, n,$$

where $\lambda_1, \ldots \lambda_n$ are the eigenvalues of $A$.

(e) Compare with the additional exercise 56.

84. We consider an $n \times n$ matrix $X$ whose entries $x_{ij}$ are independent indeterminates, meaning that the set of scalars is the ring $A := \mathbb{Z}[x_{11}, x_{12}, \ldots, x_{nn}]$. We embed $A$ into its field of fractions $k = \mathbb{Z}(x_{11}, x_{12}, \ldots, x_{nn})$.

(a) Prove that $X$ is non-singular.

(b) Let $1 \leq p \leq n-1$ be an integer. Consider the block forms

$$X = \begin{pmatrix} X_p & \cdot \\ \cdot & \cdot \end{pmatrix}, \quad X^{-1} = \begin{pmatrix} \cdot & \cdot \\ \cdot & Y_{n-p} \end{pmatrix},$$

where $X_p \in \mathbf{M}_p(A)$ and $Y_{n-p} \in \mathbf{M}_{n-p}(k)$. Prove the identity

$$\det Y_{n-p} = \frac{\det X_p}{\det X}.$$

85. Let $k$ be a field and $V$ a finite dimensional $k$-vector space. A *flag* in $V$ is a sequence $\mathcal{V} = (V_1, \dots, V_n = V)$ of subspaces with the properties $\dim V_m = m$ and $V_m \subset V_{m+1}$. In particular, $n = \dim V$. A basis $\{X_1, \dots, X_n\}$ is *adapted* to the flag if for every $m$, $\{X_1, \dots, X_m\}$ is a basis of $V_m$. Obviously, every flag admits an adapted basis, and conversely, an adapted basis determines uniquely the flag. Two adapted bases differ only by a "triangular" change of basis:

$$Y_m = a_{mm} X_m + a_{m,m-1} X_{m-1} + \cdots + a_{m1} X_1, \quad a_{mm} \neq 0.$$

Identifying $V$ to $k^n$, we deduce that the set of flags is in one-to-one correspondence with the set of right cosets $\mathbf{GL}_n(k)/\mathbf{T}_{sup}(k)$, where $\mathbf{T}_{sup}(k)$ denotes the subgroup of upper triangular matrices whose diagonal is non-singular. Therefore, questions about flags reduce to questions about $\mathbf{GL}_n(k)/\mathbf{T}_{sup}(k)$.

(a) Consider the statement

> (**B**): given two flags $\mathcal{V}$ and $\mathcal{V}'$ in $V$, there exists a basis adapted to $\mathcal{V}$, of which a permutation is adapted to $\mathcal{V}'$.

Prove that (**B**) is equivalent to

> (**M**): given $A \in \mathbf{GL}_n(k)$, there exists $T, T' \in \mathbf{T}_{sup}(k)$ and a permutation matrix $P$ such that $A = TPT'$.

(b) In the statement (**M**), prove that the permutation is necessarily unique.

(c) We turn to the existence part. Thus we give ourselves $A$ in $\mathbf{GL}_n(k)$ and we look for $T, T'$ and a permutation $\sigma$, such that

$$a_{ij} = \sum_{r=1}^{n} t_{i\sigma(r)} t'_{rj}.$$

Show the necessary condition

$$\sigma(1) = \max\{i\,;\, a_{i1} \neq 0\}.$$

(d) Show that there exists an index $i \neq \sigma(1)$ such that (we use the notation of Section 2.1 for minors)

$$A \begin{pmatrix} \sigma(1) & i \\ 1 & 2 \end{pmatrix} \neq 0.$$

**Hint**: This amounts to finding syzygies between minors taken from two given columns, here the first and the second.

Then prove that $\sigma(2)$ must be the maximum of such indices $i$.

(e) By induction, prove the necessary condition that $\sigma(j)$ is the largest index $i$ with the properties that $i \neq \sigma(1), \dots, \sigma(j-1)$ and

$$A \begin{pmatrix} \sigma(1) & \cdots & \sigma(j-1) & i \\ 1 & \cdots & j-1 & j \end{pmatrix} \neq 0.$$

(f) Deduce the theorem that for every two flags $\mathcal{V}$ and $\mathcal{V}'$, there exists a basis adapted to $\mathcal{V}$, of which a permutation is adapted to $\mathcal{V}'$.

(g) In the particular case $k = \mathbb{R}$ or $\mathbb{C}$, prove that, for every $A$ in a dense subset of $\mathbf{GL}_n(k)$, ($\mathbf{M}$) holds true, with $P$ the matrix associated with the permutation $\sigma(j) = n+1-j$.

86. Show that a complex matrix $A \in \mathbf{M}_n(\mathbb{C})$ is Hermitian if and only if $\langle Ax, x\rangle$ is real for every $x \in \mathbb{C}^n$, where $\langle \cdot, \cdot \rangle$ is the standard scalar product.

87. Let $k = \mathbb{R}$ or $\mathbb{C}$. The matrix norms that we consider here are subordinated to the $\ell^2$-norms of $k^d$.

Given three matrices $A \in \mathbf{M}_{p\times q}(k)$, $B \in \mathbf{M}_{p\times s}(k)$ and $C \in \mathbf{M}_{r\times q}(k)$, we consider the affine set $\mathcal{W}$ of matrices $W \in \mathbf{M}_{n\times m}(k)$ of the form

$$W = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $D$ runs over $\mathbf{M}_{r\times s}(k)$. Thus $n = p + r$ and $m = q + s$.

Denoting

$$P = \begin{pmatrix} I \\ 0 \end{pmatrix}, \quad Q = \begin{pmatrix} I & 0 \end{pmatrix}$$

the projection matrices, we are going to prove (Parrott'sParrott Lemma) that

$$\min\{\|W\|\,;\, W \in \mathcal{W}\} = \max\{\|QW\|, \|WP\|\}, \tag{10}$$

where the right hand side does not depend on $D$:

$$WP = \begin{pmatrix} A \\ C \end{pmatrix}, \quad QW = \begin{pmatrix} A & B \end{pmatrix}$$

(a) Check the inequality

$$\inf\{\|W\|\,;\, W \in \mathcal{W}\} \geq \max\{\|QW\|, \|WP\|\}.$$

(b) Denote $\mu(D) := \|W\|$. Show that the infimum of $\mu$ on $\mathcal{W}$ is attained.

(c) Show that it is sufficient to prove (10) when $s = 1$.

(d) From now on, we assume that $s = 1$, and we consider a matrix $D_0 \in \mathbf{M}_{r\times 1}(k)$ such that $\mu$ is minimal at $D_0$. We denote by $W_0$ the associated matrix. Let us introduce a function $D \mapsto \eta(D) = \mu(D)^2$. Recall that $\eta$ is the largest eigenvalue of $W^*W$. We denote $f_0$ its multiplicity when $D = D_0$.

   i. If $f_0 \geq 2$, show that $W_0^*W_0$ has an eigenvector $v$ with $v_m = 0$. Deduce that $\mu(D_0) \leq \|WP\|$. Conclude in this case.

ii. From now on, we suppose $f_0 = 1$. Show that $\eta(D)$ is a simple eigenvalue for every $D$ in a small neighbourhood of $D_0$. Show that $D \mapsto \eta(D)$ is differentiable at $D_0$, and that its differential is given by

$$\Delta \mapsto \frac{2}{\|y\|^2} \Re \left[ (QW_0 y)^* \Delta Q y \right],$$

where $y$ is an associated eigenvector:

$$W_0^* W_0 y = \eta(D_0) y.$$

iii. Deduce that either $Qy = 0$ or $QW_0 y = 0$.

iv. In the case where $Qy = 0$, show that $\mu(D_0) \leq \|WP\|$ and conclude.

v. In the case where $QW_0 y = 0$, prove that $\mu(D_0) \leq \|QW\|$ and conclude.

**Nota:** ParrottParrott is the name of a mathematician. Therefore, Parrott'sParrott Lemma has nothing to do with the best seller *Le Théorème du Perroquet*, written by the mathematician Denis GuedjGuedj.

88. For a Hermitian matrix $A$, denote by $P_k$ the leading principal minors:

$$P_k := \det \begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix}.$$

When $k = 0$, we also set $P_0 = 1$. Finally, we set

$$\epsilon_k := \operatorname{sign} \frac{P_k}{P_{k-1}} \in \{-1, 0, 1\}, \quad k = 1, \dots, n.$$

(a) We assume that $P_k \neq 0$ for all $k$. Prove that the number of positive eigenvalues of $A$ is precisely the number of 1's in the sequence $\epsilon_1, \dots, \epsilon_n$.

**Hint**: Argue by induction, with the help of the interlacing property (Theorem 3.3.3).

(b) We assume only that $P_n = \det A \neq 0$. Prove that the number of negative eigenvalues of $A$ is precisely the number of sign changes in the sequence $\epsilon_1, \dots, \epsilon_n$ (the zeros are not taken in account).

89. We define a HilbertHilbert space $\mathbb{H}^2$ of holomorphic functions on the unit disc $\mathbb{D}$, endowed with the scalar product

$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} \overline{f(e^{i\theta})} g(e^{i\theta}) \, d\theta.$$

Don't worry about this loosy definition. You may view $\mathbb{H}^2$ as the completion of the space of polynomials under the norm

$$\|f\| = \sqrt{\langle f, f \rangle},$$

or as the set of $L^2$-functions on the unit circle $\mathbb{T}$ that have a holomorphic extension to $\mathbb{D}$. In other words, $L^2$-functions $f : \mathbb{T} \to \mathbb{C}$ such that

$$\int_0^{2\pi} f(e^{i\theta})e^{im\theta}\, d\theta = 0, \quad m = 0, 1, 2, \dots$$

We define the SzegöSzegö kernel

$$k(\lambda, \mu) = \frac{1}{1 - \lambda\bar{\mu}}.$$

When $\lambda \in \mathbb{D}$, we define a holomorphic function

$$k_\lambda(z) := k(z, \lambda).$$

(a) If $u \in \mathbb{H}^2$, prove
$$\langle k_\lambda, u\rangle = u(\lambda).$$

(b) For $\phi$ holomorphic and bounded on $\mathbb{D}$, the operator $M_\phi : u \mapsto \phi u$ is bounded on $\mathbb{H}^2$. Prove
$$M_\phi^* k_\lambda = \overline{\phi(\lambda)}\, k_\lambda.$$

(c) Prove also that
$$\|M_\phi^*\| = \|M_\phi\| = \sup\{|\phi(z)|\,;\, z \in \mathbb{T}\}.$$

(d) We assume moreover that $\phi : \mathbb{D} \to \mathbb{D}$. Given $N$ distinct numbers $\lambda_1, \dots, \lambda_N$ in $\mathbb{D}$, we form the Hermitian matrix $A$ with entries

$$a_{ij} := (1 - w_i\bar{w}_j)k(\lambda_i, \lambda_j),$$

where $w_j = \phi(\lambda_j)$. Prove that $A$ is semi-definite positive.

**Hint**: Write that $M_\phi M_\phi^* \leq I$ on the space spanned by the $k_{\lambda_j}$'s.

**Nota**: Pick'sPick Theorem tells us that, given the $\lambda_j$'s and the $w_j$'s, $A \geq 0_N$ is equivalent to the existence of a holomorphic function $\phi : \mathbb{D} \to \mathbb{D}$ such that $\phi(\lambda_j) = w_j$. This is an interpolation problem. The space $\mathbb{H}^2$ is a *HardyHardy space.*

90. Let $A \in \mathbf{H}_n$ be a semi-definite positive Hermitian matrix, with $a_{jj} > 0$ and $a_{jk} \neq 0$ for every $(j, k)$. Let us form the Hermitian matrix $B$ such that $b_{jk} := 1/a_{jk}$. Assume at last that $B$ is semi-definite positive too.

(a) Prove that $|a_{jk}| = \sqrt{a_{jj}a_{kk}}$, using principal minors of rank 2.

(b) Using principal minors of rank 3, show that $a_{jk}a_{kl}\bar{a}_{jl}$ is real positive.

(c) Deduce that $A$ is rank-one: There exists a $v \in \mathbb{C}^n$ such that $A = vv^*$.

(d) What does it tell in the context of the previous exercise ?

91. Let $A \in \mathbf{M}_n(\mathbb{C})$ be a normal matrix. We decompose $A = L + D + U$ in strictly lower, diagonal and strictly upper triangular parts. Let us denote by $\ell_j$ the Euclidean length of the $j$-th column of $L$, and by $u_j$ that of the $j$-th row of $U$.

   (a) Show that
   $$\sum_{j=1}^{k} u_j^2 \leq \sum_{j=1}^{k} l_j^2 + \sum_{j=1}^{k}\sum_{m=1}^{j-1} u_{mj}^2, \quad k = 1, \dots, n-1.$$

   (b) Deduce the inequality
   $$\|U\|_S \leq \sqrt{n-1}\|L\|_S,$$
   where $\|\cdot\|_S$ is the Schur–FrobeniusSchurFrobenius norm.

   (c) Prove also that
   $$\|U\|_S \geq \frac{1}{\sqrt{n-1}}\|L\|_S.$$

   (d) Verify that each of these inequalities are optimal. **Hint**: Consider a circulant matrix.

92. The ground field is $\mathbb{R}$.

   (a) Let $P$ and $Q$ be two monic polynomials of respective degrees $n$ and $n-1$ ($n \geq 2$). We assume that $P$ has $n$ real and distinct roots, strictly separated by the $n-1$ real and distinct roots of $Q$. Show that there exists two real numbers $d$ and $c$, and a monic polynomial $R$ of degree $n-2$, such that
   $$P(X) = (X-d)Q(X) - c^2 R(X).$$

   (b) Let $P$ be a monic polynomial of degree $n$ ($n \geq 2$). We assume that $P$ has $n$ real and distinct roots. Build sequences $(d_j, P_j)_{1\leq j\leq n}$ and $(c_j)_{1\leq j\leq n-1}$, where $d_j, c_j$ are real numbers and $P_j$ is a monic polynomial of degree $j$, with
   $$P_n = P, \quad P_j(X) = (X-d_j)P_{j-1}(X) - c_{j-1}^2 P_{j-2}(X), \quad (2 \leq j \leq n).$$
   Deduce that there exists a tridiagonal matrix $A$, which we can obtain by algebraic calculations (involving square roots), whose characteristic polynomial is $P$.

   (c) Let $P$ be a monic polynomial. We assume that $P$ has $n$ real roots. Prove that one can factorize $P = Q_1 \cdots Q_r$, where each $Q_j$ has simple roots, and the factorization requires only finitely many operations. Deduce that there is a finite algorithm, involving no more than square roots calculations, which provides a tridiagonal symmetric matrix $A$, whose characteristic polynomial is $P$ (a *tridiagonal symmetric companion matrix*).

93. (D. KnuthKnuth.) Let $A$ be an alternate matrix in $\mathbf{M}_{n+1}(k)$, $k$ a field. We index its rows and columns from 0 to $n$ (instead of $1, \dots, n+1$), and form the matrices $A_i^j$ by removing from $A$ the $i$-th row and the $j$-th column. We denote also by $\mathrm{PF}[l_1, \dots, l_{2m}]$ the Pfaffian of the alternate matrix obtained by retaining only the rows and columns of indices $l_1, \dots, l_{2m}$.

(a) Prove the following formulas, either (if $n$ is even):

$$\det A_1^0 = \mathrm{PF}[0,2,\dots,n]\,\mathrm{PF}[1,2,\dots,n],$$

or (if $n$ is odd):

$$\det A_1^0 = \mathrm{PF}[0,1,2,\dots,n]\,\mathrm{PF}[2,\dots,n].$$

**Hint**: If $n$ is odd, expand the identity $\det(A+XB) = (\mathrm{Pf}(A+XB))^2$ where $b_{12} = -b_{21} = 1$ and $b_{ij} = 0$ otherwise. Then use Exercise 11.b) in the present list. If $n$ is even, expand the identity $\det M(X,Y) = (\mathrm{Pf}M(X,Y))^2$, where

$$M(X,Y) := \begin{pmatrix} 0 & -X & -Y & 0 & \cdots & 0 \\ X & & & & & \\ Y & & & & & \\ 0 & & & A & & \\ \vdots & & & & & \\ 0 & & & & & \end{pmatrix}.$$

(b) With the first formula, show that if $A$ is an $m \times m$ alternate matrix, then the transpose matrix of cofactors $\mathrm{adj}A$ is *symmetric* for $m$ odd! Prove that in fact $\mathrm{adj}A$ has the form $ZZ^T$ where $Z \in k^m$. Compare with the additional exercise 56.

(c) On the contrary, show that $\mathrm{adj}A$ is alternate when $m$ is even. Prove also that the entries of the matrix

$$\tilde{A} := \frac{1}{\mathrm{Pf}A}\mathrm{adj}A$$

are polynomials in the entries of $A$ (every entry of $\mathrm{adj}A$ is a multiple of the Pfaffian). This matrix $\tilde{A}$ plays an intermediate role between $\mathrm{adj}A$ and $A^{-1}$ in that the inverse of $A$ is given by

$$A^{-1} = \frac{1}{\mathrm{Pf}A}\tilde{A}.$$

94. Let $k$ be a field and $n$ an even integer. If $x, y \in k^n$, denote by $x \wedge y$ the alternate matrix $xy^T - yx^T$. Show the formula

$$\mathrm{Pf}(A + x\wedge y) = (1 + y^T A^{-1} x)\,\mathrm{Pf}A$$

for every non-singular alternate $n \times n$ matrix $A$.

**Hint**: We recall the formulæ

$$\det(M+xy^T) = (1+y^TM^{-1}x)\det M, \quad (M+xy^T)^{-1} = M^{-1} - \frac{1}{1+y^TM^{-1}x}M^{-1}xy^TM^{-1}.$$

95. Let $k$ be a field, $n$ be an even integer and $A$ be an $n \times n$ non-singular alternate matrix. Using the odd case of Exercise 93 above, prove the formula

$$A^{-1} = \frac{1}{\mathrm{Pf}A}\left(\alpha(i,j)(-1)^{i+j+1}\mathrm{Pf}A^{ij}\right)_{1,\le i,j\le n},$$

where $A^{ij}$ is obtained from $A$ be removing the $i$-th and $j$-th rows and columns, and $\alpha(i,j)$ is $\pm$ according to the sign of $j-i$. Compare this formula with Exercise 11.b) above.

In particular, show that $\mathrm{Pf}A$ divides, as a polynomial, every entry of $\mathrm{adj}A$.

96. (BanachBanach.) Let $p \in [1,+\infty]$ be such that $p \neq 2$. We consider matrices $M \in \mathbf{M}_n(\mathbb{R})$ which are isometries, namely

$$\|Mx\|_p = \|x\|_p, \quad \forall x \in \mathbb{R}^n.$$

(a) Let us begin with the case $2 < p < +\infty$. We give $x, y \in \mathbb{R}^n$ such that $x_i y_i = 0$ for every $i \leq n$. Define $u = Mx$ and $v = My$, and let $H$ be the set of indices $j$ such that $u_j v_j \neq 0$.

i. Show that the function

$$\theta(s) := \sum_{j \in H} (|su_j + v_j|^p - |su_j|^p - |v_j|^p)$$

vanishes identically.

ii. Computing the second derivative of $\theta$, show that $H$ is void.

(b) (Continued, $2 < p < +\infty$.) Let $m_k$ be the number of non-zero entries in the $k$-th column of $M$, and $E_k$ be the vector space spanned by the other columns. Using the previous question, show that $\dim E_k \leq n - m_k$. Then deduce that $m_k = 1$. At last, show that $M$ is the product of a diagonal matrix $\mathrm{diag}(\pm 1, \dots, \pm 1)$ and of a permutation matrix.

(c) If $1 < p < 2$, prove the same conclusion. **Hint**: Apply the previous result to $M^T$.

(d) If $p = 1$, prove directly that if $x_i y_i = 0$ for every index $i$, then $(Mx)_j (My)_j = 0$ for every index $j$. **Hint**: Use $\theta(1) = \theta(-1) = 0$. Conclude.

(e) If $p = +\infty$, conclude by applying the case $p = 1$ to $M^T$.

97. (LemmensLemmens & van Gaansvangaa@van Gaans.) We endow $\mathbb{R}^n$ with some norm $\|\cdot\|$. Let $M \in \mathbf{M}_n(\mathbb{R})$ be non-expansive: $\|Mx\| \leq \|x\|$ for every $x \in \mathbb{R}^n$.

(a) Let $B$ be the unit ball. Show that

$$D := \bigcap_{k \geq 1} M^k B$$

is a compact symmetric convex set. We denote by $E$ the vector space spanned by $D$.

(b) Show that $ME = E$ and that the restriction of $M$ to $E$ is an isometry.

(c) Let $(k_j)_{j \in \mathbb{N}}$ be increasing sequence such that $M^{k_j} \to A$. Prove that $AB = D$.

(d) Show that there exists an increasing sequence $(k_j)_{j \in \mathbb{N}}$ such that $M^{k_j}$ converges. Prove that $M^{k_{j+1}-k_j}$ converges towards a projector $P$ whose range is $E$, and which is non-expansive.

98. Given an alternate $4 \times 4$ matrix $A$, verify that its characteristic polynomial equals

$$X^4 + X^2 \sum_{i<j} a_{ij}^2 + \mathrm{Pf}(A)^2.$$

We define

$$\begin{aligned} R_+(A) &= (a_{12} + a_{34})^2 + (a_{23} + a_{14})^2 + (a_{31} + a_{24})^2, \\ R_-(A) &= (a_{12} - a_{34})^2 + (a_{23} - a_{14})^2 + (a_{31} - a_{24})^2. \end{aligned}$$

Factorize $P_A$ in two different ways and deduce the following formula for the eigenvalues of $A$, in characteristic different from 2:

$$\frac{i}{2}\left(\pm\sqrt{R_+(A)} \pm \sqrt{R_-(A)}\right),$$

where the signs are independent of each other.

99. (a) Verify that the characteristic polynomial $P_V$ of a real orthogonal matrix $V$ can be factorized as

$$P_V(X) = (X-1)^r(X+1)^s X^m Q\left(X + \frac{1}{X}\right)$$

where $Q \in \mathbb{R}[X]$ a monic polynomial whose roots lie in $(-2, 2)$.

(b) Conversely, we give a monic polynomial $Q \in \mathbb{R}[Y]$ of degree $m$, whose roots lie in $(-2, 2)$, and we consider

$$P(X) = X^m Q\left(X + \frac{1}{X}\right).$$

Let $A$ be a tridiagonal symmetric matrix whose characteristic polynomial is $Q$ (see Exercise 92.)

i. Prove that $I_m - \frac{1}{4}A^2$ is positive definite.

ii. Let us define

$$B := \sqrt{I_n - \frac{1}{4}A^2}.$$

Prove that

$$V := \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

is orthogonal.

iii. Prove $P_V = P$ ($V$ is a companion matrix of $P$.)

100. (Inspired by M. T. KaraevKaraev.) Let $E$ be a finite dimensional HilbertHilbert space. We are interested in nilpotent endomorphisms. Recall that $u \in \mathcal{L}(E)$ is nilpotent of order $m$ if $u^m = 0_E$ but $u^k \neq 0_E$ if $k < m$.

(a) Let $F$ be the orthogonal of $\ker u$ and let $G$ be $u(F)$. Prove that there exists an orthonormal basis of $F$, whose image is an orthogonal basis of $G$. **Hint**: This is essentially the Singular Value Decomposition.

(b) Deduce that, if $m = 2$, there exists an orthonormal basis of $E$, in which the matrix of $u$ has the "JordanJordan!Camille" form

$$\begin{pmatrix} 0 & a_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & 0 \\ \vdots & & & \ddots & a_n \\ 0 & \cdots & & \cdots & 0 \end{pmatrix}. \tag{11}$$

(c) Arguing by induction, prove the same result for every $m \geq 2$.

(d) Deduce that if $M \in \mathbf{M}_n(\mathbb{C})$ is nilpotent, then $M$ is unitarily similar to a matrix of the form (11).

(e) If $M$ and $N$ are unitarily similar, check that their numerical ranges (see Exercise 21) are equal, and that $\|M\|_2 = \|N\|_2$.

(f) Let $M \in \mathbf{M}_n(\mathbb{C})$ be nilpotent of order $m$. Prove that $\mathcal{H}(M)$ is a disk centered at the origin. Show that its radius is less than or equal to (HaagerupHaagerup–de la Harpedela@de la Harpe inequality)

$$\|M\|_2 \cos \frac{\pi}{m+1}.$$

**Hint**: It is enough to work in the case $M$ is of the form (11) and the $a_j$'s are non-zero. Then it is easy to show that $\mathcal{H}(M)$ is rotationaly invariant. Since it is convex (Exercise 21), it is a disk. The triangular inequality leads to the computation of the spectral radius of

$$\begin{pmatrix} 0 & 1/2 & 0 & \cdots & 0 \\ 1/2 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1/2 \\ 0 & \cdots & & 1/2 & 0 \end{pmatrix}.$$

101. (From de Oliveiradeol@de Oliveira.) Let $(\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ and $(\beta_1, \dots, \beta_n) \in \mathbb{C}^n$ be given. Let us form the diagonal matrices $\Delta := \operatorname{diag}\{\alpha_1, \dots, \alpha_n\}$ and $D := \operatorname{diag}\{\beta_1, \dots, \beta_n\}$.

(a) If $V$ is unitary, show that

$$\operatorname{Tr}(\Delta V D V^*) = \begin{pmatrix} \beta_1 & \cdots & \beta_n \end{pmatrix} S \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix},$$

for some orthostochastic matrix $S$.

(b) Using Birkhoff'sBirkhoff Theorem, deduce that $\mathrm{Tr}(\Delta VDV^*)$ belongs to the convex hull of the set of numbers

$$\sum_{j=1}^{n} \alpha_j \beta_{\sigma(j)}, \qquad \sigma \in \mathfrak{S}_n.$$

(c) More generally, given two normal matrices $A, B \in \mathbf{M}_n(\mathbb{C})$ whose respective spectra are $(\alpha_1, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_n)$, prove that $\mathrm{Tr}(AB)$ belongs to this convex hull. **Nota**: See Exercise 139 for a related result.

(d) With the same notations as above, prove that

$$\min_{\sigma \in \mathfrak{S}_n} \sum_{j=1}^{n} |\alpha_j - \beta_{\sigma(j)}|^2 \leq \|A - B\|_F^2 \leq \max_{\sigma \in \mathfrak{S}_n} \sum_{j=1}^{n} |\alpha_j - \beta_{\sigma(j)}|^2,$$

where $\|\cdot\|_F$ is the Frobenius norm. The first inequality constitutes Hoffman–WielandtHoffmanWielandt theorem. The case for Hermitian matrices was found earlier by Loewner (Löwner)Loewner.

102. (Y. Shizuta, S. KawashimaShizutaKawashima.) Let $A, B$ be $n \times n$ Hermitian matrices, with $B \geq 0_n$. We denote by $a_1, \ldots, a_r$ the distinct eigenvalues of $A$ and by $P_1, \ldots, P_r$ the corresponding eigenprojectors, so that $A = \sum_j a_j P_j$.

(a) Prove that $P_j$ is Hermitian and that $P_j P_k = 0_n$ when $k \neq j$.

(b) Let us assume that there exists a skew-Hermitian matrix $K$ such that $B + [K, A]$ is positive definite. Show that $\ker B$ does not contain any eigenvector of $A$.

(c) Let us assume that the intersection of the kernels of $B$, $[B, A]$, $[[B, A], A]$,... (take successive commutators with $A$) equals $\{0\}$. Prove that $\ker B$ does not contain any eigenvector of $A$.

(d) Conversely, we assume that $\ker B$ does not contain any eigenvector of $A$.

i. Define
$$K := \sum_{i \neq j} \frac{1}{a_i - a_j} P_i B P_j.$$
Show that $K$ is skew-Hermitian, and that $B + [K, A]$ is positive definite.

ii. Show that the intersection of the kernels of $B$, $[B, A]$, $[[B, A], A]$,... equals $\{0\}$.

103. Let $A \in \mathbf{M}_{n \times m}(k)$ be given, with $k = \mathbb{R}$ or $k = \mathbb{C}$. We put the singular values $\sigma_1 \geq \sigma_2 \geq \cdots$ in decreasing order. We endow $\mathbf{M}_{n \times m}(k)$ with the norm $\|\cdot\|_2$. Prove that the distance of $A$ to the set $R_l$ of matrices of rank less than or equal to $l$, equals $\sigma_{l+1}$ (if $l = \min\{n, m\}$, put $\sigma_{l+1} = 0$). **Hint**: Use Theorem 7.7.1 of Singular Value Decomposition.

104. (a) Let $T_0 \in \mathbf{M}_n(\mathbb{C})$ be such that $\|T_0\|_2 \leq 1$. Prove

$$\|(T_0 - wI_n)^{-1}\|_2 \leq \frac{1}{|w| - 1}, \qquad \forall w \in \mathbb{C}\,;\ |w| > 1 \tag{12}$$

**Hint**: Use von Neumannvonneu@von Neumann Inequality (Exercise 82), while approximating the function $f(z) := (z-w)^{-1}$ by polynomials, uniformly on a neighbourhood of $\mathrm{Sp}(T_0)$.

(b) Let $T \in \mathbf{M}_n(\mathbb{C})$ be of the form $T = aI_n + N$ where $a \in \mathbb{C}$ and $N^2 = 0_n$.

i. Prove that
$$(\|T\|_2 \le 1) \iff \left(\|N\|_2 + |a|^2 \le 1\right).$$
Deduce that $\|T\|_2$ is the largest root of the quadratic equation
$$r^2 - \|N\|_2 r - |a|^2 = 0.$$

ii. Prove that
$$\left(\|(T - wI_n)^{-1}\|_2 \le \frac{1}{|w|-1}, \qquad \forall w \in \mathbb{C}\,;\,|w| > 1\right) \iff \left(\frac{1}{2}\|N\|_2 + |a| \le 1\right).$$

iii. Deduce that the converse of (12) does not hold if $n \ge 2$. In particular, one cannot replace the assumption $\|T\|_2 \le 1$ by (12) in the inequality of von Neumannvonneu@von Neumann.

iv. However, prove that for such a $T = aI_n + N$, (12) implies $\|T\|_2 \le 2$, and the equality is achieved for some $(a, N)$.

(c) Let $T \in \mathbf{M}_n(\mathbb{C})$ satisfy Property (12). Prove that
$$\|T\|_2 \le e.$$

**Hint**: Use the formula
$$T = \frac{1}{2ik\pi}\int_{\Gamma_r} z^k (zI_n - T)^{-k} dz.$$

(Interestingly enough, this part of the exercise is true not only for the norm $\|\cdot\|_2$, but also in every BanachBanach algebra.)

105. Let $A, B$ be two positive semi-definite Hermitian matrices.

(a) Prove that for every $x \in \mathbb{C}^n$, $\|Bx\|_2^2 \le \|B\|_2 \langle Bx, x\rangle$.

(b) Deduce that $BAB \le \|A\|_2\|B\|_2 B$ in the sense of Hermitian matrices.

106. (From L. TartarTartar.)

(a) Let $f(z) = \sum_{n\ge0} a_n z^n$ be a series, converging for $|z| < R$. We denote $F(z) := \sum_{n\ge0} |a_n| z^n$. Check that whenever $\mathcal{A}$ is a BanachBanach algebra and $a, b \in \mathcal{A}$, there holds
$$\|f(b) - f(a)\| \le \frac{F(\|b\|) - F(\|a\|)}{\|b\| - \|a\|}\|b - a\|.$$

(b) Deduce that if $A, B \in \mathbf{H}_n(\mathbb{C})$, then

$$\|e^{iB} - e^{iA}\|_2 \leq \|B - A\|_2.$$

**Hint**: Choose an integer $m \geq 1$. Apply the previous result to $f(z) = e^{iz/m}$, and decompose $e^{iB} - e^{iA}$ as a sum of $m$ products having $e^{iB/m} - e^{iA/m}$ as one factor and unitary matrices otherwise. Then let $m \to +\infty$.

107. Let $k$ be $\mathbb{R}$ or $\mathbb{C}$. We consider a norm on $k^n$ such that the induced norm has the property $\|A^2\| = \|A\|^2$ for every Hermitian, positive semi-definite matrix $A$.

(a) Show that $\|A\| = \rho(A)$ for every $A \in \mathbf{H}_n^+$.

(b) Deduce that, for every $x \in k^n$, there holds $\|x\|_2^2 = \|x\| \, \|x\|_*$ (recall that $\|\cdot\|_*$ is the dual norm of $\|\cdot\|$).

(c) Let $\Sigma$ be the unit sphere of norm $\|\cdot\|$. Given $x \in \Sigma$, show that $\Sigma$ is on one side of the plane defined by $\Re(y^*x) = \|x\|^2$.

(d) Deduce that $\|\cdot\|$ is proportional to $\|\cdot\|_2$. **Hint**: Given a point $x_0 \in \Sigma$, show that the smallest convex cone with vertex $x_0$, which contains $\Sigma$, is the half-plane $\Re(y^*x) \leq \|x\|^2$ (this uses the previous question). Deduce that $\Sigma$ is a differentiable manifold of codimension one. Then conclude with the previous question.

(e) Let $\|\cdot\|$ be and induced norm on $\mathbf{M}_n(k)$ such that the square root $A \mapsto \sqrt{A}$ is 1/2-Holderian on $\mathbf{HPD}_n$ (or $\mathbf{SPD}_n$ if $k = \mathbb{R}$), with constant *one* (compare with Exercise 52):

$$\|\sqrt{B} - \sqrt{A}\| \leq \|B - A\|^{1/2}. \tag{13}$$

Prove that the norm on $k^n$ is proportional to $\|\cdot\|_2$. **Comment**: For the validity of (13) for $\|\cdot\|_2$, see Exercise 110.

108. (Ky FanKy Fan norms.) Let $s_j(M)$ $(1 \leq j \leq n)$ denote the singular values of a matrix $M \in \mathbf{M}_n(\mathbb{R})$, labelled in increasing order: $s_1(M) \leq \cdots \leq s_n(M)$.

(a) We define $\sigma_j := s_j + \cdots + s_n$. Prove the formula

$$\begin{aligned}\sigma_{n-j+1}(M) \;&= \sup\{\mathrm{Tr}(PMQ)\,;\;\; P \in \mathbf{M}_{j\times n}(\mathbb{R}), Q \in \mathbf{M}_{n\times j}(\mathbb{R}) \\ &\qquad\qquad\qquad\qquad\quad \text{s.t. } PP^T = I_j, Q^TQ = I_j\}.\end{aligned}$$

Deduce that $\sigma_j$ is a convex function, and thus is a norm. Do you recognize the norm $\sigma_n = s_n$ ?

(b) (Thanks to M. de la Sallede la Salle) Deduce that there exists two norms $N_\pm$ over $\mathbf{M}_n(\mathbb{R})$, with the property that

$$(\det M = 0) \Longleftrightarrow (\|M\|_+ = \|M\|_-).$$

(c) Let $a := (a_1, \cdots, a_n)$ be a given $n$-uplet of non-negative reals numbers with $a_1 \le \cdots \le a_n$. We define

$$E(a) := \{M \in \mathbf{M}_n(\mathbb{R})\,;\, s_j(M) = a_j, \text{ for all } 1 \le j \le n\}.$$

Verify that the set $E'(a)$ defined below is convex, and that it contains the convex hull of $E$:

$$E'(a) := \{M \in \mathbf{M}_n(\mathbb{R})\,;\, \sigma_j(M) \le a_j + \cdots + a_n, \text{ for all } 1 \le j \le n\}.$$

(d) Show that the extremal points of $E'(a)$ belong to $E(a)$, and deduce that $E'(a)$ is the convex hull of $E(a)$. **Hint**: The set $\mathrm{ext}(E'(a))$ is left- and right-invariant under multiplication by orthogonal matrices. Thus one may consider diagonal extremal points.

(e) Deduce that the convex hull of $\mathbf{O}_n(\mathbb{R})$ is the unit ball of $\|\cdot\|_2$. **Remark**: Here, the convex hull of a set of small dimension ($n(n-1)/2$) has a large dimension $n^2$. Thus it must have faces of rather large dimension ; this is precisely the contents of Corollary 5.5.1, when applied to the induced norm $\|\cdot\|_2$. See Exercise 137 below for a more accurate description.

109. (Continuation.) We say that a subset $X$ of $\mathbf{M}_n(\mathbb{R})$ is *rank-one convex* if whenever $A, B \in X$ and $B - A$ is of rank one, then the segment $(A, B)$ is included in $X$. Rank-one convexity is preserved under intersection. If $Y$ is a subset of $\mathbf{M}_n(\mathbb{R})$, its *rank-one-convex hull* is the smallest rank-one convex subset that contains $Y$. We recall that a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is rank-one convex if it is convex on every segment of the type above.

(a) Let $f : \mathbf{M}_n(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ be rank-one convex. Show that the sets defined by $f(M) \le \alpha$ are rank-one convex.

(b) Verify that $M \mapsto |\det M|$ is rank-one convex.

(c) Prove the formula for products of singular values:

$$\begin{aligned} s_n(M) \cdots s_{n-j+1}(M) \quad &= \sup\{\det(PMQ)\,;\quad P \in \mathbf{M}_{j\times n}(\mathbb{R}), Q \in \mathbf{M}_{n\times j}(\mathbb{R}) \\ &\qquad\qquad\qquad\qquad\quad \text{s.t. } PP^T = I_j, Q^TQ = I_j\}. \end{aligned}$$

(d) Deduce that $\pi_j := s_n \cdots s_j$ is rank-one convex.

(e) Deduce that the rank-one-convex hull of the set $E(a)$ (see the previous exercise) is included in

$$E''(a) := \{M \in \mathbf{M}_n(\mathbb{R})\,;\, \pi_j(M) \le a_j \cdots a_n, \text{ for all } 1 \le j \le n\}.$$

**Comment**: A result of B. DacorognaDacorogna tells us that $E''(a)$ is actually equal to the rank-one-convex hull of $E(a)$.

110. Let $A, B$ be $n \times n$ positive semi-definite Hermitian matrices.

(a) Using the formula (TaylorTaylor expansion)

$$B^2 - A^2 = (B-A)^2 + A(B-A) + (B-A)A,$$

prove that $\rho(B-A)^2 \le \|B^2 - A^2\|_2$. **Hint**: Use an eigenvector of $B-A$. The formula above might not be the useful one in some case.

(b) Deduce that (13) holds true for the operator norm $\|\cdot\|_2$. **Comment**: The resulting inequality $\|\sqrt{B} - \sqrt{A}\|_2 \le \|B-A\|_2^{1/2}$ is much more powerful than that of Exercise 52, since it does not depend on the dimension $n$. In particular, it holds true for bounded self-adjoint operators in Hilbert spaces.

(c) Let $x \mapsto S(x)$ be a map of class $\mathcal{C}^2$ from the unit ball $B_d$ of $\mathbb{R}^d$ to $\mathbf{SPD}_n$. Assume that the second derivatives are bounded over $B_d$. Prove that $x \mapsto \sqrt{S(x)}$ is Lipschitz continuous.

111. (Continuation.) Likewise, write the TaylorTaylor expansion

$$B^3 = A^3 + \cdots + H^3, \qquad H := B - A.$$

Then, using an eigenvector $e$ of $H$, associated with $\rho(H)$, show that

$$e^*(B^3 - A^3)e - \rho(H)^3\|e\|_2^2 = 2\rho(H)\|Ae\|_2^2 + e^*AHAe + 3\rho(H)^2 e^*Ae.$$

Prove the bound $|e^*AHAe| \le \rho(H)\|Ae\|_2^2$ and deduce that

$$\|B-A\|_2^3 \le \|B^3 - A^3\|_2.$$

What about the map $A \mapsto A^{1/3}$ over $\mathbf{HPD}_n$ ? Any idea about $A \mapsto A^{1/4}$ (simpler than what you think in a first instance) ?

112.


Newton, *Isaac*!(France)

**Isaac Newton.**

Given a real polynomial

$$P = X^n - a_1X^{n-1} + \cdots + (-1)^n a_n \in \mathbb{R}[X],$$

whose roots are $x_1, \dots, x_n$ (repeted with multiplicities), we define the NewtonNewton sums

$$s_k := \sum_\alpha x_\alpha^k \qquad (s_0 = n).$$

We recall that $s_k$ is a polynomial in $a_1, \dots, a_k$, with integer coefficients.
Let us form the *Hankel*Hankel matrix

$$H := \begin{pmatrix} s_0 & s_1 & \cdots & s_{n-1} \\ s_1 & s_2 & \cdots & s_n \\ \vdots & & & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-2} \end{pmatrix}.$$

Let $Q$ be the quadratic form associated with $H$. Write $Q$ as a sum of squares. Deduce that the rank of $H$ equals the number of distinct complex roots of $P$, while the index of $H$ (the number of positive squares minus the number of negative squares) equals the number of distinct *real* roots of $P$. Conclude that the roots of $P$ are all real if, and only if, $H$ is positive semi-definite.

113. (Continuation. NewellNewell (1972/73), IlyushechkinIlyushechkin (1985), LaxLax (1998), DomokosDomokos (2011) ; special thanks to L. TartarTartar.) In the previous exercise, set $P := \det(XI_n - A)$ where $A$ is a general matrix in $\mathbf{Sym}_n(\mathbb{R})$. Check that the $s_k$'s are polynomials in the entries of $A$. Show that $H$ is positive semi-definite. Deduce that every principal minor of $H$,

$$H\begin{pmatrix} i_1 & \dots & i_r \\ i_1 & \dots & i_r \end{pmatrix},$$

is a polynomial in the entries of $A$, and that it takes only non-negative values.

According to Artin'sArtin Theorem (see J. BochnakBochnak M. CosteCoste & M.-F. RoyRoy: *Real algebraic geometry*, Springer-Verlag (1998), Theorem 6.1.1), this property ensures that these minors are sums of squares of rational functions. HilbertHilbert pointed out that not every non-negative polynomial is a sum of squares of polynomials. However, it turns out that these principal minors are sums of squares of polynomials : Let us endow $\mathbf{Sym}_n(\mathbb{R})$ with the scalar product $\langle A, B\rangle := \mathrm{Tr}(AB)$.

(a) Check that $H$ is a GramGram matrix: $h_{ij} = \langle W_i, W_j\rangle$ for some $W_i \in \mathbf{Sym}_n(\mathbb{R})$.

(b) Show that the exterior algebra of $\mathbf{Sym}_n(\mathbb{R})$ is naturally endowed with a scalar product.

(c) Show that

$$H\begin{pmatrix} i_1 & \dots & i_r \\ i_1 & \dots & i_r \end{pmatrix} = \|W_{i_1} \wedge \cdots \wedge W_{i_r}\|^2$$

and conclude. *Remark*: The scalar product of a Euclidean space $E$ extends in a natural way to the exterior algebra $\Lambda^r E$.

**Nota**: This formulation, due to IlyushechkinIlyushechkin, gives the discriminant over $\mathbf{Sym}_n(\mathbb{R})$ as the sum of $n!$ squares of polynomials. This upper bound has been improved by DomokosDomokos into

$$\binom{2n-1}{n-1} - \binom{2n-3}{n-1}.$$

The minimal number of squares is not yet know, except for $n = 2$ and 3, where 2 and 5 squares suffice.

114. Let $A$ be a principal ideal domain. If $M \in \mathbf{M}_n(A)$ and $M = PDQ$ with $P, Q \in \mathbf{GL}_n(A)$ and $D$ diagonal, prove the following equality about cofactors matrices:

$$\mathrm{co}(M) = (\det P)(\det Q)P^{-T}\mathrm{co}(D)Q^{-T}.$$

Prove

$$D_\ell(\mathrm{co}(M)) = (\det M)^{\ell-1} D_{n-\ell}(M)$$

and deduce the value of $d_\ell(\mathrm{co}(M))$, the $\ell$-th invariant factor of $\mathrm{co}(M)$. Compare with the result of Exercise 56.

115. (Potter.) Let $k$ be a field and $\omega$ an element of $k$.

(a) Prove that there exists polynomials $P_{r,j} \in \mathbb{Z}[X]$ such that, for every integer $n \geq 1$, every element $\omega$ in $k$ and every pair of matrices $A, B \in M_n(k)$ such that

$$AB = \omega BA, \tag{14}$$

there holds

$$(A+B)^r = \sum_{j=0}^{r} P_{r,j}(\omega) B^j A^{r-j}. \tag{15}$$

Matrices satisfying (14) are said to *$\omega$-commute*. **Remark** that they satisfy $A^p B^q = \omega^{pq} B^q A^p$, a formula that is a discrete analogue of the *Stone–von Neumann formula*Stonevonneu@von Neumann.

(b) Define polynomials

$$\phi_l(X) = \prod_{s=1}^{l} (1 + X + \cdots + X^{s-1}).$$

Show the formula

$$\phi_j \phi_{r-j} P_{r,j} = \phi_r.$$

**Hint**: Proceed by induction over $r$.

(c) Assume that $\omega$ is a primitive root of unity, of order $r$ (then $r$ is not the characteristic of $k$). Deduce that (14) implies (Potter's TheoremPotter)

$$(A+B)^r = A^r + B^r.$$

**Remark.** It is amazing that when $r$ is prime, the identity (15) can occur in two cases : either $A$ and $B$ $\omega$-commute with respect to a primitive root of unity of order $r$, or $r$ is the characteristic of $k$ and $[A, B] = 0_n$; and both cases exclude each other !

(d) Let $B$ be a cyclic matrix in the sense of Section 5.4:

$$B := \begin{pmatrix} 0 & M_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & M_{r-1} \\ M_r & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

We recall that the diagonal blocks are square (null) matrices, of respective sizes $n_1, \dots, n_r$. Let us define $A := \mathrm{diag}(I_{n_1}, \omega I_{n_2}, \dots, \omega^{r-1} I_{n_r})$. We assume again that $\omega$ is a primitive root of unity, of order $r$. Prove that $(B, A)$ $\omega$-commute and deduce that

$$(A+B)^r = I_n + \mathrm{diag}(N_1, \dots, N_r),$$

where $N_j := M_j M_{j+1} \cdots M_{j-1}$. For instance,

$$N_1 = M_1 M_2 \cdots M_r, \qquad N_r = M_r M_1 M_2 \cdots M_{r-1}.$$

116. Let $H \in \mathbf{HPD}_n$ and $h \in \mathbf{H}_n$ be given. We recall that $M := hH^{-1}$ is diagonalizable with real eigenvalues (see Exercise 258). We recall that a matrix is normal if and only if its eigenvectors form a unitary basis. Thus the angles between eigenvectors of $M$ measure the deviation of $M$ from normality. We compute here the minimum angle $\theta_*$ as $h$ runs over $\mathbf{H}_n$.

The space $\mathbb{C}^n$ is endowed with its Hermitian norm $\|\cdot\|_2$.

(a) Let $v$ and $w$ be two eigenvectors of $M$ associated with distinct eigenvalues. Prove that

$$v \perp_H w \qquad \text{(i.e. } w^* H v = 0).$$

(b) Let $v \in \mathbb{C}^n$ be given. We define a linear form $L(w) := v^* w$ with domain the $H$-orthogonal to $v$. Check that

$$\|L\| \le \inf_{r \in \mathbb{R}} \|v + rHv\|_2.$$

Deduce that

$$\Lambda := \sup \left\{ \frac{|v^* w|}{\|v\|_2 \|w\|_2} \,;\, w \perp_H v \right\} \le \inf_{r \in \mathbb{R}} \rho(I_n + rH) = \frac{K(H)-1}{K(H)+1},$$

where $K(H)$ is the condition number of $H$.

(c) Looking for a specific pair $(v, w)$, prove that there holds actually

$$\Lambda = \frac{K(H)-1}{K(H)+1}.$$

(d) Deduce that

$$\sin \theta_* = \frac{2}{K(H)+1}.$$

117. Let us define $K_n := \Delta_n \cap \mathbf{Sym}_n(\mathbb{R})$, the set of symmetric, bistochastic matrices.

(a) Show that $K_n$ is the convex hull of the set of matrices of the form

$$Q_\sigma := \frac{1}{2}(P_\sigma + P_{\sigma^{-1}}), \qquad (\sigma \in \mathbf{S_n})$$

where $P_\sigma$ denotes the permutation matrix associated with $\sigma$.

(b) If $\sigma = (1, 2, \ldots, n)$ is a cycle, prove that $Q_\sigma$ is extremal in $K_n$ if and only if either $n$ is odd or $n = 2$. **Hint**: If $n$ is even, show that $Q_\sigma = \frac{1}{2}(Q_+ + Q_-)$ where $Q_\pm$ are permutation matrices associated with involutions, and $Q_+ \neq Q_-$ if $n \geq 4$. If $n$ is odd, consider the graph $\Gamma$ of pairs $(i, j)$ for which $q_{ij} \neq 0$ in $Q_\sigma$ ; an edge between $(i, j)$ and $(i', j')$ means that either $i = i'$ or $j = j'$. The graph $\Gamma$ is a cycle of length $2n$, and $d((i, j), (j, i)) = n$ is odd. If $Q_\sigma = \frac{1}{2}(R + S)$ with $R, S \in K_n$, show that $r_{ji} + r_{ij} = 1$ along $\Gamma$ and conclude.

(c) Deduce that $\mathrm{ext}(K_n)$, the set of extremal points of $K_n$, consists in the matrices $Q_\sigma$ for the permutations $\sigma$ that are products of disjoint cycles of lengths as above (either odd or equal to two).

118. Let $L : \mathbf{M}_n(\mathbb{C}) \to \mathbb{C}$ be a linear form with the properties that if $H$ is Hermitian, then $L(H)$ is real, and if moreover $H \geq 0_n$, then $L(H) \geq 0$. Prove that $L(A^*) = \overline{L(A)}$ for every $A \in \mathbf{M}_n(\mathbb{C})$. Then prove, for every pair of matrices,

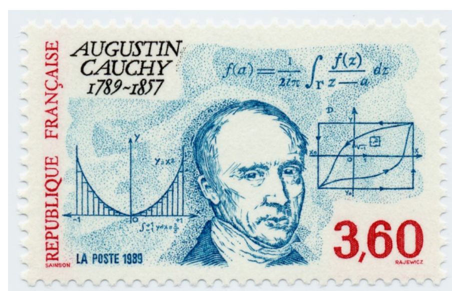
$$|L(A^*B)|^2 \leq L(A^*A)L(B^*B).$$

119. (See also Exercise 8)

(a) Prove the determinantal identity (Cauchy'sCauchy *double alternant*)

$$\left\| \frac{1}{a_i + b_j} \right\|_{1 \leq i,j \leq n} = \frac{\prod_{i<j}(a_j - a_i) \prod_{k<l}(b_k - b_l)}{\prod_{i,k}(a_i + b_k)}.$$

**Hint**: One may assume that $a_1, \ldots, a_n, b_1, \ldots, b_n$ are indeterminate, and then work in the field $\mathbb{Q}(a_1, \ldots, a_n, b_1, \ldots, b_n)$. This determinant is a homogeneous rational function whose denominator is quite trivial. Some specialisations make it vanishing ; this gives an accurate information about the numerator. There remains to find a scalar factor. That can be done by induction on $n$, with an expansion with respect to the last row and column.



Cauchy, *Augustin* (France)

**Augustin Cauchy.**

(b)

Hilbert, *David* (D. R. of Congo)

**David Hilbert.**

Define the HilbertHilbert matrix of order $n$ as the GramGram matrix

$$h_{ij} := \langle x^{i-1}, x^{j-1} \rangle_{L^2(0,1)}, \quad (1 \leq i, j \leq n)$$

where

$$\langle f, g \rangle_{L^2(0,1)} := \int_0^1 f(x)g(x)\, dx.$$

Use the formula above to compute $(H^{-1})_{nn}$.

(c) In particular, find that $\|H^{-1}\|_2 \geq \frac{1}{32n} 16^n$ (asymptotically, there holds a better bound $\frac{1}{8\pi} 16^n$.) **Hint**: The binomial $(2m)!/(m!)^2$ is larger than $2^{2m}/(2m+1)$.

(d) Deduce that the HilbertHilbert matrix is pretty much ill-conditionned:

$$\kappa(H) := \|H\|_2 \|H^{-1}\|_2 \geq \frac{1}{32n} 16^n.$$

**Remark**: Hilbert's matrix satisfies $\|H\|_2 \leq \sqrt{\pi}$, and this estimate is the best one to be uniform, in the sense that, denoting $H_n$ the Hilbert matrix of size $n \times n$, one has $\sup_n \|H\|_2 = \sqrt{\pi}$. See for instance G. PólyaPol@Pólya & G. SzegöSzegö, *Problems and theorems in analysis*, vol. I. Fourth edition (1970), Springer-Verlag. Part III, chapter 4, exercise 169.

120. (I. KovacsKovacs, D. SilverSilver & S. WilliamsWilliams) Let $A \in \mathbf{M}_{pq}(k)$ be in block form

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots \\ A_{21} & \ddots & \\ \vdots & & \end{pmatrix}$$

where the blocks are $p \times p$. Let us assume that these blocks commute pairwise. Prove that

$$\det A = \det \Delta, \qquad \Delta := \sum_{\sigma \in \mathfrak{S}_q} \epsilon(\sigma) \prod_{m=1}^{q} A_{m\sigma(m)}.$$

Notice that $\Delta$ is nothing but the determinant of $A$, considered as a $q \times q$ matrix with entries in the abelian ring $\mathcal{R}$ generated by the $A_{ij}$'s. One may therefore write

$$\det_{pq} A = \det_p \det_{\mathcal{R}} A,$$

where the subscripts indicate the meaning of each determinant. **Hint**: Use Schur'sSchur formula for the determinant of a matrix with four blocks. Argue by induction over $q$.

121. Let us define the tridiagonal and block-tridiagonal matrices

$$J_p := \begin{pmatrix} 0 & 1 & & & \\ 1 & \ddots & \ddots & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{pmatrix} \in \mathbf{M}_p(k), \quad A_{pq} := \begin{pmatrix} J_p & I_p & & & \\ I_p & \ddots & \ddots & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & \ddots & \ddots & I_p \\ & & & I_p & J_p \end{pmatrix} \in \mathbf{M}_{pq}(k).$$

Denote by $T_p$ the polynomial

$$T_p(X) := \det(XI_p + J_p).$$

Using the previous exercise, prove that the characteristic polynomial $P_{pq}(Y)$ of $A_{pq}$ is the resultant

$$\mathrm{Res}(T_q(\cdot - Y), \hat{T}_p), \qquad \hat{T}(X) := T(-X).$$

**Nota**: Since these matrices have integer coefficients, their characteristic polynomials do not depend of the scalar field $k$. It is therefore enough to consider the real case.

122. Let $A \in \mathbf{M}_n(\mathbb{C})$ have no purely imaginary eigenvalue (one says that $A$ is *hyperbolic*). The aim of this exercise is to prove the existence and uniqueness of a *Green*Green matrix. This is a matrix-valued function $G : \mathbb{R} \to \mathbf{M}_n(\mathbb{C})$ that is bounded, differentiable for $t \neq 0$, which has left and right limits $G(0\pm)$, and satisfies

$$\frac{dG}{dt}(t) = AG(t), \quad (t \neq 0), \qquad G(0+) - G(0-) = I_n.$$

   (a) We begin with the case where the eigenvalues of $A$ have negative real part. We recall that there exists a positive $\omega$ and a finite $C$ such that $\|\exp(tA)\| \leq Ce^{-t\omega}$ for $t > 0$. Prove that

$$G(t) := \begin{cases} 0_n, & t < 0, \\ \exp(tA), & t > 0 \end{cases}$$

   defines a Green matrix.

   Prove that there are constants as above such that $\|\exp(tA)\| \geq C'e^{-t\omega}$ for $t < 0$ (**Hint**: Use the inequality $1 \leq \|M\| \, \|M^{-1}\|$). Deduce that the Green matrix is unique.

   (b) Treat the case where $A = \mathrm{diag}(B, C)$, where the eigenvalues of $B$ (resp. of $C$) have negative (resp. positive) real parts.

(c) Treat the general case.

(d) Show that actually the Green matrix decays exponentially fast at infinity.

(e) Let $f : \mathbb{R} \to \mathbb{C}^n$ be bounded continuous and define

$$y(t) := \int_{\mathbb{R}} G(t-s) f(s).$$

Show that $y$ is the unique bounded solution of

$$y'(t) = Ay(t) + f(t).$$

123. Let $A \in \mathbf{M}_n(\mathbb{C})$ be given. The spectrum of $A$ is split into three parts $\sigma_-, \sigma_+, \sigma_0$ according to the sign of the real part of the eigenvalues. For instance, $\sigma_0$ is the intersection of the spectrum with the imaginary axis.

(a) Show that the following definitions of a subspace are equivalent:

- The sum of the generalized eigenspaces associated with the eigenvalues of negative real part,
- The largest invariant subspace on which the spectrum of the restriction of $A$ lies in the open left half-space of $\mathbb{C}$.
- The set of data $a \in \mathbb{C}^n$ such that the solution of the Cauchy problem

$$x'(t) = Ax(t), \quad x(0) = a$$

tends to zero as $t \to +\infty$.

This subspace is called the *stable invariant subspace* (more simply the stable subspace) of $A$ and denoted by $S(A)$. Prove that if $a \in S(A)$, the solution of the Cauchy problem above actually decays exponentially fast.

(b) Let us define the unstable subspace by $U(A) := S(-A)$. Give characterizations of $U(A)$ similar to above.

(c) Show that the following definitions of a subspace are equivalent:

- The sum of the generalized eigenspaces associated with the purely imaginary eigenvalues,
- A subspace $C(A)$, invariant under $A$ and such that

$$\mathbb{C}^n = S(A) \oplus C(A) \oplus U(A),$$

- The set of data $a \in \mathbb{C}^n$ such that the solution of the Cauchy problem above is polynomially bounded for $t \in \mathbb{R}$: there exists $m \leq n-1$ and $c_0$ such that $\|x(t)\| \leq c_0(1 + |t|^m)$.

The subspace $C(A)$ is called the *central subspace* of $A$.

(d) Prove that the spectra of the restrictions of $A$ to its stable, unstable and central subspaces are respectively $\sigma_-, \sigma_+$ and $\sigma_0$.

(e) If $A \in \mathbf{M}_n(\mathbb{R})$, prove that the stable, unstable and central subspaces are real, in the sense that they are the complexifications of subspaces of $\mathbb{R}^n$.

(f) Express $S(A^*), \dots$ in terms of $S(A), \dots$.

**Nota**: The case where $\sigma_0$ is void corresponds to a hyperbolic matrix, in the sense of the previous exercise.

124. In control theory, one meets a matrix $H \in \mathbf{M}_{2n}(\mathbb{C})$ given by the formula (notations are equivalent but not identical to the standard ones)

$$H := \begin{pmatrix} A & BB^* \\ C^*C & -A^* \end{pmatrix}.$$

Hereabove, $A$, $B$, $C$ have respective sizes $n \times n$, $n \times m$ and $m \times n$. Without loss of generality, one may assume that $B$ is one-to-one and $C$ is onto ; hence $m \leq n$.

One says that the pair $(A, B)$ is *stabilizable* if the smallest invariant subspace of $A$, containing the range of $B$, contains $C(A) \oplus U(A)$. One also says that the pair $(A, C)$ is *detectable* if the largest invariant subspace of $A$, contained in $\ker C$, is contained in $S(A)$.

(a) Prove that $(A, C)$ is detectable if and only if $(A^*, C^*)$ is stabilizable.

(b) From now on, we assume that $(A, B)$ is stabilizable and $(A, C)$ is detectable. If $\rho \in \mathbb{R}$, show that $(A - i\rho I_n, B)$ is stabilizable and $(A - i\rho I_n, C)$ is detectable.

(c) Prove that $H$ is non-singular. **Hint**: Let $(x, y)^T$ belong to $\ker H$. Find an *a priori* estimate and deduce that

$$Cx = 0, \quad By = 0, \quad Ax = 0, \quad A^*y = 0.$$

(d) Deduce that $H$ is hyperbolic (see the previous exercise).

(e) Let $(0, y_0)^T$ belong to $S(H)$, the stable subspace. Define $(x(t), y(t))^T$ the solution of the Cauchy problem $z'(t) = H(t)$, $z(0) = (0, y_0)^T$. Establish an integral estimate ; prove that $B^*y \equiv 0$, $Cx \equiv 0$ and

$$x'(t) = Ax(t), \qquad y'(t) = -A^*y(t).$$

Deduce that $x \equiv 0$. Using the fact that $(A, B)$ is stabilizable, prove that $y \equiv 0$. **Hint**: The space spanned by the values $y(t)$ is invariant under $A^*$ and annihilated by $B^*$.

(f) Likewise, if $(x_0, 0)$ belongs to $U(H)$, use the detectability of $(A, C)$ to prove that $x_0 = 0$.

(g) Deduce the LopatinskiiLopatinskiĭ condition

$$\mathbb{C}^{2n} = (\{0\} \times \mathbb{C}^n) \oplus S(H).$$

125. Let $A, B, C$ be matrices with complex entries, such that the product $ABC$ makes sense and is a square matrix. Prove

$$|\mathrm{Tr}(ABC)| \leq \|A\|_S \|B\|_S \|C\|_S,$$

with $\|A\|_S = \sqrt{\mathrm{Tr} A^* A}$ the Schur–FrobeniusSchurFrobenius norm. **Hint**: Apply repeatedly the Cauchy–SchwarzCauchySchwarz inequality.

**Nota**: This is the discrete analogue of the following inequality for functions defined on the plane:

$$\left| \int_{\mathbb{R}^3} f(x,y) g(y,z) h(z,x) dx\, dy\, dz \right| \leq \|f\|_{L^2(\mathbb{R}^2)} \|g\|_{L^2(\mathbb{R}^2)} \|h\|_{L^2(\mathbb{R}^2)}.$$

126. Let $H$ be a tridiagonal Hermitian $3 \times 3$ matrix:

$$H = \begin{pmatrix} a_1 & b_1 & 0 \\ \bar{b}_1 & a_2 & b_2 \\ 0 & \bar{b}_2 & a_3 \end{pmatrix} \qquad (a_j \in \mathbb{R},\, b_j \in \mathbb{C}).$$

We denote the characteristic polynomial of $H$ by $P$.

(a) Prove the formula $(a_3 - a_1)|b_1|^2 = P(a_1)$. Therefore, the signs of $P(a_1)$ and $P(a_3)$ are determined by that of $a_3 - a_1$.

(b) Construct a pair $(a, \lambda) \in \mathbb{R}^3 \times \mathbb{R}^3$ such that $a \succ \lambda$, with $a_1 < a_3$ and

$$\prod_j (\lambda_j - a_1) < 0.$$

(c) Deduce that there exists a pair $(a, \lambda) \in \mathbb{R}^3 \times \mathbb{R}^3$ such that $a \succ \lambda$ (and therefore there is a Hermitian matrix with diagonal $a$ and spectrum $\lambda$, from Theorem 3.4.2), but no such matrix being tridiagonal.

**Nota**: A theorem of M. AtiyahAtiyah (Sir M.) asserts that the Hermitian matrices with given diagonal and given spectrum form a connected set (not true for real symmetric matrices). A strategy could have been to show that in such a set, there exists a tridiagonal element. False alas !

(d) Prove Atiyah's result for $2 \times 2$ Hermitian matrices. Prove also that it becomes false in $2 \times 2$ real symmetric matrices.

127. If $N$ is a square matrix, we denote $\hat{N}$ the transpose of the matrix of its cofactors. Recall that $N\hat{N} = \hat{N}N = (\det N) I_n$.

Let $A, B, C, D$ be given $2 \times 2$ matrices. We form a $4 \times 4$ matrix

$$M := \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

(a) Prove that
$$\det M = \det(AD) + \det(BC) - 2\mathrm{Tr}(B\hat{D}C\hat{A}).$$

(b) Deduce that
$$\hat{M} = \begin{pmatrix} (\det D)\hat{A} - \hat{C}D\hat{B} & \cdots \\ \cdots & \cdots \end{pmatrix}.$$
Compare with formula of Corollary 8.1.1.

(c) Show that $\mathrm{rk}M \leq 2$ holds if and only if $\hat{C}D\hat{B} = (\det D)\hat{A}$, ... or equivalently $B\hat{D}C = (\det D)A$, ... Deduce that
$$(\mathrm{rk}M \leq 2) \Longrightarrow (\det(AD) = \det(BC)).$$

(d) More generally, let $A, B, C, D$ be given in $\mathbf{M}_n(k)$, such that the matrix $M$ defined blockwise as above is of rank at most $n$ (this is a $(2n) \times (2n)$ matrix). Prove that $\det(AD) = \det(BC)$. **Hint**: Use the rank decomposition.

**Nota**: This ensures that, for every matrix $M'$ equivalent to $M$ ($M' = PMQ$ with $P, Q$ non-singular), there holds $\det(A'D') = \det(B'C')$

128. Let $f : (0, +\infty) \to \mathbb{R}$ be a continous monotone matrix function (see Exercise 74). We recall that $f$ has a representation of the form (7) with $a \geq 0$, $b \in \mathbb{R}$ and $m$ a non-negative bounded measure, with $(a, m) \neq (0, 0)$.

(a) Show that $f$ is concave increasing.

(b) From now on, we assume that $f$ is continous at $s = 0$, meaning that $f(0+) > -\infty$. Prove that for every $s, t \geq 0$, there holds $f(s + t) \leq f(s) + f(t) - f(0)$. Deduce that if $A \in \mathbf{H}_n$ is non-negative, and if $s \geq 0$, then $f(A + sI_n) \leq f(A) + (f(s) - f(0))I_n$.

(c) Deduce that is $A, B \in \mathbf{H}_n$ are non-negative and if $s \geq 0$, then $A \leq B + sI_n$ implies $f(A) \leq f(B) + (f(s) - f(0))I_n$.

(d) Prove at last that for every non-negative Hermitian $A, B$, there holds
$$\|f(B) - f(A)\|_2 \leq f(\|B - A\|_2) - f(0).$$
**Hint**: Choose $s$ cleverly in the above inequality.

Compare with the results of Exercises 110 and 111. We recall (Exercise 74) that $s \mapsto s^\alpha$ is a monotone matrix function for $0 < \alpha \leq 1$.

129. (KrattenthalerKrattenthaler) Let $X_1, ..., X_n$, $A_2, ..., A_n$ and $B_1, ..., B_{n-1}$ be indeterminates. Prove the identity
$$\det_{1 \leq i,j \leq n}((X_i + A_n) \cdots (X_i + A_{j+1})(X_i + B_{j-1}) \cdots (X_i + B_1)) = \prod_{i<j}(X_i - X_j) \prod_{i<j}(B_i - A_j).$$

**Hint**: Prove that the right-hand side divides the left-hand side. Then compare the degrees of these homogeneous polynomials. At last, compute the ratio by specializing the indeterminates.

130. (R. PichéPiché.) We recall (see Exercise 49 for the case with real entries) that the Hermitian form $h(M) := (n-1)\operatorname{Tr}(M^*M) - |\operatorname{Tr} M|^2$ takes non-negative values on the cone of singular matrices in $\mathbf{M}_n(\mathbb{C})$. Use this result to prove that the spectrum of a general matrix $A \in \mathbf{M}_n(\mathbb{C})$ is contained in the disk of center $n^{-1}\operatorname{Tr} A$ and radius $n^{-1}\sqrt{|\operatorname{Tr} A|^2 + nh(A)}$.

    Check (on $n = 2$ for instance) that this result is not implied by Gershgorin'sGershgorin Theorem.

131. Given a complex $n \times n$ matrix $A$, show that there exists a unitary matrix $U$ such that $M := U^*AU$ has a constant diagonal:

$$m_{ii} = \frac{1}{n}\operatorname{Tr} A, \qquad \forall i = 1, ..., n.$$

    **Hint**: Use the convexity of the numerical range (see Exercise 21).

    In the Hermitian case, compare with Schur'sSchur Theorem 3.4.2.

132. We aim at computing the number of connected components in the subsets of $\mathbf{M}_n(\mathbb{R})$ or of $\mathbf{Sym}_n(\mathbb{R})$ made of matrices with simple eigenvalues. The corresponding sets are denoted hereafter by $\mathbf{sM}_n(\mathbb{R})$ and $\mathbf{sSym}_n(\mathbb{R})$. We recall that $\mathbf{GL}_n(\mathbb{R})$ has two connected components, each one characterized by the sign of the determinant. We denote by $\mathbf{GL}_n^+$ the connected set defined by $\det M > 0$.

    (a) Let $A \in \mathbf{sM}_n(\mathbb{R})$ be given. Show that there exists a matrix $P \in \mathbf{GL}_n^+$ such that $PAP^{-1}$ is block-diagonal, with the diagonal blocks being either scalars (distinct real numbers), or $2 \times 2$ matrices of rotations of distinct nonzero angles.

    (b) Let $A, B$ be given in $\mathbf{sM}_n(\mathbb{R})$, with the same number of pairs of complex conjugate eigenvalues.

        i. Find a path from the spectrum of $A$ to that of $B$, such that each intermediate set is made of distinct real numbers and distinct pairs of complex conjugate numbers.

        ii. By using the connectedness of $\mathbf{GL}_n^+$, prove that $A$ and $B$ belong to the same connected component of $\mathbf{sM}_n(\mathbb{R})$.

    (c) Deduce that $\mathbf{sM}_n(\mathbb{R})$ has exactly $\left[\frac{n+1}{2}\right]$ connected components.

    (d) Following the same procedure, prove that $\mathbf{sSym}_n$ is connected. **Comment**: Here is a qualitative explanation of the discrepancy of both results. The complement of the set of matrices with simple eigenvalues is the zero set of a homogeneous polynomial $A \mapsto \Delta(A)$, the discriminant of the characteristic polynomial. In the general case, this polynomial takes any sign, and the complement of $\mathbf{sM}_n(\mathbb{R})$ is an algebraic hypersurface. It has codimension one and splits $\mathbf{M}_n(\mathbb{R})$ into several connected components. In the symmetric case, $\Delta$ takes only non-negative values, because the spectra remain real. It therefore degenerates along its zero set, which turns out to be of algebraic codimension two (see V. I. ArnoldArnold, *Chapitres supplémentaires de la théorie des équations différentielles ordinaires*, Mir (1980)). Consequently $\mathbf{sSym}_n$ is connected.

133. In paragraph 6.3.1, one shows that the minimal polynomial of a companion matrix equals its characteristic polynomial. On another hand, Proposition 10.1.1 tells us that eigenvalues of an irreducible Hessenberg matrix are geometrically simple. Show that these results are variants of the more general one: The minimal polynomial of an irreducible Hessenberg matrix equals its characteristic polynomial.

134. Given $A \in \mathbf{M}_{n\times m}(k)$ and $B \in \mathbf{M}_{m\times n}(k)$, we form the matrix $M \in \mathbf{M}_{2n+m}(k)$:

$$M = \begin{pmatrix} I_n & A & 0 \\ 0 & I_m & B \\ 0 & 0 & I_n \end{pmatrix}.$$

Compute the inverse $M^{-1}$. Deduce that if we need $dN^\alpha$ ($d, \alpha$ independent of $N$) operations to invert an $N \times N$ matrix, we can multiply two matrices such as $A$ and $B$ in $d(2n+m)^\alpha$ operations. In particular, the converse of Proposition 8.1.3 holds true.

135. Let $n$ be an even integer ($n = 2m$). You check easily that given a row $x$ and a column $y$ of $n$ scalars, their product satisfies the relation

$$xy = \sum_{k=1}^{m} (x_{2k-1} + y_{2k})(x_{2k} + y_{2k-1}) - \sum_k x_{2k}x_{2k-1} - \sum_k y_{2k}y_{2k-1}.$$

Deduce a way to compute a matrix product in $\mathbf{M}_n(k)$ in $\frac{n^3}{2} + O(n^2)$ multiplications and $n^3 + O(n^2)$ additions, instead of $n^3 + O(n^2)$ of each by the naive method. **Comment**: This is Winograd'sWinograd calculation. In the 60's, the computational cost of a multiplication was two or three times that of an addition. Thus it was valuable to divide the number of multiplications by some factor (here by two), keeping the number of additions roughly the same.

Can this idea be used recursively, as for Strassen'sStrassen multiplication ?

136. Let $k$ be $\mathbb{R}$ or $\mathbb{C}$ and $\|\cdot\|$ be a unitary invariant norm on $\mathbf{M}_n(k)$. Prove that for every matrix $A \in \mathbf{M}_n(k)$, there holds $\|A^*\| = \|A\|$.

137. Given a norm $\|\cdot\|$ on $\mathbb{R}^n$, we denote by $S$ the unit sphere of the corresponding induced norm on $\mathbf{M}_n(\mathbb{R})$.

   (a) Given $e, f \in \mathbb{R}^n$ such that $\|f\| = \|e\| \neq 0$, we define

   $$K(e,f) := \{M \in \mathbf{M}_n(\mathbb{R})\,;\, \|M\| \leq 1 \text{ and } Me = f\}.$$

   Prove that $K(e,f)$ is a convex subset of $S$.

   (b) Conversely, let $K$ be a convex subset of $S$. We assume that the norm $\|\cdot\|$ in $\mathbb{R}^n$ is strictly convex, meaning that if $\|x+y\| = \|x\| + \|y\|$, then $x$ and $y$ are proportional (with the same sense of course). Prove that there exists a pair $(e,f)$ as above, such that $K = K(e,f)$. **Hint**: Consider an internal point $N$ of $K$ and a unit vector $e$ for which $\|Ne\| = 1$.

(c) In this question, the norm of $\mathbb{R}^n$ is $\|\cdot\|_2$ and the sphere $S$ is denoted by $S_2$.

i. Prove that every $K(f', f)$ is linearly isometric to $K(e, e)$ with $e = (1, \dots, 1)^T$.

ii. Show that $K(e, e)$ is the set of matrices such that $Me = e$, $M^T e = e$ and the restriction of $M$ to $e^\perp$ (an endomorphism of course) has operator norm at most one. Hence $K(e, e)$ is linearly isometric to the unit ball of $\mathbf{M}_{n-1}(\mathbb{R})$ equipped with the induced norm $\|\cdot\|_2$. Compare this result with Corollary 5.5.1.

iii. In particular, show that $K(e, e)$, and therefore each $K(e, f)$, is maximal among the convex subsets of $S_2$.

138. (Continuation.) We keep $e = (1, \dots, 1)^T$ but consider the norm $\|\cdot\|_p$, where $1 \le p \le \infty$. The corresponding set $K(e, e)$ is denoted by $K_p$.

(a) For $p = 1$, show that $K_1$ reduces to the set $\Delta_n$ of bistochastic matrices.

(b) For $p = \infty$, show that $K_\infty$ is the set of stochastic matrices, defined by $Me = e$ and $M \ge 0$.

(c) If $r \in (p, q)$, show that $K_p \cap K_q \subset K_r$.

(d) Assume that $1 < p < \infty$. Making a Taylor expansion of $\|e + \epsilon y\|_p$ and of $\|e + \epsilon My\|_p$, as $\epsilon \to 0$, prove that $K_p \subset K_2$.

(e) Prove that $K_\infty$ is not a subset of $K_2$. **Hint**: Compare the dimensions of these convex sets.

(f) Show that $p \mapsto K_p$ is non-decreasing on $[1, 2]$ and non-increasing on $[2, \infty)$ (we have seen above that the monotonicity fails on $[2, \infty]$).

(g) If $1 \le p < 2$, prove the "right-continuity"

$$K_p := \bigcap_{p<q\le 2} K_q.$$

(h) If $1 < p < \infty$, show that $M \mapsto M^T$ is an isometry from $K_p$ onto $K_{p'}$, where $p'$ is the conjugate exponent (the calculations above show that this is false for $p = 1$ or $p = \infty$). Deduce that $q \mapsto K_q$ is "left-continuous" on $(2, \infty)$ (of course it is not at $q = \infty$ since $K_\infty$ is much too big).

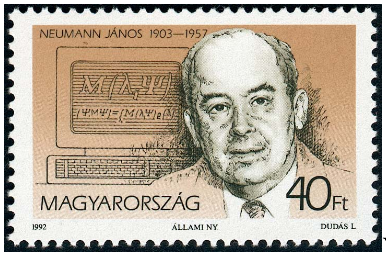
(i) Deduce that

$$\bigcap_{2\le q<\infty} K_q = \Delta_n.$$

139. Let $d, \delta \in \mathbb{R}^n$ be given, together with unitary matrices $Q, R$. We form the diagonal matrices $D = \operatorname{diag}(d)$ and $\Delta = \operatorname{diag}(\delta)$.

(a) Show that $\operatorname{Tr}(DQ\Delta R)$ equal $d^T S\delta$, where the matrix of moduli $|S|$ is majorized by a bi-stochastic matrix $M$ (see also Exercise 101).

(b)



vonneu@von Neumann, *John*!(Hungary)

Deduce von Neumann's inequalityvonneu@von Neumann in $\mathbf{M}_n(\mathbb{C})$:

$$|\mathrm{Tr}(AB)| \le \sum_i s_i(A)s_i(B), \tag{16}$$

where $s_1(C) \le \cdots \le s_n(C)$ denote the singular values of $C$.

**John von Neumann.**

**Nota**: B. DacorognaDacorogna and P. MaréchalMaréchal have proven the more general inequality

$$\mathrm{Tr}(AB) \le (\mathrm{sign}\det(AB))s_1(A)s_1(B) + \sum_{i\ge 2} s_i(A)s_i(B). \tag{17}$$

140. Let $k$ be $\mathbb{R}$ or $\mathbb{C}$. Given a bounded subset $F$ of $\mathbf{M}_n(k)$, let us denote by $F_k$ the set of all possible products of $k$ elements in $F$. Given a matrix norm $\|\cdot\|$, we denote $\|F_k\|$ the supremum of the norms of elements of $F_k$.
   (a) Show that $\|F_{k+l}\| \le \|F_k\| \cdot \|F_l\|$.
   (b) Deduce that the sequence $\|F_k\|^{1/k}$ converges, and that its limit is the infimum of the sequence.
   (c) Prove that this limit does not depend on the choice of the matrix norm.
   This limit is called the *joint spectral radius* of the family $F$, and denoted $\rho(F)$. This notion is due to G.-C. RotaRota and G. StrangStrang.
   (d) Let $\hat\rho(F)$ denote the infimum of $\|F\|$ when $\|\cdot\|$ runs over all matrix norms. Show that $\rho(F) \le \hat\rho(F)$.
   (e) Given a norm $N$ on $k^n$ and a number $\epsilon > 0$, we define for every $x \in k^n$
   $$\|x\| := \sum_{l=0}^{\infty}(\rho(F)+\epsilon)^{-l}\max\{N(Bx)\,;\, B \in F_l).$$
      i. Show that the series converges, and that it defines a norm on $k^n$.
      ii. For the matrix norm associated with $\|\cdot\|$, show that $\|A\| \le \rho(F)+\epsilon$ for every $A \in F$.
      iii. Deduce that actually $\rho(F) = \hat\rho(F)$. Compare with Householder'sHouseholder Theorem.

141. (G.-C. RotaRota & G. StrangStrang.) Let $k$ be $\mathbb{R}$ or $\mathbb{C}$. Given a subset $F$ of $\mathbf{M}_n(k)$, we consider the semi-group $\mathcal{F}$ generated by $F$. It is the union of sets $F_k$ defined in the previous exercise, as $k$ runs over $\mathbb{N}$. We have $F_0 = \{I_n\}$, $F_1 = F$, $F_2 = F \cdot F$,...

If $\mathcal{F}$ is bounded, prove that there exists a matrix norm $\|\cdot\|$ such that $\|A\| \leq 1$ for every $A \in F$. **Hint**: In the previous exercise, take a sup instead of a series.

142. Let define the two matrices

$$A_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Given a map $s : \{1, ..., r\} \to \{0, 1\}$ (i.e. a word in two letters), we define

$$A(s) := A_{s(1)}A_{s(2)} \cdots A_{s(r)}, \qquad \hat{A}(s) := A_{s(r)}A_{s(r-1)} \cdots A_{s(1)}$$

($\hat{A}(s)$ is the *palindrome* of $A(s)$). Show that

$$\hat{A}(s) - A(s) = \begin{pmatrix} m(s) & 0 \\ 0 & -m(s) \end{pmatrix},$$

where $m(s)$ is an integer.

143. Let $T \in \mathbf{Sym}_n(\mathbb{R})$ be ToeplizToepliz, meaning that

$$T = \begin{pmatrix} a_1 & a_2 & a_3 & & a_n \\ a_2 & a_1 & a_2 & \ddots & \\ a_3 & a_2 & a_1 & \ddots & \\ & \ddots & \ddots & \ddots & \\ a_n & & & & a_1 \end{pmatrix}.$$

We denote by $\Delta_j$ the principal minors, in particular $\Delta_1 = a$ and $\Delta_n = \det T$.

(a) Prove the inequality

$$\Delta_n \Delta_{n-2} \leq \Delta_{n-1}^2.$$

**Hint:** Use Desnanot–JacobiDesnanotJacobi formula of Exercise 24.

(b) If $\Delta_{n-2} \neq 0$ and $a_1, \dots, a_{n-1}$ are given, show that there exists a unique value $a_n$ such that the above inequality becomes an equality.

(c) When $T$ is positive definite, deduce the inequalities

$$\Delta_n^{1/n} \leq \Delta_{n-1}^{1/(n-1)} \leq \cdots \leq \Delta_1.$$

(d) Let us assume that $T$ is positive definite. Prove that for every $N > n$, $T$ may be completed into an $\mathbf{SDP}_N$ Toepliz matrix.

144. Let $A \in \mathbf{M}_n(k)$ be given, with $n = p + q$. For $1 \leq i, j \leq p$, let us define the minor

$$d_{ij} := A\begin{pmatrix} i & p+1 & \cdots & n \\ j & p+1 & \cdots & n \end{pmatrix}.$$

With the entries $d_{ij}$, we form a matrix $D \in \mathbf{M}_p(k)$. Prove the Desnanot–JacobiDesnanotJacobi formula (see Exercise 24 for the case p=2)

$$\det D = \delta^{p-1} \det A, \qquad \delta := A \begin{pmatrix} p+1 & \cdots & n \\ p+1 & \cdots & n \end{pmatrix}.$$

**Hint**: Develop $d_{ij}$ with the help of Schur'sSchur determinant formula. Then apply once more Schur's formula to $\det A$.

145. (B. Perthame and S. Gaubert.)PerthameGaubert Given a non-negative matrix $N \in \mathbf{M}_n(\mathbb{R})$ that is irreducible, let denote $M := N - \rho(N) I_n$. From Perron–Frobenius Theorem, $\lambda = 0$ is a simple eigenvalue of $M$, associated with a positive eigenvector $X$. Let also $Y$ denote a positive eigenvector of $M^T$, again for the zero eigenvalue.

Given an initial data $x^0 \in \mathbb{R}^n$, let $t \mapsto x(t)$ be the solution of the ODE $\dot{x} = Mx$, such that $x(0) = x^0$.

(a) Show that

$$Y \cdot x(t) \equiv Y \cdot x^0. \tag{18}$$

(b) Let $H$ be a $\mathcal{C}^1$-function over $\mathbb{R}$. Show that

$$\begin{aligned} \frac{d}{dt} \sum_j Y_j X_j H\left(\frac{x_j}{X_j}\right) \quad &= \quad \sum_{j,k} X_k Y_j \left( H\left(\frac{x_j}{X_j}\right) - H\left(\frac{x_k}{X_k}\right) \right. \\ &\qquad \left. + \left(\frac{x_k}{X_k} - \frac{x_j}{X_j}\right) H'\left(\frac{x_j}{X_j}\right)\right). \end{aligned} \tag{19}$$

(c) Show that the kernel of the quadratic form

$$(x, z) \mapsto \sum_{j,k} m_{jk} \frac{Y_j}{X_j} (x_j z_k - x_k z_j)^2$$

is exactly the line spanned by $z$.

(d) Let $s := x \cdot Y / X \cdot Y$. Deduce that the expression

$$D(t) := \sum_j \frac{Y_j}{X_j} (x_j - s X_j)^2$$

satisfies a differential inequality of the form

$$\dot{D} + \epsilon D \leq 0,$$

where $\epsilon > 0$ depends only on $M$. **Hint**: Use both (18) and (19).

(e) Verify that $x(t)$ converges towards $sX$, exponentially fast.

(f) State a result for the solutions of $\dot{x} = Nx$.

146. To do this exercise, you need to know about the exterior algebra $\Lambda E$. Recall that if $E$ is a $K$-vector space of dimension $n$, the exterior algebra $\Lambda E$ is the direct sum of the subspaces $\Lambda^k E$ of the tensor algebra, spanned by the vectors $x^1 \wedge \cdots \wedge x^k$, where $x \wedge y := x \otimes y - y \otimes x$ whenever $x, y \in E$, and $\wedge$ is associative. We have

$$\dim \Lambda^k E = \begin{pmatrix} n \\ k \end{pmatrix}.$$

If $\{e^1, \ldots, e^n\}$ is a basis of $E$, then a basis of $\Lambda^k E$ is given by the vectors $e^{j_1} \wedge \cdots \wedge e^{j_k}$ as $1 \leq j_1 < \cdots < j_k \leq n$.

(a) Let $u$ be an endomorphism in $E$. Prove that there exists a unique endomorphism in $\Lambda^k E$, denoted by $u^{(k)}$, such that

$$u^{(k)}(x^1 \wedge \cdots \wedge x^k) = u(x^1) \wedge \cdots \wedge u(x^k), \qquad \forall x^1, \ldots, x^k \in E.$$

(b) Let $\{e^1, \ldots, e^n\}$ be a basis of $E$ and $A$ be the matrix of $u$ in this base. Show that the entries of the matrix associated with $u^{(k)}$ are the minors

$$A \begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix}$$

with $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$.

(c) Let $x^1, \ldots, x^k$ be linearly independent vectors of $u$, associated with the eigenvalues $\lambda_1, \ldots, \lambda_k$. Prove that $x^1 \wedge \cdots \wedge x^k$ is an eigenvector of $u^{(k)}$. What is the corresponding eigenvalue ?

(d) If $K = \mathbb{C}$, show that the spectrum of $u^{(k)}$ is made of the products

$$\prod_{i \in I} \lambda_i, \qquad |I| = k,$$

where $\{\lambda_1, \ldots, \lambda_n\}$ is the spectrum of $u$. **Hint**: Consider first the case where $\lambda_1, \ldots, \lambda_n$ are distinct. Then proceed by density.

(e) Prove that the above property is true for every scalar field $K$. **Hint**: The case $K = \mathbb{C}$ provides an algebraic identity with integral coefficients.

147. (Continuation.) We now take $K = \mathbb{R}$. Recall that a matrix $M \in \mathbf{M}_n(\mathbb{R})$ is totally positive if all the minors

$$M \begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix}$$

with $k \leq n$, $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$ are positive. Total positiveness implies positiveness.

(a) If $M$ is positive, prove that $\rho(M)$ is a positive simple eigenvalue with the property that $\rho(M) > |\lambda|$ for every other eigenvalue of $M$.

(b) Let $M \in \mathbf{M}_n(\mathbb{R})$ be totally positive. Prove that its eigenvalues are real, positive and pairwise distinct (thus simple). **Hint**: Proceed by induction on $n$. Use the fact that $\lambda_1 \cdots \lambda_n$ is the unique eigenvalue of $M^{(n)}$.

(c) Likewise, show that the singular values of $M$ are pairwise distinct.

148. (LoewnerLoewner.)

(a) Let $A$ be a tridiagonal matrix with non-negative off-diagonal entries ($i \neq j$).

i. Let us consider a minor
$$A\begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix}$$
with $k \geq 2$, $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$. Show that it is either the product of minors of smaller sizes (reducible case), or it is a principal minor ($j_1 = i_1, \ldots, j_k = i_k$).

ii. Deduce that there exists a real number $x$ such that $xI_n + A$ is totally non-negative. **Hint**: Choose $x$ large enough. Argue by induction on $n$.

(b) Let $B \in \mathbf{M}_n(\mathbb{R})$ be tridiagonal with non-negative off-diagonal entries. Deduce from above and from Trotter'sTrotter formula
$$\exp M = \lim_{k \to +\infty} \left( I_n + \frac{1}{k} M \right)^k,$$
that the semi-group $(\exp(tB))_{t \geq 0}$ is made of totally non-negative matrices.

(c) Conversely, let $A \in \mathbf{M}_n(\mathbb{R})$ be given, such that the semi-group $(\exp(tA))_{t \geq 0}$ is made of totally non-negative matrices.

i. Show that the off-diagonal entries of $A$ are non-negative.

ii. If $j > i + 1$, show that $a_{ij} \leq 0$. **Hint**: consider the minor
$$M_t \begin{pmatrix} i & i+1 \\ i+1 & j \end{pmatrix}$$
of $M_t := \exp(tA)$.

iii. Deduce that $A$ is tridiagonal, with non-negative off-diagonal entries.

149. We denote by $e$ the vector of $\mathbb{R}^n$ whose every component equals one.

(a) Let $A$ be a Hermitian matrix, semi-positive definite, and denote $a \in \mathbb{R}^n$ the vector whose components are the diagonal entries of $A$. Show that $B := ae^T + ea^T - 2A$ is a Euclidean distance matrix (see Exercise 62). **Hint**: Use a factorization $A = M^T M$.

(b) Conversely, show that every Euclidean distance matrix is of this form for some semi-positive definite Hermitian matrix.

(c) Deduce that the set of Euclidean distance matrices is a convex cone. Find also a direct proof of this fact.

(d) Let $s \in \mathbb{R}^n$ be such that $e^T s = 1$. Let $F(s)$ be the cone of semi-positive definite Hermitian matrices $T$ such that $Ts = 0$. Show that the restriction to $F(s)$ of the map $A \mapsto ae^T + ea^T - 2A$ is injective, and that its inverse is given by

$$M \mapsto -\frac{1}{2}(I_n - es^T)M(I_n - se^T).$$

150. Let $A \in \mathbf{M}_{n\times m}(\mathbb{R})$ and $b \in \mathbb{R}^n$ be given. Define two sets

$$X := \{x \in \mathbb{R}^m\,;\, Ax \le b \text{ and } x \ge 0\}, \quad Y := \{y \in \mathbb{R}^n\,;\, A^T y \ge 0,\, y \ge 0 \text{ and } b \cdot y < 0\},$$

where the inequalities stand for vectors, as in Chapter 5.

Prove that exactly one of both sets is void (Farkas'Farkas Lemma).

151. Consider the homogeneous polynomial $p_r(X) := X_0^2 - X_1^2 - \cdots - X_r^2$ for some integer $r \ge 1$.

(a) For $r = 1$ and $r = 2$, show that there exist real symmetric matrices $S_0, \dots, S_r$ such that

$$p_r(X) = \det(X_0 S_0 + \cdots + X_r S_r).$$

(b) When $r \ge 3$ show that there does not exist matrices $A_0, \dots, A_r \in \mathbf{M}_2(\mathbb{R})$ such that

$$p_r(X) = \det(X_0 A_0 + \cdots + X_r A_r).$$

**Hint**: Consider a vector $v \in \mathbb{R}^{r+1}$ such that $v_0 = 0$ and the first row of $v_1 A_1 + \cdots + v_r A_r$ vanishes.

152. Here is another proof of Birkhoff'sBirkhoff Theorem. Let $A$ be a bistochastic matrix of size $n$.

(a) Prove that there does not exist a submatrix of size $k \times l$ with null entries and $k+l > n$. **Hint**: Count the sum of all entries of $A$.

Then Exercise 10 of Chapter 2, page 32, tells you that there exists a permutation $\sigma$ such that $a_{i\sigma(i)} \neq 0$ for every $i = 1, ..., n$ (this result bears the name of Frobenius–König TheoremFrobeniusKonig@König). In the sequel, we denote by $P$ the permutation matrix with entries $p_{ij} = \delta_j^{\sigma(i)}$.

(b) Let $a$ be the minimum of the numbers $a_{i\sigma(i)} \neq 0$, so that $a \in (0,1]$. If $a = 1$, prove that $A = P$. **Hint**: Again, consider the sum of all entries.

(c) If $a < 1$, let us define

$$B = \frac{1}{1-a}(A - aP).$$

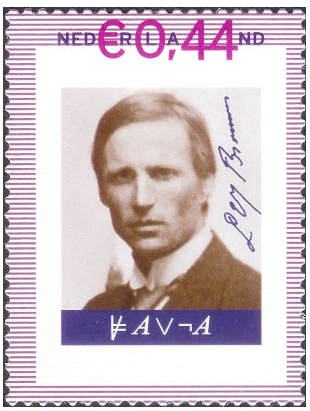
Show that $B$ is bistochastic. Deduce that if $A$ is extremal in $\Delta_n$, then $A = P$.

153. This is a sequel of Exercise 26, Chapter 4 (#23, Chap. 7 in the second edition ; this exercise does not exist in the French edition). We recall that $\Sigma$ denotes the unit sphere of $\mathbf{M}_2(\mathbb{R})$ for the induced norm $\|\cdot\|_2$. Also recall that $\Sigma$ is the union of the segments $[r,s]$ where $r \in \mathcal{R} := \mathbf{SO}_2(\mathbb{R})$ and $s \in \mathcal{S}$, the set of orthogonal symmetries. Both $\mathcal{R}$ and $\mathcal{S}$ are circles. At last, two distinct segments may intersect only at an extremity.

(a) Show that there is a unique map $\rho : \Sigma \setminus \mathcal{S} \to \mathcal{R}$, such that $M$ belongs to some segment $[\rho(M), s)$ with $s \in \mathcal{S}$. For which $M$ is the other extremity $s$ unique ?

(b) Show that the map $\rho$ above is continuous, and that $\rho$ coincides with the identity over $\mathcal{R}$. We say that $\rho$ is a *retraction* from $\Sigma \setminus \mathcal{S}$ onto $\mathcal{R}$.

(c) Let $f : D \to \Sigma$ be a continuous function, where $D$ is the unit disk of the complex plane, such that $f(\exp(i\theta))$ is the rotation of angle $\theta$. Show that $f(D)$ contains an element of $\mathcal{S}$.
**Hint**: Otherwise, there would be a retraction of $D$ onto the unit circle, which is impossible (an equivalent statement to BrouwerBrouwer Fixed Point Theorem).
**Meaning.** Likewise, one finds that if a disk $D'$ is immersed in $\Sigma$, with boundary $\mathcal{S}$, then it contains an element of $\mathcal{R}$. We say that the circles $\mathcal{R}$ and $\mathcal{S}$ of $\Sigma$ are *linked*.




**Luitzen Brouwer.**

154. Recall that $\Delta_3$ denotes the set of $3\times 3$ bistochastic matrices. As the convex hull of a finite set (the permutation matrices), it is a polytope. It thus has $k$-faces for $k = 0,1,2,3$. Of course, 0-faces are vertices. Justify the following classification:

- There are 6 vertices,
- 15 1-faces, namely all the segments $[P,Q]$ with $P,Q$ permutation matrices,
- 18 2-faces, all of them being triangles. Each one is characterized by an inequality $m_{ij} + m_{i'j'} \leq 0$, where $i \neq i'$ and $j \neq j'$,
- 9 3-faces, all of them being 3-simplex. Each one is characterized by an inequality $m_{ij} \leq 0$ for some pair $(i,j)$.

Euler, *Leonhard* (Switzerland)

**Hint**: To prove that a convex subset of dimension $k \leq 3$ is a face, it is enough to characterize it by a linear inequality within $\Delta_3$. Notice that the alternate sum $6-15+18-9$ vanishes, as the EulerEuler–PoincaréPoincaré characteristics of the sphere $\mathbf{S}^3$ is zero. Be cautious enough to prove that there is not any other face.

**Leonhard Euler.**

155. (From V. Blondel & Y. Nesterov.) Let $F = \{A_1, \dots, A_m\}$ be a finite subset of $M_n(k)$ ($k = \mathbb{R}$ or $k = \mathbb{C}$). We denote by $\rho(A_1, \dots, A_m)$ the joint spectral radius of $F$ (see Exercise 140 for this notion). Prove that

$$\frac{1}{m}\rho(A_1 + \cdots + A_m) \leq \rho(A_1, \dots, A_m).$$

Suppose that $k = \mathbb{R}$ and that $A_1, \dots, A_m$ are non-negative. Prove that

$$\rho(A_1, \dots, A_m) \leq \rho(A_1 + \cdots + A_m).$$

156. Let $n = lm$ and $A^1, \dots, A^m \in \mathbf{M}_l(k)$ be given matrices. Let us form the matrix

$$A := \begin{pmatrix} 0_l & \cdots & \cdots & 0_l & A^1 \\ A^2 & \ddots & & & 0_l \\ 0_l & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_l & \cdots & 0_l & A^m & 0_l \end{pmatrix}.$$

Prove that

$$\det(I_n - A) = \det(I_l - A^m \cdots A^1).$$

Deduce the formula involving characteristic polynomials:

$$P_A(X) = P_{A^m \cdots A^1}(X^m).$$

157. Let $H$ be a Hermitian matrix, given in block form as

$$H = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}.$$

Assume that $\mathrm{rk}(H) = \mathrm{rk}(A)$ (we say that $H$ is a *flat extension* of $A$). Prove that the number of positive (resp. negative) eigenvalues of $A$ and $H$ are equal. In particular:

$$(H \geq 0) \Longleftrightarrow (A \geq 0).$$

158. Let $A \in \mathbf{M}_n(\mathbb{C})$ be a normal matrix. We define $B \in \mathbf{M}_{n-1}(\mathbb{C})$ by deleting the last row and the last column from $A$. Let $\{\lambda_1, \dots, \lambda_n\}$ be the spectrum of $A$, and $\{\mu_1, \dots, \mu_{n-1}\}$ be that of $B$. Finally, denote $e := (0, \dots, 0, 1)^T$.

    (a) Show the identity
    $$\langle (\lambda - A)^{-1} e, e \rangle = \frac{\det(\lambda I_{n-1} - B)}{\det(\lambda I_n - A)}.$$

    (b) Deduce that the rational function
    $$R(\lambda) := \frac{\det(\lambda I_{n-1} - B)}{\det(\lambda I_n - A)}$$
    has simple poles with non-negative residues.

    (c) Conversely, let $\{\lambda_1, \dots, \lambda_n\}$ and $\{\mu_1, \dots, \mu_{n-1}\}$ be given tuples of complex numbers, such that the rational function
    $$R(\lambda) := \frac{\prod_{j=1}^{n-1}(\lambda - \mu_j)}{\prod_{k=1}^{n}(\lambda - \lambda_k)}$$
    has simple poles with non-negative residues. Prove that there exists a normal matrix $A$ such that $\{\lambda_1, \dots, \lambda_n\}$ is the spectrum of $A$, and $\{\mu_1, \dots, \mu_{n-1}\}$ is that of $B$.

159. Let $n = lm$ and $A^1, \dots, A^m, B^1, \dots, B^m \in \mathbf{M}_l(k)$ be given matrices. Let us form the matrix
$$A := \begin{pmatrix} 0_l & \cdots & \cdots & 0_l & B^1 & A^1 \\ A^2 & \ddots & & & 0_l & B^2 \\ B^3 & \ddots & \ddots & & & 0_l \\ 0_l & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0_l & \cdots & 0_l & B^m & A^m & 0_l \end{pmatrix}.$$
We define also the product
$$M(X) := \begin{pmatrix} 0_l & XI_l \\ B^m & A^m \end{pmatrix} \cdots \begin{pmatrix} 0_l & XI_l \\ B^1 & A^1 \end{pmatrix}.$$

(a) Let $\lambda \in k$ and $r := \begin{pmatrix} s \\ t \end{pmatrix} \in k^{2l}$ be such that $M(\lambda)r = \lambda^m r$. Construct an eigenvector $T$ of $A$, for the eigenvalue $\lambda$, such that we have in block form

$$T = \begin{pmatrix} t^1 \\ \vdots \\ t^{m-2} \\ t^{m-1} = s \\ t^m = t \end{pmatrix}.$$

(b) Conversely, let $\lambda$ be an eigenvalue of $A$. Assume that $\lambda$ is not zero. Prove that $\lambda^m$ is an eigenvalue of $M(\lambda)$.

(c) Let us define the multivariate polynomial

$$Q(X_1, \dots, X_m) := \det \left( X_1 \cdots X_m I_{2l} - \begin{pmatrix} 0_l & X_m I_l \\ B^m & A^m \end{pmatrix} \cdots \begin{pmatrix} 0_l & X_1 I_l \\ B^1 & A^1 \end{pmatrix} \right).$$

Show that if at least one of the $X_j$'s vanishes, then $Q$ vanishes. Therefore $X_1 \cdots X_m$ factorizes in $Q(X_1, \dots, X_m)$.

(d) In the case where $A$ has $n$ distinct non-vanishing eigenvalues, deduce that

$$X^n P_A(X) = \det(X^m I_{2l} - M(X)), \tag{20}$$

where $P_A$ is the characteristic polynomial of $A$.

(e) Using the principle of algebraic identities, show that (20) holds for every scalar field $k$ and every matrices $A^1, \dots, B^m$.

160. Let $S$ be a set and $m, n$ be positive integers. Let $(A_s)_{s \in S}$ and $(B_s)_{s \in S}$ be two families indexed by $S$, with $A_s \in \mathbf{M}_m(k)$, $B_s \in \mathbf{M}_n(k)$. We assume that the only subspaces of $k^m$ (respectively $k^n$) invariant by every $A_s$ (respectively $B_s$), that is $A_s E \subset E$, are $\{0\}$ and $k^m$ (respectively $k^n$) itself.

    Let $M \in \mathbf{M}_{n \times m}(k)$ be such that $B_s M = M A_s$ for every $s$. Prove (Schur's LemmaSchur) that either $M = 0_{n \times m}$, or $m = n$ and $M$ is non-singular. **Hint**: Consider the range and the kernel of $M$.

161. Let $n = p + q$ with $0 < p < q$ be given. We denote by $\mathcal{A}$ the subset of $\mathbf{M}_n(k)$ made of the matrices with block form

$$\begin{pmatrix} 0_p & 0_{p \times q} \\ A & 0_q \end{pmatrix}.$$

Likewise, $\mathcal{B}$ is made of the matrices

$$\begin{pmatrix} 0_q & 0_{q \times p} \\ B & 0_p \end{pmatrix}.$$

Both $\mathcal{A}$ and $\mathcal{B}$ are subalgebras of $\mathbf{M}_n(k)$, with dimension $pq$ and the property that $MN = 0_n$ for every two elements (of the same algebra). Prove that $\mathcal{A}$ and $\mathcal{B}$ are not conjugated in $\mathbf{M}_n(k)$. Show however that $\mathcal{B}$ is conjugated to $\mathcal{A}^T$ in $\mathbf{M}_n(k)$.

162. (von Neumannvonneu@von Neumann)


vonneu@von Neumann, *John*!(Tatarstan)

Recall that a norm $\|\cdot\|$ on $\mathbf{M}_n(\mathbb{C})$ is *unitarily invariant* if $\|UAV\| = \|A\|$ for every $U, V \in \mathbf{U}_n$ and $A \in \mathbf{M}_n(\mathbb{C})$. In this exercise, we define $\sigma : \mathbf{M}_n(\mathbb{C}) \to \mathbb{R}^n$ by $\sigma(A) = (s_1(A), \dots, s_n(A))$, where $s_1(A) \le \cdots \le s_n(A)$ are the singular values of $A$.

**John von Neumann.**

(a) Let $\|\cdot\|$ be a unitarily invariant norm. Prove that there exists a unique function $g : \mathbb{R}^n \to \mathbb{R}$, even with respect to each coordinates and invariant under permutations, such that $\|\cdot\| = g \circ \sigma$.

In the sequel, such an invariant norm on $\mathbb{R}^n$ is called a *gauge*.

(b) What are the gauges associated with $\|\cdot\|_2$ and to the Frobenius norm $(\mathrm{Tr}(A^*A))^{1/2}$ ?

(c) Conversely, let $g$ be a gauge on $\mathbb{R}^n$. We denote by $g_*$ its dual norm on $\mathbb{R}^n$. Verify that $g_*$ is a gauge. Then prove that

$$g_* \circ \sigma(A) \le \sup_{B \neq 0_n} \frac{\Re \,\mathrm{Tr}(A^*B)}{g(\sigma(B))}. \tag{21}$$

**Hint**: First consider the case where $A$ is diagonal and non-negative, and use only the $B$'s that are diagonal an non-negative.

(d) Prove that (21) is actually an equality. **Hint**: Use von Neumann'svonneu@von Neumann inequality of Exercise 139.

(e) Deduce that if $G$ is a gauge, then $G \circ \sigma$ is a unitarily invariant norm on $\mathbf{M}_n(\mathbb{C})$. **Hint**: Apply the results above to $g = G_*$.

163. Given $A \in \mathbf{M}_n(k)$, we define a linear map $T_A$ by $X \mapsto T_A X := A^T X A$ for $X \in \mathbf{Sym}_n(k)$. The goal of this exercise is to compute the determinant of $T_A$.

(a) If $A = Q^T D Q$ with $Q$ orthogonal, prove that $T_A$ and $T_D$ are conjugate to each other.

(b) Compute $\det T_D$ when $D$ is diagonal.

(c) Verify $T_{QS} = T_S \circ T_Q$, whence $\det T_{QS} = \det T_Q \det T_S$.

(d) Consider the case $k = \mathbb{R}$. Deduce from above that if $A$ is itself symmetric, then $\det T_A = (\det A)^2$.

(e) (Case $k = \mathbb{R}$, continuing). Let $Q$ be real orthogonal. Show that $I_n$ is a cluster point of the sequence $(Q^m)_{m \in \mathbb{N}}$. Deduce that the identity of $\mathbf{Sym}_n(\mathbb{R})$ is a cluster point of the sequence $(T_Q^m)_{m\in\mathbb{N}}$. Thus $\det T_Q = \pm 1$. Using the connectedness of $\mathbf{SO}_n$, show that actually $\det T_Q = 1$.

(f) (Case $k = \mathbb{R}$, continuing). Using the polar decomposition of $A \in \mathbf{GL}_n(\mathbb{R})$, prove that $\det T_A = (\det A)^2$. Show that this formula extends to every $A \in \mathbf{M}_n(\mathbb{R})$.

(g) Check that the formula $\det T_A = (\det A)^2$ is a polynomial identity with integer coefficients, thus extends to every scalar field.

164. (BoydBoyd, DiaconisDiaconis, SunSun, Jun & XiaoXiao, Lin.) Let $P$ be a symmetric stochastic $n \times n$ matrix:

$$p_{ij} = p_{ji} \geq 0, \qquad \sum_j p_{ij} = 1 \quad (i = 1, \dots, n).$$

We recall that $\lambda_1 = 1$ is an eigenvalue of $P$, which is the largest in modulus (Perron–Frobenius). We are interested in the second largest modulus $\mu(P) = \max\{\lambda_2, -\lambda_n\}$ where $\lambda_1 \geq \cdots \geq \lambda_n$ is the spectrum of $P$ ; $\mu(P)$ is the second singular value of $P$.

(a) Let $y \in \mathbb{R}^n$ be such that $\|y\|_2 = 1$ and $\sum_j y_j = 0$. Let $w, z \in \mathbb{R}^n$ be such that

$$(p_{ij} \neq 0) \Longrightarrow \left(\frac{1}{2}(z_i + z_j) \leq y_i y_j \leq \frac{1}{2}(w_i + w_j)\right).$$

Show that $\lambda_2 \geq \sum_j z_j$ and $\lambda_n \leq \sum_j w_j$. **Hint**: Use Rayleigh ratio.

(b) Taking

$$y_j = \sqrt{\frac{2}{n}} \cos\frac{(2j-1)\pi}{2n}, \quad z_j = \frac{1}{n}\left(\cos\frac{\pi}{n} + \frac{\cos\frac{(2j-1)\pi}{n}}{\cos\frac{\pi}{n}}\right),$$

deduce that $\mu(P) \geq \cos\frac{\pi}{n}$ for every tridiagonal symmetric stochastic $n \times n$ matrix.

(c) Find a tridiagonal symmetric stochastic $n \times n$ matrix $P^*$ such that

$$\mu(P^*) = \cos\frac{\pi}{n}.$$

**Hint**: Exploit the equality case in the analysis, with the $y$ and $z$ given above.

(d) Prove that $P \mapsto \mu(P)$ is a convex function over symmetric stochastic $n \times n$ matrices. **Comment**: S.-G. HwangHwang, Suk-Geun and S.-S. PyoPyo, Sung-Soo prove conversely that, given real numbers $\lambda_1 = 1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ satisfying

$$\frac{1}{n} + \frac{\lambda_2}{n(n-1)} + \frac{\lambda_3}{(n-1)(n-2)} + \cdots + \frac{\lambda_n}{2 \cdot 1} \geq 0,$$

there exists a symmetric bistochastic matrix whose spectrum is $\{\lambda_1, \dots, \lambda_n\}$. This matrix is not necessarily tridiagonal, since $\mu$ may be smaller than the bound $\cos\frac{\pi}{n}$. It may even vanish, for the matrix $\frac{1}{n}\mathbf{1}\mathbf{1}^T$.

165. We deal with non-negative matrices in the sense of Perron–Frobenius theory. We say that a non-negative matrix $A \in \mathbf{M}_n(\mathbb{R})$ is *primitive* if it is irreducible – thus the spectral radius is a simple eigenvalue – and this eigenvalue is the only one of maximal modulus.

    Let us denote by $x$ and $y$ positive eigenvectors of $A$ and $A^T$, according to Perron–Frobenius TheoremPerronFrobenius. We normalize them by $y^T x = 1$.

    (a) Assume first that $\rho(A) = 1$. Remarking that $\rho(A - xy^T)$ is less than one, prove that $A^m - xy^T$ tends to zero as $m \to +\infty$. Deduce that $A^m > 0$ for $m$ large enough.

    (b) Deduce the same result without any restriction over $\rho(A)$.
    Wielandt'sWielandt Theorem asserts that $A^m > 0$ for $m \geq n^2 - 2n + 2$.

    (c) Conversely, prove that if $A$ is non-negative and irreducible, and if $A^m > 0$ for some integer $m$, then $A$ is primitive.

166. Let $A, B, C$ be complex matrices of respective sizes $n \times r$, $s \times m$ and $n \times m$. Prove that the equation
$$AXB = C$$
is solvable if, and only if,
$$AA^\dagger CB^\dagger B = C.$$
In this case, verify that every solution is of the form
$$A^\dagger CB^\dagger + Y - A^\dagger AYBB^\dagger,$$
where $Y$ is an arbitrary $r \times s$ matrix. We recall that $M^\dagger$ is the Moore–PenroseMoorePenrose inverse of $M$.

167. Let $k$ be a field and $A \in \mathbf{M}_n(k)$, $B \in \mathbf{M}_m(k)$ be given matrices.

    (a) Suppose that the spectra of $A$ and $B$ are disjoint : $\sigma(A) \cap \sigma(B) = \emptyset$. Prove that, given $C \in \mathbf{M}_{n\times m}(k)$, the equation $AX - XB = C$ is uniquely solvable in $\mathbf{M}_{n\times m}(k)$ (SylvesterSylvester–RosenblumRosenblum Theorem).

    (b) We go back to a general pair $(A, B)$ and we assume that the equation $AX - XB = C$ is solvable. Prove that the following $(n + m) \times (n + m)$ matrices are similar :
$$D := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \qquad T := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

    (c) We now prove the converse, following FlandersFlanders & WimmerWimmer. This constitutes Roth'sRoth Theorem.

        i. We define two homomorphisms $\phi_j$ of $\mathbf{M}_{n+m}(k)$:
$$\phi_0(K) := DK - KD, \qquad \phi_1(K) := TK - KD.$$
        Prove that the kernels of $\phi_1$ and $\phi_0$ are isomorphic, hence of equal dimensions. **Hint**: This is where we use the assumption.

ii. Let $E$ be the subspace of $\mathbf{M}_{m\times(n+m)}(k)$, made of matrices $(R,S)$ such that

$$BR = RA, \qquad BS = SB.$$

Verify that if

$$K := \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in \ker\phi_j \quad (j = 0 \text{ or } 1),$$

then $(R,S) \in E$. This allows us to define the projections $\mu_j(K) := (R,S)$, from $\ker\phi_j$ to $E$.

iii. Verify that $\ker\mu_0 = \ker\mu_1$, and therefore $R(\mu_0)$ and $R(\mu_1)$ have equal dimensions.

iv. Deduce that $\mu_1$ is onto.

v. Show that there exists a matrix in $\ker\phi_1$, of the form

$$\begin{pmatrix} P & X \\ 0 & -I_m \end{pmatrix}.$$

Conclude.

**Remark**. With the theory of elementary divisors, there is a finite algorithm which computes a matrix conjugating given similar matrices. However, the knowledge of such a conjugator between $D$ and $K$ does not give an explicit construction of a solution.

168. (WielandtWielandt.) As in Exercise 146, we use the exterior algebra $\Lambda E = \oplus_{k=0}^n \Lambda^k E$, where now $E = \mathbb{C}^n$. For a given matrix $M \in \mathbf{M}_n(\mathbb{C})$, we define $M_{(k)} \in \mathrm{End}(\Lambda^k E)$ by

$$M_{(k)} x^1 \wedge \cdots \wedge x^k = (Mx^1) \wedge x^2 \wedge \cdots \wedge x^k + \cdots + x^1 \wedge \cdots \wedge x^{k-1} \wedge (Mx^k).$$

(a) Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $M$. Verify that the spectrum of $M_{(k)}$ consists of the numbers $\lambda_I$, $|I| = k$, where

$$\lambda_I := \sum_{j\in I} \lambda_j.$$

(b) The canonical scalar product in $E$ extends to $\Lambda^k E$ ; its definition over decomposable vectors is

$$\langle x^1 \wedge \cdots \wedge x^k, y^1 \wedge \cdots \wedge y^k \rangle = \det(\langle x_i, y_j \rangle)_{1\le i,j\le k}.$$

If $M$ is Hermitian, show that $M_{(k)}$ is Hermitian.

(c) From now on, we assume that $A$ and $B$ are Hermitian, with respective eigenvalues (they must be real) $\mu_1 \ge \cdots \ge \mu_n$ and $\nu_1 \ge \cdots \ge \nu_n$. We from $C := A + B$, whose eigenvalues are $\lambda_1 \ge \cdots \ge \lambda_n$.

i. What is the largest eigenvalue of $B_{(k)}$ ?

ii. Show that there is a permutation $I \mapsto I'$ of subsets of $\{1, \dots, n\}$ of cardinal $k$, such that

$$\lambda_I \leq \mu_{I'} + \sum_{i=1}^{k} \nu_i.$$

iii. We now assume that $A$ has simple eigenvalues: $\mu_1 > \cdots > \mu_n$, and that $B$ is *small*, in the sense that

$$\max_i |\nu_i| < \min_j (\mu_j - \mu_{j+1}). \tag{22}$$

Show that the permutation mentionned above is the identity. Deduce the set of inequalities

$$\lambda_I \leq \mu_I + \sum_{i=1}^{k} \nu_i, \qquad \forall I\,;\, |I| = k. \tag{23}$$

(d) Conversely, we give ourselves two lists of real numbers $\mu_1 > \cdots > \mu_n$ and $\nu_1 \geq \cdots \geq \nu_n$, satisfying the smallness assumption (22). Let $\lambda_1 \geq \cdots \geq \lambda_n$ be given, satisfying the list of inequalities (23), together with

$$\sum_i \lambda_i = \sum_i \mu_i + \sum_i \nu_i. \tag{24}$$

We define $\kappa'_i := \lambda_I - \mu_i$, and $\kappa$ is a re-ordering of $\kappa'$, so that $\kappa_1 \geq \cdots \geq \kappa_n$.

i. Show that $\kappa \prec \nu$. Deduce that there exists a orthostochastic matrix $M$, such that $\kappa = M\nu$. We recall that an orthostochastic matrix is of the form $m_{ij} = |u_{ij}|^2$, where $U$ is unitary.

ii. Show that $\lambda$ is the spectrum of $C = A + B$, where $A := \text{diag}(\mu)$ and $B := U^* \text{diag}(\nu) U$, where $U$ is as above. Therefore the inequalities (23), together with (24), solve the A. Horn'sHorn!Alfred problem when $B$ is "small".

169. Let $k$ be a field, $A = k[Y, Z]$ be the ring of polynomials in two indeterminates and $K = k(Y, Z)$ be the corresponding field of rational fractions.

(a) We consider the matrix

$$M(X, Y, Z) := \begin{pmatrix} X + Y & Z \\ Z & X - Y \end{pmatrix} \in \mathbf{M}_2(A[X]).$$

From Theorem 6.2.1, we know that there exist $P, Q \in \mathbf{GL}_2(K[X])$ such that

$$PM = \begin{pmatrix} 1 & 0 \\ 0 & X^2 - Y^2 - Z^2 \end{pmatrix} Q.$$

Show that one cannot find such a pair with $P, Q \in \mathbf{GL}_2(A[X])$, namely with polynomial entries in $(X, Y, Z)$. **Hint**: The top row entries would vanish at the origin.

(b) Let us now consider the matrix

$$N(X,Y) := \begin{pmatrix} X+Y & 1 \\ 1 & X-Y \end{pmatrix} \in \mathbf{M}_2(A'[X]), \qquad A' := k[Y].$$

Show that there exist $S, T \in \mathbf{GL}_2(A'[X])$ such that

$$SN = \begin{pmatrix} 1 & 0 \\ 0 & X^2 - Y^2 - 1 \end{pmatrix} T.$$

**Hint**: multiply $N$ left and right by appropriate elementary matrices.

(c) One has $S, T \in \mathbf{GL}_2(k[X,Y])$. Explain why this does not contradict the previous result.

170. (HeisenbergHeisenberg Inequality.) Let $A$ and $B$ be two Hermitian matrices of size $n$. We employ the canonical Hermitian product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^n$ and the associated norm.

(a) For every $x \in \mathbb{C}^n$, prove

$$|\langle [A,B]x, x\rangle|^2 \leq 4\|Ax\|^2 \, \|Bx\|^2.$$

Heisenberg, *Werner* (Germany)

**Werner Heisenberg.**

(b) If $\|x\| = 1$ and $C$ is Hermitian (quantum physicists say that $C$ is an *observable*), we define the *expectation* and the *variance* of $C$ by

$$E(C;x) := \langle Cx, x\rangle, \qquad V(C;x) := \|Cx - E(x)x\|^2.$$

Prove the Heisenberg Inequality: for $x$, $A$ and $B$ as above,

$$|E(i[A,B];x)|^2 \leq 4V(A;x)\,V(B;x).$$

The Heisenberg inequality is therefore a manifestation of the non-commutativity between operators.

171. A *lattice* in $\mathbb{R}^m$ is a discrete subgroup of maximal rank. Equivalently, it is the set of vectors of which the coordinates over a suitable basis of $\mathbb{R}^m$ are integers. It is thus natural to study the bases of $\mathbb{R}^m$.

We consider the case where $m = 2n$ and $\mathbb{R}^m$ is nothing but $\mathbb{C}^n$. Given a family $B := \{v^1, \dots, v^{2n}\}$ of vectors in $\mathbb{C}^n$, we form the $n \times (2n)$ matrix $\Pi := (v^1, \dots, v^{2n})$. Prove that $B$ is an $\mathbb{R}$-basis if, and only if, the $(2n) \times (2n)$ matrix

$$\begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}$$

is non-singular, where $\overline{\Pi}$ denotes the matrix with complex conjugate entries.

172. Prove Schur'sSchur Pfaffian identity

$$\mathrm{Pf}\left(\left(\frac{a_j - a_i}{a_i + a_j}\right)\right)_{1\le i,j\le 2n} = \prod_{i<j} \frac{a_j - a_i}{a_j + a_i}.$$

See Exercise 119 for a hint.

173.

Hua, *Loo-Keng* (China)

**Loo-Keng Hua.**

Check the easy formula valid whenever the inverses concern regular $n \times n$ matrices:

$$(I_n + A^{-1})^{-1} + (I_n + A)^{-1} = I_n.$$

Deduce Hua IdentityHua, Loo-Keng

$$(B + BA^{-1}B)^{-1} + (A + B)^{-1} = B^{-1}.$$

**Hint**: transport the algebra structure of $\mathbf{M}_n(k)$ by the linear map $M \mapsto BM$. This procedure is called *isotopy* ; remark that the multiplicative identity in the new structure is $B$. Then apply the easy formula.

174. In $\mathbf{M}_n(k)$, we define the JordanJordan!Pascual product by

$$A \bullet B := \frac{1}{2}(AB + BA).$$

Of course, we assume that the characteristic of the field $k$ is not equal to 2. We warn the reader that the bullet is not an associative product. We notice that the square $A \bullet A$ coincides with $A^2$.

(a) Prove Jordan Identity
$$A^2 \bullet (A \bullet B) = A \bullet (A^2 \bullet B).$$

(b) Deduce that there is no ambiguity in the definition of $A^{\bullet m}$ when $m \in \mathbb{N}$. In other words, $A$ generates an associative as well as commutative bullet-algebra.

(c) For every matrices $A, B$, we define two linear maps $U_A$ and $V_{A,B}$ by

$$U_A(B) := 2A \bullet (A \bullet B) - A^2 \bullet B$$

and
$$V_{A,B} := 4U_{A\bullet B} - 2(U_A \circ U_B + U_B \circ U_A).$$

Prove Thedy'sThedy Identity:

$$U_{V_{A,B}(C)} = V_{A,B} \circ U_C \circ V_{A,B}.$$

**Note**: One should not confuse Pascual Jordan, a German physicist, with the French mathematician Camille JordanJordan!Camille, whose name has been given to a canonical form of matrices over an algebraically closed field. Amazingly, simple Euclidean (P.) Jordan algebras obey a spectral theorem, where every element is "diagonalizable with real eigenvalues" ; hence their (C.) Jordan's form is trivial. Somehow, we can say that there is an exclusion principle about Jordan concepts.

Besides Jordan and Thedy identities, there are two more complicated identities, due to GlennieGlennie, valid in $\mathbf{M}_n(k)$. It has been an important discovery, in the theory of Jordan algebras, that Glennie's and Thedy's identities do not follow from Jordan's. As a matter of fact, there exists a Jordan algebra, namely that of $3 \times 3$ Hermitian matrices over Cayley'sCayley *octonions* (a non-associative division algebra), in which all these three identities are violated. This exceptional object is called Albert algebraAlbert.

175. Let $P, Q \in \mathbf{M}_n(\mathbb{R})$ be rank-one projectors. Prove that the matrices $xP + yQ$ are diagonalizable with real eigenvalues for every $x, y \in \mathbb{R}$ if, and only if, either $0 < \mathrm{Tr}(PQ) < 1$ or $PQ = QP = 0_n$ or $Q = P$.

176. In $\mathbf{M}_n(\mathbb{C})$, we define endomorphisms $L_A$ and $P_A$ (the linear and quadratic representations of the underlying JordanJordan!Pascual algebra) when $A \in \mathbf{M}_n(\mathbb{C})$ by

$$L_A(M) := \frac{1}{2}(AM + MA), \qquad P_A(M) := AMA.$$

Show that

$$P_{\exp(tA)} = \exp(2tL_A).$$

177. Let $A \in \mathbf{M}_n(k)$ and $p \in k[X]$ be given. Show that the minimal polynomial divides $p$ (that is $p(A) = 0_n$) if, and only if, there exists a matrix $C \in \mathbf{M}_n(k[X])$ such that $p(X)I_n = (XI_n - A)C(X)$.

178. (DelvauxDelvaux, Van Barelvanbar@Van Barel) Let $m, n \geq 1$ and $A \in \mathbf{GL}_n(k)$, $B \in \mathbf{GL}_m(k)$, $G, H \in \mathbf{M}_{m\times n}(k)$ be given. Let us define $R := A - G^TBH$ and $S := B^{-1} - HA^{-1}G^T$.

    (a) Show that the following matrices are equivalent within $\mathbf{M}_{m+n}(k)$:

    $$\begin{pmatrix} B^{-1} & H \\ G^T & A \end{pmatrix}, \qquad \begin{pmatrix} B^{-1} & 0 \\ 0 & R \end{pmatrix}, \qquad \begin{pmatrix} S & 0 \\ 0 & A \end{pmatrix}.$$

    (b) Deduce the equality

    $$\mathrm{rk}R - \mathrm{rk}S = n - m.$$

179. (Higham & al.)Higham Let $M \in \mathbf{GL}_n(k)$ be given. We define the classical group $G \subset \mathbf{GL}_n(k)$ by the equation
$$A^T M A = M.$$
Let $p \in k[X]$ be given, with $p \not\equiv 0$. Let us form the rational function $f(X) := p(X^{-1})p(X)^{-1}$.

(a) Prove (again) that for every $A \in G$, the following identities hold true:
$$Mp(A) = p(A^{-1})^T M, \qquad p(A^T)M = Mp(A^{-1}).$$

(b) Deduce that
$$(A \in G) \Longrightarrow (f(A) \in G),$$
whenever $p(A)$ is non-singular.

(c) Exemple: every MœbiusMœbius (Möbius) function preserves a complex group defined by an equation
$$A^* M A = M.$$

180. A matrix *pencil* is a polynomial
$$L(X) := X^m A_0 + X^{m-1} A_1 + \cdots + X A_{m-1} + A^m,$$
whose coefficients are $n \times n$ matrices. It is a monic pencil if $A_0 = I_n$. If in addition the scalar field is $\mathbb{C}$ and the matrices $A_k$ are Hermitian, we say that the pencil is *Hermitian*. Finally, a Hermitian pencil is *hyperbolic* if the roots of the polynomial
$$P_u(X) := \langle L(X)u, u\rangle$$
are real and simple (notice that $P_u \in \mathbb{R}[X]$) for every non-zero vector $u \in \mathbb{C}^n$.

(a) Let $A \in \mathbf{H}_n$ and $B \in \mathbf{M}_n(\mathbb{C})$ be given. We assume that there exist $x, y \in \mathbb{C}^n$ such that
$$\langle Ax, x\rangle < 0 < \langle Ay, y\rangle \qquad \text{and} \qquad \langle Bx, x\rangle = \langle By, y\rangle = 0.$$
Show that there exists a non-zero vector $z \in \mathbb{C}^n$ such that $\langle Az, z\rangle = \langle Bz, z\rangle = 0$. **Hint**: Apply the Toeplitz–HausdorffToeplizHausdorff Theorem from Exercise 21 to $A + iB$.

(b) Let $L(X)$ be a hyperbolic Hermitian pencil. We denote $\lambda_1(u) < \cdots < \lambda_n(u)$ the roots of $P_u$ when $u \neq 0$. Recall that the $\lambda_j$'s are smooth functions. We denote also
$$\Delta_j := \lambda_j(S^{n-1})$$
the image of the unit sphere under $\lambda_j$. This is obviously a compact interval $[\lambda_j^-, \lambda_j^+]$ of $\mathbb{R}$.

i. Check that $\lambda_j^- \leq \lambda_{j+1}^-$ and $\lambda_j^+ \leq \lambda_{j+1}^+$.

ii. Show that $P'_u(\lambda_j(u))$ is not equal to zero, that its sign $\epsilon_j$ does not depend on $u$, and that $\epsilon_j\epsilon_{j+1}=-1$.

iii. Assume that $\lambda_{j+1}^-\le\lambda_j^+$. Let us choose $t\in[\lambda_{j+1}^-,\lambda_j^+]$. Prove that there exists a unit vector $z$ such that $\langle L(t)z,z\rangle=\langle L'(t)z,z\rangle=0$. Reach a contradiction.

iv. Deduce that the root intervals $\Delta_j$ are pairwise disjoint.

**Nota**: A Hermitian pencil is *weakly* hyperbolic if the roots of every $P_u$ are real, not necessarily disjoint. For such pencils, two consecutive root intervals intersect at most at one point, that is

$$\lambda_j^+\le\lambda_{j+1}^-.$$

181. We prove here the converse of Exercise 127, when the scalar field $k$ is infinite. We recall that in an infinite field, a polynomial function vanishes identically on $k^n$ if, and only if, the associated polynomial is zero. Actually, if a product of polynomial functions vanishes identically, one of the polynomials at least is zero.

We thus assume that, for every matrix $M'$ equivalent to $M$ ($M'=PMQ$ with $P,Q$ non-singular), there holds $\det(A'D')=\det(B'C')$, where

$$M'=\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}.$$

Using the rank decomposition, we may assume that $M=\mathrm{diag}\{I_n,J\}$, where $J$ is a quasi-diagonal $n\times n$ matrix.

(a) Choosing $P$ an $Q$ appropriately, show that for every $X,Y\in\mathbf{M}_n(k)$, there holds $\det(XY+J)=(\det X)(\det Y)$. Deduce that $\det(I_n+JR)=1$ for every non-singular $R$.

(b) Deduce that the polynomial $X\mapsto\det(I_n+JX)-1$ (a polynomial in $n^2$ indeterminates) is zero, and thus that the polynomial $X\mapsto\mathrm{Tr}(JX)$ is zero.

(c) Show at last that the rank of $M$ is at most $n$.

**Nota**: This, together with Hilbert'sHilbert *Nullstellensatz*, implies that for every minor $\mu$ of $M$ of size $r\in(n,2n]$, viewed as a polynomial of the entries, there exist an integer $m$ such that $\mu^m$ belongs to the ideal spanned by the polynomials $\Delta_{P,Q}:=(\det A')(\det D')-(\det B')(\det C')$ with $A',\ldots$ the blocks of $M':=PMQ$. Clearly, $m\ge 2$, but the least integer $m(r)$ is not known, to our knowledge.

182. Recall that if $V$ is a vector space of dimension $n$, a *complete flag* in $V$ is a set of subspaces $F_0=\{0\}\subset F_1\cdots\subset F_n=V$, with $\dim F_k=k$ for each index $k$. Let $P$ be a subset of $\{1,\ldots,n\}$, say that $P=\{p_1<\cdots<p_r\}$. Given a complete flag $F$ and an index subset $P$ of cardinality $r$, we define

$$\Omega_P(F)=\{L\in X_r\mid \dim F_k\cap L\ge k\},$$

with $X_r$ the set of subspaces of $V$ of dimension $r$. Let $A$ be an $n\times n$ Hermitian matrix, and take $V=\mathbb{C}^n$.

(a) Let $L \in X_r$ be given. Show that the quantity

$$\sum_{k=1}^{r} \langle Ax_k, x_k \rangle$$

does not depend on the choice of a unitary basis $\{x_1, \ldots, x_m\}$ of $L$. We call it the *trace* of $A$ over $L$, and denote $\mathrm{Tr}_L A$.

(b) Let $a_1 \leq \cdots \leq a_n$ be the eigenvalues of $A$, counting with multiplicities. Let $\mathcal{B} = (u_1, \ldots, u_n)$ the corresponding unitary basis and $F_*(A)$ be the complete flag spanned by $\mathcal{B}$: $F_k = u_k \oplus F_{k-1}$. Let $P := (p_1, \ldots, p_r)$ be as above. Show that

$$\sum_{p \in P} a_p = \sup_{L \in \Omega_P(F_*(A))} \mathrm{Tr}_L A.$$

**Hint**: To show that every such $\mathrm{Tr}_L A$ is less than or equal the left-hand side, find an adapted basis of $L$. To show the inequality left$\leq$right, choose an appropriate $L$.

183. (From G. PisierPisier).

Let $Z \in \mathbf{M}_n(\mathbb{C})$ be given, whose entries are unit numbers: $|z_{jk}| = 1$. We define the linear map over $\mathbf{M}_n(\mathbb{C})$

$$F : M \mapsto F(M) := Z \circ M, \qquad (F(M))_{jk} := z_{jk} m_{jk}.$$

(a) Show that

$$\|F(M)\|_1 = \|M\|_1, \quad \|F(M)\|_\infty = \|M\|_\infty, \quad \|F(M)\|_2 \leq \sqrt{n} \|M\|_2.$$

(b) Deduce that

$$\|F(M)\|_p \leq n^\alpha \|M\|_p, \quad p \in [1, \infty], \quad \alpha := \min\{1/p, 1/p'\}.$$

**Hint**: Use Riesz–ThorinRieszThorin Interpolation Theorem.

(c) Given $N \in \mathbf{M}_n(\mathbb{C})$, let $Abs(N)$ be its *absolute value*: $a_{ij} := |m_{ij}|$. Prove that

$$\|Abs(N)\|_p \leq n^\alpha \|N\|_p, \quad p \in [1, \infty].$$

**Hint**: Find a $Z$ adapted to this $N$.

(d) Define $\xi := \exp(2i\pi/n)$ and $\Omega \in \mathbf{M}_n(\mathbb{C})$ by $\omega_{jk} := \xi^{jk}$. Show that $\Omega^* \Omega = nI_n$. Deduce that

$$\|Abs(\Omega)\|_2 = \sqrt{n} \|\Omega\|_2,$$

and therefore the constant $\sqrt{n}$ is optimal in the inequality

$$\|Abs(M)\|_2 \leq C_2 \|M\|_2.$$

**Nota**: The analogous problem in $\mathbf{M}_n(\mathbb{R})$ is less trivial. One can show that $\sqrt{n}$ is optimal if and only if there exists a unitary matrix $U$ with entries of constant modulus, namely $n^{-1/2}$. In the real case, this means that the entries are $\pm n^{-1/2}$. This amounts to the existence of a HadamardHadamard matrix. Such matrices exist only for a few values of $n$. For instance, they do not for $n = 3, 5, 6, 7$. The determination of sizes $n$ for which a Hadamard matrix exists is still an open question.

(e) More generally, we show now that $n^\alpha$ is the optimal constant in the inequality

$$\|Abs(M)\|_p \le C_p \|M\|_p, \qquad \forall M \in \mathbf{M}_n(\mathbb{C}).$$

i. Check that if $p \le 2$, then

$$\|x\|_2 \le \|x\|_p \le n^{\frac{1}{p}-\frac{1}{2}} \|x\|_2,$$

while if $p \ge 2$, then

$$\|x\|_2 \ge \|x\|_p \ge n^{\frac{1}{p}-\frac{1}{2}} \|x\|_2.$$

ii. Deduce that $\|\Omega\|_p \le n^{1-\alpha}$, and therefore

$$\|Abs(\Omega)\|_p \ge n^\alpha \|\Omega\|_p.$$

Conclude.

iii. Deduce in particular that $\|\Omega\|_p = n^{1-\alpha}$.

184. Let $A, B \in \mathbf{H}_n$ be given, such that $A + B$ is positive definite.

(a) Show that there exists a Hermitian matrix, denoted by $A \square B$, such that

$$\inf_{y \in \mathbb{C}^n} \{\langle A(x-y), x-y\rangle + \langle By, y\rangle\}$$

equals $\langle A \square Bx, x\rangle$ for every $x \in \mathbb{C}^n$.

We call $A \square B$ the *inf-convolution* of $A$ and $B$.

(b) Check that $\square$ is symmetric and associative.

(c) Compute $A \square B$ in closed form. Check that

$$A \square B = B(A+B)^{-1}A$$

(surprizingly enough, this formula is symmetric in $A$ and $B$, and it defines a Hermitian matrix).

(d) If $A$ and $B$ are non-degenerate, show that

$$A \square B = (A^{-1} + B^{-1})^{-1}.$$

185. (T. YamamotoYamamoto.)

Given $A \in \mathbf{M}_n(\mathbb{C})$, we denote by $s_1(A) \geq \cdots \geq s_n(A)$ its singular values, and by $\lambda_1(A), \ldots$ its eigenvalues ordered by non-increasing modulus:

$$|\lambda_1(A)| \geq \cdots \geq |\lambda_n(A)|.$$

In this list, it is useful that an eigenvalue with multiplicity $\ell$ corresponds to $\ell$ consecutive eigenvalues of the list. In this exercise, we prove

$$\lim_{m\to+\infty} s_k(A^m)^{1/m} = |\lambda_k(A)|, \qquad \forall k = 1, \ldots, m.$$

Remark that for $k = 1$, this is nothing but the fundamental Lemma of Banach algebras, namely

$$\lim_{m\to+\infty} \|A^m\|^{1/m} = \rho(A),$$

where the operator norm is $\|\cdot\|_2$.

(a) Prove that there exist subspaces $G$ (of dimension $n-k+1$) and $H$ (of dimension $k$) that are invariant under $A$, such that the spectrum of the restriction of $A$ to $G$ (respectively to $H$) is $\lambda_k(A), \ldots, \lambda_n(A)$ (resp. $\lambda_1(A), \ldots, \lambda_k(A)$). **Hint**: Use Jordan reduction.

(b) Verify the formula

$$s_k(A) = \sup_{\dim F=k} \inf_{x\in^* F} \frac{\|Ax\|_2}{\|x\|_2}, \tag{25}$$

where $x \in^* F$ means that $x$ runs over the non-zero vectors of $F$. **Hint**: Both sides are unitarily invariant. Use the decomposition in singular values.

(c) Deduce the bounds

$$\left\|(A|_H)^{-1}\right\|^{-1} \leq s_k(A) \leq \|A|_G\|.$$

(d) Apply these bounds to $A^m$ and conclude.

186. (R. C. ThompsonThompson.) Let $R$ be a principal ideal domain. Let $M \in \mathbf{M}_{n\times m}(R)$ be given, with $r := \min\{n, m\}$. Let $M' \in \mathbf{M}_{p\times q}(R)$ be a submatrix of $M$, with $p + q > r$. Show that $D_{p+q-r}(M')$ divides $D_r(M)$ (recall that $D_k(M)$ is the g.c.d. of all minors of size $k$ of $M$). **Hint**: If $p + q = r + 1$, have a look to Exercise 10 of Chapter 2.

Let $k$ be a field and $A \in \mathbf{M}_n(k)$ be given. Let $M' \in \mathbf{M}_{p\times q}(k[X])$ be a submatrix of $XI_n - A$, with $p + q > n$. We denote the invariant factors of $M'$ by $\alpha_1|\alpha_2|\cdots$. Deduce that the product $\alpha_1 \cdots \alpha_{p+q-n}$ divides the characteristic polynomial of $A$.

187. The first part is an exercise about polynomials. The second part is an application to matrices, yielding DunfordDunford decomposition. The field $k$ has characteristic zero.

(a) Let $P \in k[X]$ be monic of degree $n$. Suppose first that $P$ splits, say

$$P(X) = \prod_{j=1}^{s} (X - a_j)^{m_j}, \qquad .$$

i. Show that there exists a unique polynomial $Q$ of degree $n$, such that for every $j = 1, \dots, s$, one has

$$Q(a_j) = a_j \quad \text{and} \quad Q^{(\ell)}(a_j) = 0, \quad \forall 1 \le \ell \le m_j - 1.$$

ii. Define the polynomial

$$\pi(X) := \prod_{j=1}^{s} (X - a_j).$$

Show that $a_j$ is a root of $\pi \circ Q$ of order $m_j$ at least. Deduce that $P$ divides $\pi \circ Q$.

iii. If $P$ does not split, let $K$ be an extension of $k$ in which $P$ splits. Let $Q$ and $\pi$ be defined as above. Show that $\pi \in k[X]$ and $Q \in k[X]$.

(b) Let $M \in \mathbf{M}_n(k)$ be given. We apply the construction above to $P = P_M$, the characteristic polynomial. We let $R[X] := X - Q(X)$, and we define $D := Q(M)$ and $N := R(M)$.

i. What is the spectrum of $N$ ? Deduce that $N$ is nilpotent.

ii. Show that $\pi(D) = 0_n$. Deduce that $D$ is diagonalisable in a suitable extension of $k$.

iii. Deduce the Dunford decomposition: $M$ writes as $D + N$ for some diagonalisable $D$ and nilpotent $N$, with $[D, N] = 0_n$. Both $D$ and $N$ have entries in $k$, though $D$ could be diagonalisable only in a suitable extension, that one containing all the eigenvalues of $M$.

188.


*Leonhard* (Switzerland)

**Leonhard Euler.**

Let $\phi$ denote the EulerEuler indicator, $\phi(m)$ is the number of integers less than $m$ that are prime to $m$. We recall the formula

$$\sum_{d|n} \phi(d) = n.$$

In the sequel, we define the $n \times n$ matrices $G$ (for Euler, *gcd*), $\Phi$ and $D$ (for *divisibility*) by

$$g_{ij} := \gcd(i, j), \qquad \Phi := \mathrm{diag}\{\phi(1), \dots, \phi(n)\}, \qquad d_{ij} := \begin{cases} 1 & \text{if } i|j, \\ 0 & \text{else.} \end{cases}$$

(a) Prove that $D^T \Phi D = G$.

(b) Deduce the SmithSmith determinant formula:

$$\det((\gcd(i,j)))_{1\le i,j\le n} = \phi(1)\phi(2)\cdots\phi(n).$$

(c) Compute the invariant factors of $G$ as a matrix of $\mathbf{M}_n(\mathbb{Z})$, for small values of $n$. Say up to $n = 10$.

189. (from D. FerrandFerrand.) Let $\sigma$ be a permutation of $\{1,\dots,n\}$ and $P^\sigma$ be the associated permutation matrix.

(a) We denote by $c_m$ the number of cycles of length $m$ in $\sigma$. Show that the characteristic polynomial of $P^\sigma$ is

$$\prod_{m\ge 1}(X^m - 1)^{c_m}.$$

(b) Let $\sigma$ and $\tau$ be two permutations. We assume that $P^\sigma$ and $P^\tau$ are similar in $M_n(k)$, the field $k$ having characteristic zero. Show that for every $m$, there holds $c_k(\sigma) = c_k(\tau)$. Deduce that $\sigma$ and $\tau$ are conjugated as permutations.

190. Let $M \in \mathbf{M}_{n\times m}(A)$ be given. Verify that $\det M^T M$ equals the sum of the squares of the minors of $M$ of size $m$.

191. One begins with the following observation: if $A, B \in \mathbf{GL}_n(k)$ are such that $A^2 = B^2$, then $M := A^{-1}B$ is conjugated to its inverse $M^{-1}$. Verify!

We prove now the converse, and even slightly better; namely, if $M$ and $M^{-1}$ are similar, then there exist $A, B \in \mathbf{GL}_n(k)$ such that $A^2 = B^2 = I_n$ and $M := A^{-1}B$.

(a) Show that both the assumption and the conclusion are invariant under conjugation.

(b) Assume $n = 2m$. We define $S, T \in \mathbf{GL}_n(k)$ by

$$S = \begin{pmatrix} I_m & 0_m \\ J(4;m) & -I_m \end{pmatrix}, \qquad T = \begin{pmatrix} I_m & I_m \\ 0_m & -I_m \end{pmatrix},$$

where $J(a;m)$ stands for the Jordan block with eigenvalue $a$. Check that $S$ and $T$ are involutions, and show that $S^{-1}T$ has only one eigenvalue and one eigendirection. Conclude that the result holds true for $J(1;2m)$ (notice that $J(1;n)$ is always similar to its inverse).

(c) We keep the notations of the previous question. Show that the intersection of $\ker(S^T + I_n)$ and $\ker(T^T + I_n)$ is a line. Deduce that there is a hyperplane $H$ that is stable under both $S$ and $T$. Show that the restrictions of $S$ and $T$ to $H$ are involutions, and that of $S^{-1}T$ is similar to $J(1;2m-1)$. **Hint**: the latter is the restriction of $J(1;2m)$ to a stable hyperplane; it cannot be something else than $J(1;2m-1)$.

(d) We have thus proven that for every $n$, $J(1;n)$ satisfies the claim. Use this to prove that $J(-1;n)$ satisfies it too.

(e) Let $a \in k$ be given, with $a \neq 0, \pm 1$. Check that

$$\begin{pmatrix} 0_m & J(a;m) \\ J(a;m)^{-1} & 0_m \end{pmatrix}$$

is an involution. Deduce that $\mathrm{diag}\{J(a;m), J(a;m)^{-1}\}$ satisfies the claim. Conclude that $\mathrm{diag}\{J(a;m), J(a^{-1};m)\}$ satisfies it too.

(f) If the characteristic polynomial of $M$ splits on $k$ (for instance if $k$ is algebraically closed), prove the claim. **Hint**: Apply JordanJordan!Camille decomposition.

(g) In order to solve the general case, we use the second canonical form in FrobeniusFrobenius reduction. Let $P$ be a power of an irreducible monic polynomial over $k$.

i. Show that the inverse of the companion matrix $B_P$ is similar to the companion matrix of $\hat{P}$, the polynomial defined by

$$\hat{P}(X) = \frac{1}{P(0)} X^{\deg P} P(1/X).$$

ii. Show that $\mathrm{diag}\{B_P, B_P^{-1}\}$ is the product of two involutions. **Hint**: Mimic the case of $J(a;m)$.

iii. Conclude. **Hint**: Verify that if the list of elementary divisors of an invertible matrix $M$ is $p_1, \dots, p_r$, then the elementary divisors of $M^{-1}$ are $\hat{p}_1, \dots, \hat{p}_r$. Mind that one must treat the cases $(X \pm 1)^s$ apart.

192. Let $M \in \mathbf{M}_n(\mathbb{R})$ have the following properties:

- $M$ is irreducible,
- For every pair $i \neq j$, one has $m_{ij} \geq 0$,
- $e^T M = 0$, where $e^T = (1, \dots, 1)$.

(a) Show that $\lambda = 0$ is a simple eigenvalue of $M$, associated with a positive eigenvector $V > 0$, and that the other eigenvalues have a negative real part.

(b) Let us denote $D := \mathrm{diag}\{1/v_1, \dots, 1/v_n\}$. Prove that the symmetric matrix $DM + M^T D$ is negative semi-definite, its kernel being spanned by $V$.

193. (Tan LeiLei, Tan). The following is an easy but not so well-known consequence of Perron–FrobeniusPerronFrobenius Theorem.

Let $A$ be a non-negative $n \times n$ matrix. Then $\rho(A) < 1$ if, and only if, there exists a positive vector $x > 0$, such that $Ax < x$.

194. In Le Verrier'slever@Le Verrier method, and even in Fadeev'sFadeev variant (see Exercise 67), the complexity of the computation of the characteristic polynomial of an $n \times n$ matrix $M$ in characteristic 0 is $n^4$ if we use the naive way to multiply matrices. It is $n^{\alpha+1}$ if we now multipling matrices in $O(n^\alpha)$ operations. Here is an improvement, found by F. PreparataPreparata and D. SarwateSarwate in 1978.

We still compute the NewtonNewton sums of the eigenvalues, which are equal to the traces of the powers $M^k$, $k = 0, ..., m-1$. However we do not compute all the matrix powers. Let $r$ be the smallest integer larger than or equal to $\sqrt{m}$ ($r = \sqrt{m}$ if $m$ is a square).


**Urbain Le Verrier.**
Le Verrier, *Urbain* (France)

(a) What is the complexity of the calculation of the powers $M, \ldots, M^r$ ?

(b) What is the complexity of the calculation of the powers $M^{2r}, \ldots, M^{r(r-1)}$ ?

(c) How many operations do we need to compute the Newton sums $S_0, \ldots, S_{m-1}$ once we now the powers computed above ? **Hint**: To compute $\mathrm{Tr}(BC)$, one does not need to compute $BC$.

(d) Prove that the complexity of the computation of the characteristic polynomial is at most $O(n^{\alpha+1/2})$.

195.

In the theory of differential equations, one says that a matrix $A \in \mathbf{M}_n(\mathbb{R})$ is *stable* if its eigenvalues lie in the open left half-plane $\{z\,;\,\Re z < 0\}$. It is *strongly stable* if $A - D$ is stable for every diagonal non-negative matrix $D$. The lack of strong stability yields what is called a *Turing instability*Turing. We propose here necessary conditions for $A$ to be strongly stable.
Let us define $M = -A$, so that the spectrum of $M + D$ has a positive real part for all $D$ as above.


**Alan Turing (with von Neumann).**
Turing, *Alan* (Portugal)vonneu@von Neumann, *John*!(Portugal)

(a) Let $i_1 < \cdots < i_r$ be indices between 1 and $n$. Prove that the principal submatrix

$$M\begin{bmatrix} i_1 & \cdots & i_r \\ i_1 & \cdots & i_r \end{bmatrix}$$

has its spectrum in $\{\Re z \geq 0\}$. **Hint**: take $d_j = 0$ if $j$ is an index $i_s$, $d_j = y$ otherwise. Take the limit as $y \to +\infty$ and use Schur'sSchur complement formula.

(b) Deduce that every principal minor of $M$ must be non-negative.

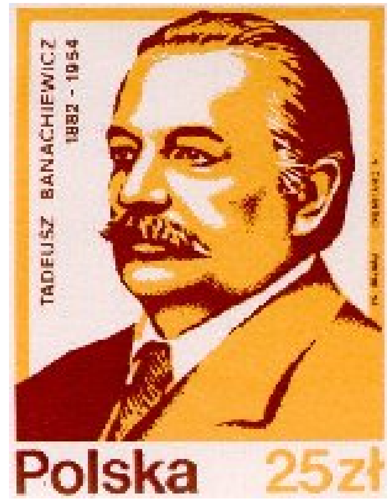
(c) Show that the polynomial $P(X) := \det(XI_n + M)$ has positive coefficients. **Hint**: This real polynomial has roots of positive real parts. Deduce that for every size $1 \leq r \leq n$, there exists a principal minor of order $r$ in $M$ which is positive.

(d) What does all that mean for the principal minors of $A$ ?

R. A. SatnoianuSatnoianu and P. van den Driesschevanden@van den Driessche (2005) provided an example which shows that these necessary conditions are not sufficient when $n \geq 4$. They turn out to be so for $n = 1, 2, 3$.

Schur's complement formula is associated with the *Banachiewicz's inversion formula* (Corollary 8.1.1).Banachiewicz

Left: **Tadeusz Banachiewicz.**




Banachiewicz, *Tadeusz* (Poland)

196. Let $A \in \mathbf{M}_n(\mathbb{C})$ be given, with characteristic polynomial $X^n - a_0 X^{n-1} - \cdots - a_{n-1}$. Let $\phi$ be the solution of the differential equation

$$y^{(n)} = a_0 y^{(n-1)} + \cdots + a_{n-1} y,$$

with the initial conditions $\phi(0) = \phi'(0) = \cdots = \phi^{(n-2)}(0) = 0$ and $\phi^{(n-1)}(0) = 1$ (this is the *fundamental solution* of the ODE). Finally, define matrices $A_0, \ldots, A_{n-1}$ by $A_j = h_j(A)$

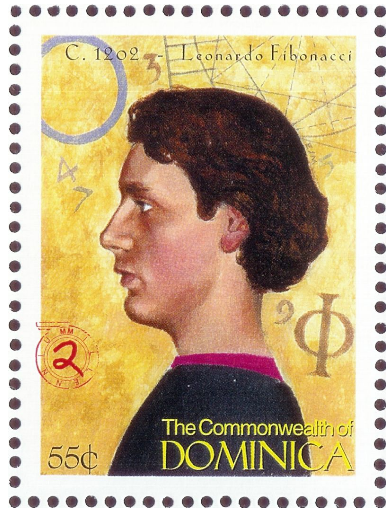
with

$$\begin{aligned} h_0(z) &= 1, \quad h_1(z) = z - a_0, \quad \dots \\ h_j(z) &= z h_{j-1}(z) - a_{j-1}, \quad \dots \quad h_r(z) = P_A(z). \end{aligned}$$

Prove that

$$\exp(tA) = \phi^{(n)}(t)A_0 + \phi^{(n-1)}(t)A_1 + \cdots + \phi(t)A_{n-1}.$$

**Nota**. The sequence $A_0, \dots, A_{n-1}$ is the *Fibonacci–Horner basis*FibonacciHorner of the algebra spanned by $A$. It is actually a basis only if $P_A$ is the minimal polynomial of $A$. The interest of the formula is that it is valid even if the minimal polynomial has a lower degree.




**Leonardo Fibonacci.**
Fibonacci, *Leonardo* (Dominica)

197. We apply the JacobiJacobi method to a real $3 \times 3$ matrix $A$. Our strategy is that called "optimal choice".

(a) Let $(p_1, q_1)$, $(p_2, q_2)$, $\dots$, $(p_k, q_k)$, $\dots$ be the sequence of index pairs that are chosen at consecutive steps (recall that one vanishes the off-diagonal entry of largest modulus). Prove that this sequence is cyclic of order three: It is either one of the sequences $\dots, (1,2), (2,3), (3,1), (1,2), \dots$, or $\dots, (1,3), (3,2), (2,1), (1,3), \dots$

(b) Assume now that $A$ has simple eigenvalues. At each step, one of the three off-diagonal entries is null, while the two other ones are small, since the method converges. Say that they are $0, x_k, y_k$ with $0 < |x_k| \le |y_k|$ (if $x_k$ vanishes then we are gone because one diagonal entry is an eigenvalue). Show that $y_{k+1} \sim x_k$ and $x_{k+1} \sim 2x_k y_k/\delta$, where $\delta$ is a gap between two eigenvalues. Deduce that the method is of order $\omega = (1+\sqrt{5})/2$, the golden ratio (associated notably with the Fibonacci sequence), meaning that the error $\epsilon_k$ at step $k$ satisfies

$$\epsilon_{k+1} = O(\epsilon_k \epsilon_{k-1}).$$

This behaviour ressembles that of the secant method for the resolution of algebraic equations.


Illustration for the golden ratio (left) and the Fibonacci sequence (right).

golden ratio (Macao)

198. (PuszPusz and WoronowiczWoronowicz). Let $A, B \in \mathbf{H}_n^+$ two given positive semi-definite matrices. We show here that among the positive semi-definite matrices $X \in \mathbf{H}_n^+$ such that

$$H(X) := \begin{pmatrix} A & X \\ X & B \end{pmatrix} \geq 0_{2n},$$

there exists a maximal one. The latter is called the *geometric mean* of $A$ and $B$, and is denoted by $A\#B$. Then we extend properties that were well-known for scalars.

(a) We begin with the case where $A$ is positive definite.

i. Prove that $H(X) \geq 0_{2n}$ is equivalent to $XA^{-1}X \leq B$ (see Exercise 6, Chapter 8).

ii. Deduce that $A^{-1/2}XA^{-1/2} \leq (A^{-1/2}BA^{-1/2})^{1/2}$. **Hint**: The square root is operator monotone over $\mathbb{R}^+$. See the Additional Exercise 74.

iii. Deduce that among the matrices $X \in \mathbf{H}_n^+$ such that $H(X) \geq 0_{2n}$, there exists a maximal one, denoted by $A\#B$. Write the explicit formula for $A\#B$.

iv. If both $A, B$ are positive definite, prove that $(A\#B)^{-1} = A^{-1}\#B^{-1}$.

(b) We now consider arbitrary elements $A$, in $\mathbf{H}_n^+$.

i. Let $\epsilon > 0$ be given. Show that $H(X) \geq 0_{2n}$ implies $X \leq (A + \epsilon I_n)\#B$.

ii. Prove that $\epsilon \mapsto (A + \epsilon I_n)\#B$ is non-decreasing.

iii. Deduce that $A\#B := \lim_{\epsilon \to 0^+}(A + \epsilon I_n)\#B$ exists, and that it is the largest matrix in $\mathbf{H}_n^+$ among those satisfying $H(X) \geq 0_{2n}$. In particular,

$$\lim_{\epsilon \to 0^+}(A + \epsilon I_n)\#B = \lim_{\epsilon \to 0^+} A\#(B + \epsilon I_n).$$

The matrix $A\#B$ is called the *geometric mean* of $A$ and $B$.

(c) Prove the following identities. **Hint**: Don't use the explicit formula. Use instead the definition of $A\#B$ by means of $H(X)$.

- $A\#B = B\#A$;
- If $M \in \mathbf{GL}_n(\mathbb{C})$, then $M(A\#B)M^* = (MAM^*)\#(MBM^*)$.

(d) Prove the following inequality between harmonic, geometric and arithmetic mean:

$$2(A^{-1} + B^{-1})^{-1} \leq A\#B \leq \frac{1}{2}(A + B).$$

**Hint**: Just check that

$$H\left(2(A^{-1} + B^{-1})^{-1}\right) \leq 0_{2n} \qquad \text{and} \qquad H\left(\frac{1}{2}(A + B)\right) \geq 0_{2n}.$$

In the latter case, use again the fact that $s \mapsto \sqrt{s}$ is operator monotone.

(e) Prove that the geometric mean is "operator monotone":

$$(A_1 \leq A_2 \text{ and } B_1 \leq B_2) \to (A_1\#B_1 \leq A_2\#B_2),$$

and that it is a "operator concave", in the sense that for every $\theta \in (0, 1)$, there holds

$$(\theta A_1 + (1 - \theta)A_2)\#(\theta B_1 + (1 - \theta)B_2) \geq \theta(A_1\#B_1) + (1 - \theta)(A_2\#B_2).$$

Note that the latter property is accurate, since the geometric mean is positively homogeneous of order one. Note also that the concavity gives another proof of the arithmetico-geometric inequality, by taking $A_1 = B_2 = A$, $A_2 = B_1 = B$ and $\theta = 1/2$.

(f) Prove the identity between arithmetic, harmonic and geometric mean:

$$\left(2(A^{-1} + B^{-1})^{-1}\right)\#\frac{A + B}{2} = A\#B.$$

**Hint**: Use the fact that $M\#N$ is the unique solution in $\mathbf{H}_n^+$ of the RicattiRicatti equation $XM^{-1}X = N$. Use it thrice.

199. (Continuation.) Each positive definite Hermitian matrix $A$ defines a norm $\|x\|_A := \sqrt{x^*Ax}$. If $M \in \mathbf{M}_{p\times q}(\mathbb{C})$ and $A_1 \in \mathbf{HDP}_q$, $A_2 \in \mathbf{HDP}_q$, we denote by $\|M\|_{A_1 \leftarrow A_2}$ the norm

$$\sup\{\|Mx\|_{A_1}\,;\ \|x\|_{A_2} = 1\}.$$

(a) Show that the dual norm of the Hermitian norm $\|\cdot\|_A$ is $\|\cdot\|_{A^{-1}}$.

(b) Prove the following interpolation result (compare with the Riesz–ThorinRieszThorin Theorem): For every matrix $M \in \mathbf{M}_{p\times q}(\mathbb{C})$ and every $A_1, B_1 \in \mathbf{HDP}_q$, $A_2, B_2 \in \mathbf{HDP}_q$, we have

$$\|M\|_{A_1\#B_1 \leftarrow A_2\#B_2} \leq \|M\|^{1/2}_{A_1 \leftarrow A_2}\|M\|^{1/2}_{B_1 \leftarrow B_2}.$$

**Hint**: Again, use the definition of the geometric mean, not the formula.

**Comment**: In the terminology of interpolation theory, one writes for $A, B \in \mathbf{HDP}_n$

$$(\mathbb{C}^n; A\#B) = [(\mathbb{C}^n; A), (\mathbb{C}^n; B)]_{1/2},$$

where 1/2 is the interpolation parameter. Recall that

$$[(\mathbb{C}^n; A), (\mathbb{C}^n; B)]_0 = (\mathbb{C}^n; B), \qquad [(\mathbb{C}^n; A), (\mathbb{C}^n; B)]_1 = (\mathbb{C}^n; A).$$

More generally, $[(\mathbb{C}^n; A), (\mathbb{C}^n; B)]_\theta$ can be computed for every diadic $\theta = m2^{-k}$ by means of iterated geometric mean. For instance

$$[(\mathbb{C}^n; A), (\mathbb{C}^n; B)]_{3/4} = [(\mathbb{C}^n; A), [(\mathbb{C}^n; A), (\mathbb{C}^n; B)]_{1/2}]_{1/2} = [(\mathbb{C}^n; A), (\mathbb{C}^n; A\#B)]_{1/2}.$$

(c) Following the idea presented above, show that there exists a unique continuous curve $s \mapsto H(s)$ for $s \in [0, 1]$, with the property that $H(0) = B$, $H(1) = A$ and

$$H\left(\frac{s+t}{2}\right) = H(s)\#H(t), \qquad \forall 0 \le s, t \le 1.$$

This curve is defined by the formula

$$H(s) = A^{1/2}(A^{-1/2}BA^{-1/2})^{1-s}A^{1/2}.$$

We denote $[A, B]_s = H(s)$. Verify that $[A, B]_{1-s} = [B, A]_s$.

200. In periodic homogenization (a chapter of Applied Partial Differential Equations) of elliptic PDEs, one has a continuous map $x \mapsto A(x)$ from $\mathbb{R}^n$ into $\mathbf{SDP}_n$, which is $\Lambda$-periodic, $\Lambda$ being a lattice. The homogenized matrix $\bar{A}$ is defined as follow. For every vector $e \in \mathbb{R}^n$, we admit that there exists a unique (up to an additive constant) solution $w : \mathbb{R}^n \to \mathbb{R}$ of the problem

$$\operatorname{div}(A(x)\nabla w) = 0 \text{ in } \mathbb{R}^n, \qquad w(x+\omega) = w(x) + e\cdot\omega, \qquad \forall \omega \in \Lambda.$$

(Notice that the last property implies that $\nabla w$ is periodic, and its average equals $e$.) Then we define

$$\bar{A}e := < A\nabla w >,$$

where $< \cdot >$ denotes the average over $Y$.

In this exercise we consider the simple case where $n = 2$ and $A$ depends only on the first coordinate $x_1$. In particular, one can take $\Lambda$ spanned by $P\vec{e}^1$ ($P$ the period of $x_1 \mapsto A(x_1)$) and by $\vec{e}^2$.

(a) Let $w$ be as above. Prove that $\partial w/\partial x_2$ is a constant. More precisely, show that $\partial w/\partial x_2 \equiv e_2$.

(b) Likewise, prove that

$$a_{11}\frac{\partial w}{\partial x_1} + a_{12}e_2$$

is a constant and compute that constant.

(c) Finally, prove the following formula

$$\bar{A} = \begin{pmatrix} [a_{11}] & [a_{12}] \\ [a_{12}] & < (\det A)/a_{11} > + \frac{<a_{12}/a_{11}>^2}{<1/a_{11}>} \end{pmatrix}$$

where $[f] :=< f/a_{11} > / < 1/a_{11} >$. In particular, verify that $\det \bar{A} = [\det A]$.

201. We recall that the *symplectic group* $\mathbf{Sp}_n(k)$ is defined as the set of matrices $M$ in $\mathbf{M}_{2n}(k)$ which satisfy $M^T J_n M = J_n$, where

$$J_m := \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}.$$

(a) Let

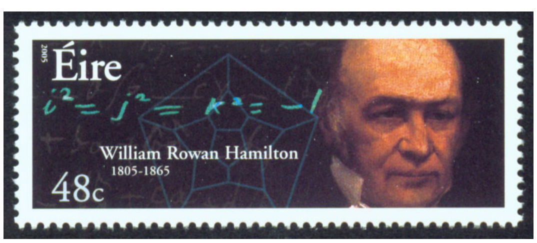
$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be a symplectic matrix. Check that $AB^T$ and $A^T C$ are symmetric, and that $A^T D - C^T B = I_n$.

(b) Let $X \in \mathbf{Sym}_n(k)$ be given and $M$ be as above. Show that $(CX + D)^T (AX + B)$ is symmetric. Deduce that if $CX + D$ is non-singular, then $M \cdot X := (AX + B)(CX + D)^{-1}$ is symmetric.

(c) Show that the matrices $R(X) := C^T (AX + B) A^T C$ and $S(X) := A^T (CX + D) A^T C$ are symmetric for every symmetric $X$.

(d)




From now on, we choose $k = \mathbb{R}$. We say that $M$ is a *Hamiltonian matrix* if $AB^T$ and $A^T C$ are positive definite.

**Rowan Hamilton.**
Hamilton, *Rowan* (Eire)

i. Let $X \in \mathbf{SPD}_n$ be given. Show that $R(X)$ and $S(X)$ are positive definite.

ii. Show also that $A^{-1} C \in \mathbf{SPD}_n$.

iii. Deduce that $Y$ is similar to the product of three positive definite symmetric matrices.

iv. Conclude that $Y$ is positive definite (see Exercise 6 of Chapter 7).

To summarize, a Hamiltonian matrix $M$ acts over $\mathbf{SPD}_n$ by $X \mapsto M \cdot X$ (SiegelSiegel, Bougerol)Bougerol.

(e) Prove that the set of Hamiltonian matrices is a semi-group: The product of two Hamiltonian matrices is Hamiltonian (WojtkowskiWojtkowski).

202. For every $A \in \mathbf{M}_{n\times m}(\mathbb{C})$ and every $t \in \mathbb{R}$, show that

$$n - m = \operatorname{Tr}\exp(-tAA^*) - \operatorname{Tr}\exp(-tA^*A).$$

**Comment**: This is a special case of a formula giving the index of a FredholmFredholm operator $T$, thus in infinite dimension:

$$\operatorname{ind}T = \operatorname{Tr}\exp(-tTT^*) - \operatorname{Tr}\exp(-tT^*T), \qquad \forall t > 0.$$

Notice that in general the difference $\exp(-tTT^*) - \exp(-tT^*T)$ does not make sense.

203. Let $n, r \geq 2$ be two integers. If $A_1, \dots, A_r \in \mathbf{M}_n(k)$ are given, one defines

$$T_r(A_1, \dots, A_r) := \sum_{\sigma \in S_r} \epsilon(\sigma) A_{\sigma(1)} \cdots A_{\sigma(r)},$$

where $S_r$ is the group of permutations of $\{1, \dots, r\}$ and $\epsilon : S_r \to \{-1, +1\}$ is the signature.

(a) Verify that $T_r : \mathbf{M}_n(k)^r \to \mathbf{M}_n(k)$ is an alternate $r$-linear map.

(b) We consider the case $k = \mathbb{R}$ and we endow $\mathbf{M}_n(\mathbb{R})$ with the Frobenius norm. We thus have a Euclidean structure, with scalar product $\langle A, B\rangle = \operatorname{Tr}(B^T A)$.

i. Show that the supremum $\tau(r, n)$ of

$$\frac{\|T_r(A_1, \dots, A_r)\|}{\|A_1\| \cdots \|A_r\|}$$

over $A_1, \dots, A_r \neq 0_n$ is reached. We choose an $r$-uplet $(M_1, \dots, M_r)$ at which this maximum is obtained. Check that one is free to set $\|M_j\| = 1$ for all $j$.

ii. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{R})$ be given. Show that

$$\|T_r(M_1, M_2, M_3, \dots, M_r)\| = \|T_r(aM_1 + bM_2, cM_1 + dM_2, M_3, \dots, M_r)\|.$$

Deduce that if $\tau(r, n) \neq 0$, then $\|aM_1 + bM_2\|\,\|cM_1 + dM_2\| \geq 1$.

iii. Derive from above that $\tau(r, n) \neq 0$ implies that $\langle M_i, M_j\rangle = 0$ for every pair $i \neq j$.

iv. Conclude that $T_r \equiv 0_n$ for every $r \geq n^2 + 1$

(c) We go back to a general field of scalars $k$. Prove that for every $r \geq n^2 + 1$ and every $A_1, \dots, A_r \in \mathbf{M}_n(k)$, one has

$$T_r(A_1, \dots, A_r) = 0_n.$$

**Hint**: Apply the principle of algebraic identities. Use the fact that $\mathbb{R}$ is infinite.

**Comment.** The vanihing of $T_r$ is called a *polynomial identity* over $\mathbf{M}_n(k)$. The above result is far from optimal. The Theorem of AmitsurAmitsur and LevitzkiLevitzki tells us that $T_r$ vanishes identically over $\mathbf{M}_n(k)$ if, and only if, $r \geq 2n$. See Exercise 289.

204. This exercise yields a lemma of MinkowskiMinkowski.

(a) Show that, if either $p = 2$ and $\alpha \geq 2$, or $p \geq 3$ and $\alpha \geq 1$, then $p^\alpha \geq \alpha + 2$.

(b) Define $q := p^\beta$, with $p$ a prime number and $\beta \geq 1$ if $p$ is odd, $\beta \geq 2$ if $p = 2$. Deduce from above that for every $2 \leq j \leq k$, $p^{2\beta+v_p(k)}$ ($v_p(k)$ is the power to which $p$ divides $k$, its *p-valuation*) divides

$$\left( \begin{array}{c} k \\ j \end{array} \right) p^{j\beta}.$$

(c) Let $q$ be as above and $B \in \mathbf{M}_n(\mathbb{Z})$ be such that $p$ does not divide $B$ in $\mathbf{M}_n(\mathbb{Z})$. Let $k \geq 2$ be an integer and form $A := I_n + qB$. Show that

$$A^k \equiv I_n + kqB \qquad \mod p^{2\beta+v_p(k)}.$$

(d) Deduce that if $A$ is as above and if $A^k = I_n$, then $A = I_n$.

(e) More generally, let $A \in \mathbf{M}_n(\mathbb{Z})$ be given. Prove that if $m$ divides $A - I_n$ and if $A^k = I_n$ for some integers $m \geq 3$ and $k \geq 1$, then $A = I_n$. In other words, the kernel of the homomorphism $\mathbf{GL}_n(\mathbb{Z}) \to \mathbf{GL}_n(\mathbb{Z}/m\mathbb{Z})$ is torsion free.

(f) Show that the statement is false when $m = 2$. Find a matrix $A \in I_2 + 2\mathbf{M}_2(\mathbb{Z})$ such that $A^2 = I_2$ and $A \neq I_2$.

205. Let $A \in \mathbf{M}_n(k)$, $B \in \mathbf{M}_m(k)$ and $M \in \mathbf{M}_{n\times m}(k)$ be such that $AM = MB$. It is well-known that if $n = m$ and $M$ is non-singular, then the characteristic polynomials of $A$ and $B$ are equal: $P_A = P_B$. Prove that $\gcd\{P_A, P_B\}$ has a factor of degree $r = \operatorname{rk} M$. **Hint**: Reduce to the case where $M$ is quasidiagonal.

206. Let $k$ be a field. Given two vectors $X, Y$ in $k^3$, we define the vector product as usual:

$$X \times Y := \left( \begin{array}{c} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{array} \right).$$

Prove the following identity in $\mathbf{M}_3(k)$:

$$X(Y \times Z)^T + Y(Z \times X)^T + Z(X \times Y)^T = \det(X, Y, Z)\, I_3, \qquad \forall X, Y, Z \in k^3.$$

207. Let $k$ be a field and $1 \leq p \leq m, n$ be integers.

(a) Let $M \in \mathbf{M}_{n\times m}(k)$ be given, with $\mathrm{rk}M = p$. Show that there exist two matrices $X \in \mathbf{M}_{n\times p}(k)$ and $Y \in \mathbf{M}_{p\times m}(k)$ such that $M = XY$.

(b) We write such a rank$-p$ matrix in block form:

$$M = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right),$$

with $A \in \mathbf{M}_{p\times p}(k)$. If $A$ is non-singular, show that $D = CA^{-1}B$.

(c) We assume that $M$ as rank $p$ and that $\det(J+M)=0$, where

$$J=\begin{pmatrix} 0_p & 0 \\ 0 & I_{n-p} \end{pmatrix}.$$

Show that the block $A$ is singular. Deduce that there exists a non-zero vector $z\in k^{n-p}$ such that either $(J+M)Z=0$, or $(J+M^T)Z=0$, where

$$Z:=\begin{pmatrix} 0 \\ z \end{pmatrix}.$$

208. (Continuation.) We assume that $n=kp$ and $M$ has rank $p$ at most. We can therefore factorize $M$ in the form

$$M=\begin{pmatrix} A_1 \\ \vdots \\ A_k \end{pmatrix}\begin{pmatrix} B_1 & \cdots & B_k \end{pmatrix}.$$

Let $X_1,\dots,X_k$ be indeterminates, and define $X:=\operatorname{diag}\{X_1I_p,\dots,X_kI_p\}$. Show that

$$\det(X+M)=\det\left(X_1\cdots X_kI_p+\sum_j \hat{X}_jB_jA_j\right),$$

where $\hat{X}_j$ denotes $X_1\cdots X_{j-1}X_{j+1}\cdots X_k$. **Hint**: As usual, Schur formula is useful.

209. We show here that $H\mapsto(\det H)^{1/n}$ is concave over $\mathbf{HPD}_n$.

(a) Recall that, given $H$ and $K$ in $\mathbf{HPD}_n$, the product $HK$ is diagonalizable with real, positive, eigenvalues (see Exercise 258).

(b) Deduce that

$$(\det H)^{1/n}(\det K)^{1/n}\le\frac{1}{n}\operatorname{Tr}HK.$$

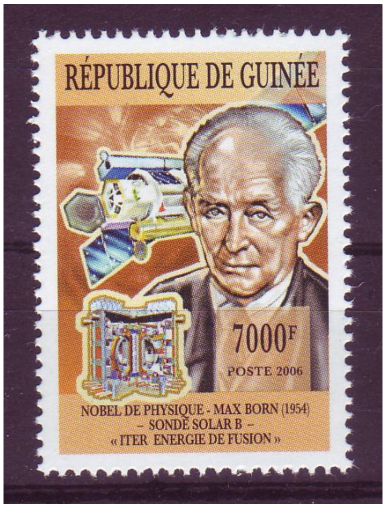
**Hint**: Use the arithmetic-geometric inequality.

(c) Show that

$$(\det H)^{1/n}=\min\left\{\frac{1}{n}\operatorname{Tr}HK\,;\;K\in\mathbf{HPD}_n\text{ and }\det K=1\right\}.$$

(d) Deduce concavity.

210. The notation comes from a nonlinear electrodynamics called the *Born–Infeld model*BornInfeld.




**Max Born.**
Born, *Max* (Rép. de Guinée)

In the canonical Euclidian space $\mathbb{R}^3$ (but $\mathbb{R}^n$ works as well), we give ourselves two vectors $E$ and $B$, satisfying

$$\|E\|^2 + (E \cdot B)^2 \leq 1 + \|B\|^2.$$

Prove the following inequality between symmetric matrices

$$EE^T + BB^T \leq (1 + \|B\|^2) I_3.$$

211. Fix two integers $0 \leq m \leq n-1$. We give ourselves complex numbers $a_{jk}$ for every $1 \leq j, k \leq n$ such that $|k-j| \leq m$ ($2m+1$ diagonals). We assume that $a_{kj} = \overline{a_{jk}}$.

Prove that we can complete this list of entries so as to make a matrix $A \in \mathbf{HDP}_n$ if, and only if, every principal submatrix of size $m+1$,

$$\begin{pmatrix} a_{jj} & \cdots & a_{j,j+m} \\ \vdots & \ddots & \vdots \\ a_{j+m,j} & \cdots & a_{j+m,j+m} \end{pmatrix}$$

is positive definite.

212. We denote

$$X := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad Y := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

(a) Let $M \in \mathbf{SL}_2(\mathbb{Z})$ be a non-negative matrix (that is an element of $\mathbf{SL}_2(\mathbb{N})$). If $M \neq I_2$, show that the columns of $N$ are ordered: $(m_{11} - m_{12})(m_{21} - m_{22}) \geq 0$.

(b) Under the same assumption, deduce that there exists a matrix $M' \in \mathbf{SL}_2(\mathbb{N})$ such that either $M = M'X$ or $M = M'Y$. Check that $\operatorname{Tr} M' \leq \operatorname{Tr} M$. Under which circumstances do we have $\operatorname{Tr} M' < \operatorname{Tr} M$ ?

(c) Let $M \in \mathbf{SL}_2(\mathbb{N})$ be given. Arguing by induction, show that there exists a word $w_0$ in two letters, and a triangular matrix $T \in \mathbf{SL}_2(\mathbb{N})$, such that $M = Tw_0(X, Y) \in \mathbf{SL}_2(\mathbb{N})$.

(d) Conclude that for every $M \in \mathbf{SL}_2(\mathbb{N})$, there exists a word $w$ in two letters, such that $M = w(X,Y)$.

**Comment**. One can show that every element of $\mathbf{SL}_2(\mathbb{Z})$, whose trace is larger than 2, is conjugated in $\mathbf{SL}_2(\mathbb{Z})$ to a word in $X$ and $Y$. This word is not unique in general, since if $M \sim w_0(X,Y)w_1(X,Y)$, then $M \sim w_1(X,Y)w_0(X,Y)$ too.

213. Let $M, N \in \mathbf{M}_n(k)$ be given.

(a) Show that there exists a non-zero pair $(a,b) \in \bar{k}^2$ ($\bar{k}$ the algebraic closure of $k$) such that $\det(aM + bN) = 0$.

(b) Let $(a,b)$ be as above and $x$ be an element of $\ker(aM + bN)$. Show that $(M \otimes N - N \otimes M)x \otimes x = 0$. Deduce that $\det(M \otimes N - N \otimes M) = 0$ for every $M, N \in \mathbf{M}_n(k)$.

(c) We assume that $M$ and $N$ commute to each other. Show that $\det(M \otimes N - N \otimes M)$ can be computed as a nested determinant (see exercise 120).

(d) When $N = I_n$, show that $\det(M \otimes I_n - I_n \otimes M) = 0$ also follows from the Cayley–HamiltonCayleyHamilton theorem. **Hint**: Use the previous question.

214. Let $M \in \mathbf{M}_m(k)$ and $N \in \mathbf{M}_n(k)$ be given. Prove that

$$\det M \otimes N = (\det M)^n (\det N)^m.$$

**Hint**: Use again Exercise 120.

Likewise, let $M \in \mathbf{M}_{p\times q}(k)$ and $N \in \mathbf{M}_{r\times s}(k)$ be given, with $pr = qs$. Check that $M \otimes N$ is a square matrix. Show that $\mathrm{rk}(M \otimes N) \le (\min\{p,q\})(\min\{r,s\})$, and deduce that $\det(M \otimes N) = 0$ when $p \neq q$. The case $p = q$ is covered by the identity above.

215. (P. FinslerFinsler, J. MilnorMilnor.) Let $F : \mathbf{Sym}_n(\mathbb{R}) \to \mathbb{R}$ be a $C^1$-function. Let us define the symmetric matrix $M$ by

$$m_{ij} := \frac{\partial F}{\partial a_{ij}}(0_n).$$

Let $\mu_-$ be the smallest eigenvalue of $M$. Prove that

$$\mu_- = \liminf \left\{ \frac{F(A) - F(0)}{\mathrm{Tr} A} \mid A > 0,\ \mathrm{Tr} A \to 0 \right\}.$$

216. Let $H$ be a Hermitian matrix, such that for every $K \in \mathbf{HPD}_n$, there holds

$$\det(H + K) \ge \det H.$$

Prove that $H$ is positive semi-definite.

217. If $A \in \mathbf{Sym}_n(\mathbb{R})$, we denote $q_A$ the associated quadratic form. In the sequel, $A$ and $B$ are too real symmetric matrices.

(a) Assume that there exists a positive definite matrix among the linear combinations of $A$ and $B$ (the *pencil* spanned by $A$ and $B$). Prove that there exists a $P \in \mathbf{GL}_n(\mathbb{R})$ such that both $P^T AP$ and $P^T BP$ are diagonal. **Hint**: A classical result if $B$ itself is positive definite.

(b) We assume instead that $q_A(x) = q_B(x) = 0$ implies $x = 0$, and that $n \geq 3$.

i. Show that $R(B) \cap A(\ker B) = \{0\}$.

ii. Let $\Delta(\lambda)$ be the determinant of $A + \lambda B$. Using a basis of $\ker B$, completed as a basis of $\mathbb{R}^n$, show that the degree of $\Delta$ equals the rank of $B$. Deduce that there exists a non-degenerate matrix in this pencil.

(c) We keep the assumption that $q_A = q_B = 0$ implies $x = 0$. From the previous question, we may assume that $B$ is non-degenerate.

i. Let us define

$$Z(x) := \frac{q_A(x) + iq_B(x)}{\sqrt{q_A(x)^2 + q_B(x)^2}} \in \mathbb{C}, \qquad \forall x \neq 0.$$

Show that there exists a differentiable real-valued map $x \mapsto \theta(x)$ over $\mathbb{R}^n \setminus \{0\}$, such that $Z(x) = \exp(i\theta(x))$ for every $x \neq 0$ in $\mathbb{R}^n$.

ii. Let $x$ be a critical point of $\theta$. Show that $q_A(x)Bx = q_B(x)Ax$. Show that there exists such a critical point $x^1$. Show that $q_B(x^1) \neq 0$.

iii. We define $E_1 := (Bx^1)^\perp$. Show that the restriction of $B$ to $E_1$ is non-degenerate. Prove that a critical point $x^2$ of the restriction of $\theta$ to $E_1 \setminus \{0\}$ again satisfies $q_A(x^2)Bx^2 = q_B(x^2)Ax^2$, as well as $(x^1)^T Bx^2 = 0$ and $q_B(x^2) \neq 0$.

iv. Arguing by induction, construct a sequence $x^1, \dots, x^n$ of vectors of $\mathbb{R}^n$ with the properties that $Ax_j \parallel Bx_j$, and $(x^j)^T Bx^k$ vanishes if and only if $j \neq k$.

v. Conclusion: There exists a $P \in \mathbf{GL}_n(\mathbb{R})$ such that both $P^T AP$ and $P^T BP$ are diagonal.

(d) We are now in the position that both $A$ and $B$ are diagonal, and still $q_A = q_B = 0$ implies $x = 0$. We wish to show that there exists a linear combination of $A$ and $B$ that is positive definite

i. We argue by contradiction. Suppose that none of the vectors $\lambda \mathrm{diag} A + \mu \mathrm{diagB}$ is positive (in the sense of Chapter 5) when $(\lambda, \mu)$ run over $\mathbb{R}^2$. Show that there exists a hyperplane in $\mathbb{R}^n$, containing $\mathrm{diag}A$ and diagB, but no positive vector. **Hint**: Apply Hahn–BanachHahnBanach.

ii. Deduce that there exists a non-negative, non-zero vector $y \in \mathbb{R}^n$ such that $\sum_j y_j a_{jj} = \sum_j y_j b_{jj} = 0$.

iii. Show that this implies $y = 0$.

In conclusion, we have found the equivalence (as long as $n \geq 3$) of the conditions:

- If $q_A(x) = q_B(x) = 0$, then $x = 0$,
- There exists a positive definite linear combination of $q_A$ and $q_B$,

and they imply a third one

- There exists a basis of $\mathbb{R}^n$, orthogonal for both $q_A$ and $q_B$.

(e) Provide a counter-example when $n = 2$: Find $A$ and $B$ such that $q_A = q_B = 0$ implies $x = 0$, but there does not exist a basis simultaneously orthogonal for $A$ and $B$. In particular, combinations of $A$ and $B$ cannot be positive definite.

218. This may be a new proof of Gårding'sGaa@Gårding Theorem.

(a) Let $F : \Omega \to \mathbb{R}^+$ be a positive, homogeneous function of degree $\alpha$, over a convex cone $\Omega$ of $\mathbb{R}^N$. We assume that $F$ is quasi-concave over $\Omega$; by this we mean that at every point $x \in \Omega$, the restriction of the Hessian $\mathrm{D}^2F(x)$ to $\ker \mathrm{d}F(x)$ is non-positive. Prove that $F^{1/\alpha}$ is concave over $\Omega$. **Hint**: Use repeatedly EulerEuler Identity for homogenous functions.

(b) We now focus to $\mathbb{R}^N \sim \mathbf{H}_n$ (and thus $N = n^2$), with $F(M) = \det M$ and $\Omega = \mathbf{HPD}_n$.

i. If $M \in \mathbf{HPD}_n$ is diagonal, compute explicitly $\mathrm{d}F(M)$ and $\mathrm{D}^2F(M)$. Then check that $F$ is quasi-concave at $M$.

ii. Extend this property to all $M \in \mathbf{HPD}_n$, using the diagonalisability through a unitary conjugation.

iii. Deduce a particular case of Gårding's Theorem: $M \mapsto (\det M)^{1/n}$ is concave over $\mathbf{HPD}_n$.

Gårding's Theorem is that if $F$ is a hyperbolic polynomial, homogeneous of degree $n$, and $\Omega$ is its forward cone, then $F^{1/n}$ is concave over $\Omega$. This may be proved in full generality with the argument above, together with the fact that the quadratic form $Z \mapsto \mathrm{D}^2F(X)Z^{\otimes 2}$ is of signature $(1, N-1)$. See an other proof below.

Notice that the first concavity result given above can be written even for non-smooth functions $F$, thus without invoquing first- an second-order differential. Prove that if $G : \Omega \to \mathbb{R}^+$ is homogeneous of degree one, and if $K := \{x \in \Omega \,|\, G(x) \geq 1\}$ is convex (by homogeneity, this amounts to quasi-concavity), then $G$ is concave.

219. Here is an other proof of the concavity of $\det^{1/n}$ over $\mathbf{HPD}_n$.

(a) Given non-negative real numbers $a_1, \dots, a_n$, we denote $\sigma_k(a)$ the $k$-th elementary symmetric polynomial, which is a sum of $\binom{n}{k}$ monomials. Prove that

$$\sigma_k(a) \geq \binom{n}{k} \sigma_n(a)^{k/n}.$$

**Hint**: Use the arithmetico-geometric inequality.

(b) Prove that for every $K \in \mathbf{HPD}_n$, one has

$$\det(I_n + K) \geq \left(1 + (\det K)^{1/n}\right)^n.$$

(c) Deduce the inequality

$$(\det(H+K))^{1/n} \geq (\det H)^{1/n} + (\det K)^{1/n}$$

for every $H, K \in \mathbf{HPD}_n$. Conclude.

220. We consider the Hermitian norm $\|\cdot\|_2$ over $\mathbb{C}^p$ and $\mathbb{C}^q$. We denote by $\mathcal{B}$ the unit ball (the set of linear contractions) in $\mathbf{M}_{p\times q}(C)$. Recall that a contraction is a map satisfying $\|f(x)-f(y)\|_2 < \|x-y\|_2$ whenever $y \neq x$.

   (a) Show that $M \in \mathbf{M}_{p\times q}(\mathbb{C})$ is a contraction if, and only if, $\|M\|_2 < 1$. Deduce that $M^*$ is also a contraction.

   (b) Let $H \in \mathbf{H}_q$ and $P \in \mathbf{GL}_q(\mathbb{C})$ be given. Show that $P^{-*}HP^{-1} < I_q$ is equivalent to $H < P^*P$.

   (c) Given a matrix $U \in \mathbf{U}(p,q)$, written in block form

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

   we define a map $F$ over $\mathbf{M}_{p\times q}(\mathbb{C})$ by

$$F(Z) := (AZ+B)(CZ+D)^{-1}.$$

   Show that $F$ maps $\mathcal{B}$ into itself.

   (d) Show that the set of maps $F$ form a group (denoted by $\Gamma$) as $U$ runs over $\mathbf{U}(p,q)$, and that the map $U \mapsto F$ is a group homomorphism.

   (e) Show that for every $Z$ given in $\mathcal{B}$, there exists an $F$ as above, such that $F(Z) = 0_{p\times q}$. Deduce that the group $\Gamma$ acts transitively over $\mathcal{B}$.

221. (Loo-Keng HuaHua, Loo-Keng.) This exercise uses the previous one, and in particular has the same notations. We define the following function over $\mathcal{B}\times\mathcal{B}$:

$$\phi(W,Z) := \frac{|\det(I_q - W^*Z)|^2}{\det(I_q - W^*W)\det(I_q - Z^*Z)}.$$

   (a) Of course, $\phi(0_{p\times q}, Z) \geq 1$ for every contraction $Z$, with equality only if $Z = 0_{p\times q}$.

   (b) Show that if $U \in \mathbf{U}(p,q)$ and $F$ is defined as above, then

$$\phi(F(W), F(Z)) = \phi(W,Z).$$

   We say that $\phi$ is invariant under the action of $\Gamma$.

   (c) Deduce Hua's Inequality: For any two contractions $W$ and $Z$, one has

$$\det(I_q - W^*W)\det(I_q - Z^*Z) \leq |\det(I_q - W^*Z)|^2,$$

   with equality if and only if $W = Z$. **Hint**: Use transitivity ; then it is enough to treat the case $W = 0_{p\times q}$.

In other words, $\phi(W,Z) \geq 1$, with equality only if $W = Z$. The quantity

$$K(W,Z) := \frac{\det(I_q - W^*Z)}{\det(I_q - W^*W)^{1/2}\det(I_q - Z^*Z)^{1/2}},$$

whose square modulus is $\phi(W,Z)$, is the *Bergman kernel*Bergman of the symmetric domain $\mathcal{B}$.

222. We use the notations of Exercise 146. We assume $K = \mathbb{C}$. The exterior algebra $\Lambda^k E$ is naturally endowed with a Hermitian structure, in which a unitary basis is given by the vectors $e_{i_1} \wedge \cdots \wedge e_{i_k}$ with $1 \leq i_1 < \cdots < i_k \leq n$, $\{e_1, \dots, e_n\}$ being the canonical basis of $E = \mathbb{C}^n$.

    (a) Prove that $A^{(k)}B^{(k)} = (AB)^{(k)}$.

    (b) Prove that $\left(A^{(k)}\right)^* = (A^*)^{(k)}$.

    (c) Deduce that if $U \in \mathbf{U}_n$, then $U^{(k)}$ is unitary too.

    (d) Let $s_1, \dots, s_n$ denote the singular values of a matrix $A$. Prove that $|\mathrm{Tr} A| \leq s_1 + \cdots + s_n$.

    (e) Let $\lambda_1, \dots, \lambda_n$ denote the eigenvalues of $A$. Deduce for every $1 \leq k \leq n$ the inequality

    $$|\sigma_k(\lambda)| \leq \sigma_k(s),$$

    where $\sigma_k$ is the elementary symmetric polynomial if degree $k$ in $n$ arguments. The case $k = 1$ has been established in the previous question. The case $k = n$ is trivial. **Hint**: Apply the case $k = 1$ to $A^{(k)}$, and use Exercise 146.

    (f) Use this result to prove that

    $$|\det(I_n + A)| \leq \det(I_n + |A|),$$

    where $|A| := \sqrt{A^*A}$ is the non-negative symmetric part in the polar decomposition.

223. (L. DinesDines.) Let $A, B \in \mathbf{Sym}_n(\mathbb{R})$ be given. Show that the range of the map $x \mapsto (x^T Ax, x^T Bx)$ is a convex subset of $\mathbb{R}^2$.

    Compare with the Toeplitz–Hausdorff Lemma (Exercise 21).

224. Let $A \in \mathbf{M}_n(k)$ be a block triangular matrix. Show that $A$ is nilpotent if, and only if, its diagonal blocks are nilpotent.

225. A subspace $V$ of $\mathbf{M}_n(k)$ is *nilpotent* if every element of $V$ is nilpotent. Let $V$ be a nilpotent subspace.

    (a) If $A \in V$, prove that $\mathrm{Tr}\, A^2 = 0$.

        i. If $\mathrm{char} k \neq 2$, deduce that $\mathrm{Tr}(AB) = 0$ for every $A, B \in V$.

ii. More generally, check that

$$\operatorname{Tr}(AB) = (\operatorname{Tr} A)(\operatorname{Tr} B) + \sigma_2(A) + \sigma_2(B) - \sigma_2(A+B),$$

and deduce that $\operatorname{Tr}(AB) = 0$ for every $A, B \in V$, even in characteristic 2. Hereabove, $\sigma_2(M)$ denotes the second elementary symmetric polynomial in the eigenvalues of $M$.

(b) Let $U$ denote the subspace of upper triangular matrices, and $U^+$ that of strictly upper triangular. Likewise, we denote $L$ and $L^-$ for lower triangular matrices. One has

$$\mathbf{M}_n(k) = L^- \oplus U = L \oplus U^+.$$

Let $\phi$ be the projection over $L^-$, parallel to $U$. We denote $K := V \cap \ker\phi$ and $R := \phi(V)$.

i. Show that $K \subset U^+$.

ii. If $M \in R$ and $B \in K$, prove that $\operatorname{Tr}(MB) = 0$. In other words, $R^T$ and $K$ are orthogonal subspaces of $U^+$, relatively to the form $\langle M, N\rangle := \operatorname{Tr}(M^T N)$.

iii. Deduce that

$$\dim R + \dim K \leq \dim U^+.$$

(c) In conclusion, show that the dimension of a nilpotent subpace of $\mathbf{M}_n(k)$ is not larger than

$$\frac{n(n-1)}{2}.$$

**Nota.** Every nilpotent subspace of dimension $n(n-1)/2$ is conjugated to $U^+$. However, it is not true that every nilpotent subspace is conjugated to a subspace of $U^+$. For instance, let $n = 3$: Prove that the space of matrices

$$\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ y & -x & 0 \end{pmatrix}$$

is nilpotent, but is not conjugated to a space of triangular matrices. **Hint**: the kernels of these matrices intersect trivially.

226. We use the scalar product over $\mathbf{M}_n(\mathbb{C})$, given by $\langle M, N\rangle = \operatorname{Tr}(M^*N)$. We recall that the corresponding norm is the Schur–Frobenius norm $\|\cdot\|_F$. If $T \in \mathbf{GL}_n(\mathbb{C})$, we denote $T = U|T|$ the polar decomposition, with $|T| := \sqrt{T^*T}$ and $U \in \mathbf{U}_n$. The AluthgeAluthge transform $\Delta(T)$ is defined by

$$\Delta(T) := |T|^{1/2} U |T|^{1/2}.$$

(a) Check that $\Delta(T)$ is similar to $T$.

(b) If $T$ is normal, show that $\Delta(T) = T$.

(c) Show that $\|\Delta(T)\|_F \le \|T\|_F$, with equality if, and only if, $T$ is normal.

(d) We define $\Delta^n$ by induction, with $\Delta^n(T) := \Delta(\Delta^{n-1}(T))$.

i. Given $T \in \mathbf{GL}_n(\mathbb{C})$, show that the sequence $\left(\Delta^k(T)\right)_{k\in\mathbb{N}}$ is bounded.

ii. Show that its limit points are normal matrices with the same characteristic polynomial as $T$ (Jung, Jung, Il Bong KoKo, Eungil & PearcyPearcy, or AndoAndo).

iii. Deduce that when $T$ has only one eigenvalue $\mu$, then the sequence converges towards $\mu I_n$.

**Comment**: The sequence does converge for every initial $T \in \mathbf{M}_n(\mathbb{C})$, according to J. AntezanaAntezana, E. R. PujalsPujals and D. Strojanoff.Strojanoff

(e) If $T$ is not diagonalizable, show that these limit points are not similar to $T$.

227. (After C. de Lellisdelel@de Lellis & L. SzékelyhidiSzékelyhidi Jr.)

If $x \in \mathbb{R}^n$, we denote $x \otimes x := xx^T$. We also use the standard Euclidian norm. The purpose of this exercise is to prove that the convex hull of

$$K := \left\{(v, S) \,|\, v \in \mathbb{R}^n,\ |v| = 1 \text{ and } S = v \otimes v - \frac{1}{n} I_n\right\}$$

equals

$$C = \left\{(v, S) \,|\, S \in \mathbf{Sym}_n(\mathbb{R}),\ \operatorname{Tr} S = 0 \text{ and } v \otimes v - \frac{1}{n} I_n \le S\right\}.$$

dele@de Lellis, *Camillo* (Vatican state)

Left: **Mmmh ! This Camillo de Lellis is not the mathematician.**

(a) Show that every $(v, S)$ in $C$ satisfies

$$|v| \le 1 \text{ and } S \le \left(1 - \frac{1}{n}\right) I_n.$$

(b) Check that $K \subset C$.

(c) Prove that $C$ is a convex compact subset of $\mathbb{R}^n \times \mathbf{Sym}_n(\mathbb{R})$.

(d) Let $(v, S) \in C$ be given, such that

$$v \otimes v - \frac{1}{n} I_n \neq S.$$

i. Show that $|v| < 1$.

ii. Let $\mu$ be the largest eigenvalue of $S$. Show that $\mu \leq 1 - 1/n$. In case of equality, show that there exists a unit vector $w$ such that

$$S = w \otimes w - \frac{1}{n} I_n.$$

iii. In the latter case ($\mu = 1 - 1/n$), show that $v = \rho w$ for some $\rho \in (-1, 1)$. Deduce that $(v, S)$ is not an extremal point of $C$.

iv. We now assume on the contrary that $\mu < 1 - 1/n$. Let $N$ denote the kernel of

$$S - v \otimes v + \frac{1}{n} I_n.$$

If $N \subset \ker S'$ for a symmetric, trace-less $S'$, show that $(v, S + \epsilon S') \in C$ for $|\epsilon|$ small enough.
If $\dim N \leq n - 2$, deduce again that $(v, S)$ is not an extremal point of $C$.

v. We still assume $\mu < 1 - 1/n$, and we now treat the case where $N$ is a hyperplane. Show that there exists a vector $z \neq 0$ such that

$$S = v \otimes v + z \otimes z - \frac{1}{n} I_n.$$

Show that there exists a non-zero pair $(\alpha, \beta) \in \mathbb{R}^2$ such that, defining $w = \alpha z$ and $s = z \otimes w + w \otimes z + \beta z \otimes z$, one has $(v, S) \pm (w, s) \in C$. Deduce that $(v, S)$ is not an extremal point in $C$.

(e) Deduce that every extremal point of $C$ belongs to $K$.

(f) Conclude, with the help of KreinKrein-MilmanMilman's Theorem.

(g) Show that in particular, $0_n$ is a relative interior point of the convex set $C$ (that is, an interior point of $C$ as a subset of the affine space spanned by $C$).

228. We recall that the Pfaffian of a $4 \times 4$ alternate matrix $A$ is $a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23}$. Show that $\mathbf{Alt}_4(\mathbb{R}) \cap \mathbf{GL}_4(\mathbb{R})$ has two connected components, each one homeomorphic to $S^2 \times S^2 \times \mathbb{R}^2$, with $S^2$ the two-dimensional sphere.

229. Let $M \in \mathbf{M}_n(k)$ be given in block form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where the diagonal blocks have square shape. Recall that if $A$ is non-singular, then its SchurSchur complement is defined as

$$A^c := D - CA^{-1}B.$$

Let us restrict ourselves to the case $k = \mathbb{C}$ and to Hermitian matrices. If $M$ is positive definite, then so is $A$, and the Schur complement is well-defined.

Is instead $M$ is positive semi-definite, show that $R(B) \subset R(A)$, and thus there exists a (not necessarily unique) rectangular matrix $X$ such that $B = AX$. Show that the product $X^*AX$ does not depend on the particular choice of $X$ above. Deduce that the map $A \mapsto A^c$ extends continuously to the closure of $\mathbf{HPD}_n$, that is to the cone of positive semi-definite matrices. Finally, show that $\det M = \det A \cdot \det A^c$.

230. After R. B. BapatBapat, we define a *doubly stochastic $n$-tuple* as an $n$-uplet $A = (A^1, \dots, A^n)$ of Hermitian semi-positive definite matrices $A^j \in \mathbf{H}_n^+$ (yes, the same $n$) with the properties that

- for every $j = 1, \dots, n$, $\operatorname{Tr} A^j = 1$,
- and moreover, $\sum_{j=1}^n A^j = I_n$.

The set of doubly stochastic $n$-tuples is denoted by $\mathcal{D}_n$.

(a) Given $A$ as above and $V \in \mathbf{U}_n$, we define $A(V) \in \mathbf{M}_n(\mathbb{R})$ by

$$A(V)_{jk} := \left(v^j\right)^* A_k v^j,$$

where $v^1, \dots, v^n$ are the columns of $V$.

Show that $A(V)$ is a doubly-stochastic matrix.

(b) Conversely, let $A = (A^1, \dots, A^n)$ be an $n$-uplet of $n \times n$ complex matrices. Define $A(W)$ as above. Show that, if $A(W)$ is doubly stochastic for every unitary $V$, then $A$ is a doubly stochastic $n$-uplet.

(c) Check that the subset of $\mathcal{D}_n$ made of those $A$'s such that every $A^j$ is diagonal, is isomorphic to the set of doubly stochastic matrices.

(d) Remark that $\mathcal{D}_n$ is a convex compact subset of $(\mathbf{H}_n)^n$. If each $A^j$ has rank one, prove that $A$ is an extremal point of $\mathcal{D}_n$.

**Nota**: there exist other extremal points if $n \geq 4$. However, the determination of all the extremal points of $\mathcal{D}_n$ is still an open question.

231. (Continuation.) We examine some of the extremal points $A = (A^1, \dots, A^n)$ of $\mathcal{D}_n$.

(a) If $A = \frac{1}{2}(B + C)$ with $B, C \in \mathcal{D}_n$, prove that $R(B^j)$ and $R(C^j)$ are contained in $R(A^j)$ for every $j$.

(b) If $A$ is not extremal, show that one may choose $B, C$ as above, such that $R(B^i) \neq R(A^i)$ for some index $i$.

(c) In the situation described above, assume that $A^i$ has rank two. Prove that $B^i = x_i x_i^*$ and $C^i = y_i y_i^*$, where $\{x_i, y_i\}$ is a unitary basis of $R(A^i)$.

(d) We now assume that every $A^j$ has rank two, and that for every pair $j \neq k$, the planes $R(A^j)$ and $R(A^k)$ are *not* orthogonal. Prove that *all* the matrices $B^j$ and $C^j$ have the forms $x_j x_j^*$ and $y_j y_j^*$ respectively, with $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ two unitary bases of $\mathbb{C}^n$.

(e) We set $n = 4$ and choose two unitary bases $\{v_1, \dots, v_4\}$ and $\{w_1, \dots, w_4\}$ of $\mathbb{C}^4$. We define

$$A^1 = \frac{1}{2}(v_1 v_1^* + w_3 w_3^*), \qquad A^2 = \frac{1}{2}(v_2 v_2^* + w_4 w_4^*),$$
$$A^3 = \frac{1}{2}(w_1 w_1^* + w_2 w_2^*), \qquad A^4 = \frac{1}{2}(v_3 v_3^* + v_4 v_4^*).$$

Check that $A \in \mathcal{D}_4$. Find a choice of the $v$'s and $w$'s such that there does not exist a unitary basis $\{x_1, \dots, x_4\}$ with $x_j \in R(A^j)$. Deduce that $A$ is an extremal point.

232. (From M. H. MehrabiMehrabi.) Let $A, B \in \mathbf{M}_n(\mathbb{R})$ be such that $[A, B]$ is non-singular and verify the identity
$$A^2 + B^2 = \rho[A, B],$$
for some $\rho \in \mathbb{R}$.

Show that
$$\omega := \frac{\rho - i}{\sqrt{1 + \rho^2}}$$
is a $(2n)$-th root of unity.

233. We are given three planes $E_1$, $E_2$ and $E_3$ in the Euclidian space $\mathbb{R}^3$, of respective equations $z_j \cdot x = 0$. We are searching an orthogonal basis $\{v_1, v_2, v_3\}$ such that $v_j \in E_j$ for each $j$.

Prove that such a basis exists if, and only if,
$$\Delta := (\det(z_1, z_2, z_3))^2 - 4(z_1 \cdot z_2)(z_2 \cdot z_3)(z_3 \cdot z_1)$$
is non-negative. When $\Delta$ is positive, there exist two such bases, up to scaling.

234. Let $M \in \mathbf{M}_n(k)$ be given. Use Theorem 2.3.1 of the book to calculate the Pfaffian of the alternate matrix
$$\begin{pmatrix} 0_n & M \\ -M^T & 0_n \end{pmatrix}.$$

**Warning**. Mind that the Pfaffian of the same matrix when $M = I_n$ is $(-1)^{n(n-1)/2}$.

235. In a vector space $V$ of dimension $n$, we are given $n$ subspaces $E_j$, $j = 1, \dots, n$. We examine the question whether there exists a basis $\{v_1, \dots, v_n\}$ of $V$ with $v_j \in E_j$ for every $j$.

(a) Check the obvious necessary condition: for every index subset $J$,

$$\dim\left(\underset{j\in J}{+} E_j\right) \geq |J|. \tag{26}$$

In the sequel, we want to prove that it is also a sufficient condition. We shall argue by induction over $n$.

(b) Let us assume that the result is true up to the dimension $n-1$.
From now on we assume that $\dim V = n$ and that the property (26) is fulfilled.

i. Prove the claim for the given set $E_1, \dots, E_n$, in the case where there exists a index subset $J$ such that

$$\dim\left(\underset{j\in J}{+} E_j\right) = |J|, \qquad 1 \leq |J| \leq n-1.$$

**Hint**: Apply the induction hypothesis to both

$$W := \underset{j\in J}{+} E_j \qquad \text{and} \qquad Z := V/W.$$

ii. There remains the case where, for every $J$ with $1 \leq |J| \leq n-1$, one has

$$\dim\left(\underset{j\in J}{+} E_j\right) \geq |J| + 1.$$

Select a non-zero vector $v_1 \in E_1$. Define $F_j = (kv_1 + E_j)/kv_1$, a subspace of $W := V/kv_1$. Check that $F_2, \dots, F_n$ and $W$ satisfy the assumption (26). Apply the induction hypothesis.

(c) Conclude that a necessary and sufficient condition for the existence of a basis $\{v_1, \dots, v_n\}$ of $V$ with $v_j \in E_j$ for every $j = 1, \dots, n$, is that (26) is fulfilled for every index subset $J$.

236. We consider differential equations over $\mathbf{M}_n(\mathbb{C})$, of the form

$$\frac{dM}{dt} = [A(t), M]. \tag{27}$$

Hereabove, $A(t)$ can be given *a priori*, or be determined by $M(t)$. As usual, the bracket denotes the commutator.

(a) Show that

$$\frac{d}{dt}\det M = \operatorname{Tr}\left(\hat{M}^T \frac{dM}{dt}\right),$$

where $\hat{M}$ is the adjugate of $M$. Deduce that $t \mapsto \det M(t)$ is constant.

(b) Define $P(t)$, the solution of the Cauchy problem

$$\frac{dP}{dt} = -PA, \qquad P(0) = I_n.$$

Show that $M(t) = P(t)^{-1}M(0)P(t)$. In particular, the spectrum of $M(t)$ remains equal to that of $M(0)$ ; one speaks of an *isospectral* flow.

(c) First example: take $A := M^*$. Show that $t \mapsto \|M(t)\|_F$ (Frobenius norm) is constant.

(d) Second example: take $A := [M^*, M]$. Show that $t \mapsto \|M(t)\|_F$ is monotonous non increasing. Deduce that the only rest points are the normal matrices.

237. This problem and the following one examine the dimension of the *commutant* of a given matrix $A \in \mathbf{M}_n(k)$ when $k$ is algebraically closed. We begin with the equation

$$BY = YC, \tag{28}$$

where $B = J(0; p)$ and $C = J(0 : q)$ are nilpotent matrices in Jordan form. The unknown is $Y \in \mathbf{M}_{p\times q}(k)$. We denote by $\{\mathbf{e}_1, \dots, \mathbf{e}_q\}$ and $\{\mathbf{f}_1, \dots, \mathbf{f}_p\}$ the canonical bases of $k^q$ and $k^p$, respectively. We thus have $C\mathbf{e}_j = \mathbf{e}_{j-1}$, and likewise with $B$ and $\mathbf{f}_j$.

(a) Show that for every solution $Y$ of (28), there exist scalars $y_1, \dots, y_q$ such

$$Y\mathbf{e}_j = y_1\mathbf{f}_j + y_2\mathbf{f}_{j-1} + \cdots + y_j\mathbf{f}_1.$$

(b) If $p < q$, explain why $y_1, \dots, y_{q-p}$ must vanish.

(c) Deduce that the solution space of (28) has dimension $\min(p, q)$.

238. (Continuation.) Let $A \in \mathbf{M}_n(k)$ be given. We denote $\mathrm{com}(A)$ the space of matrices $X \in \mathbf{M}_n(k)$ such that $AX = XA$.

(a) If $A'$ is similar to $A$, show that $\mathrm{com}(A)$ and $\mathrm{com}(A')$ are conjugated, and thus have the same dimension.

(b) Assuming that $k$ is algebraically closed, we thus restrict to the case where $A = \mathrm{diag}\{A_1, \dots, A_r\}$, where $A_j$ has only one eigenvalue $\lambda_j$ and $j \neq k$ implies $\lambda_j \neq \lambda_k$. We decompose $X$ blockwise accordingly:

$$X = (X_{jk})_{1\le j,k\le r}.$$

Show that $X$ commutes with $A$ if, and only if, $A_jX_{jk} = X_{jk}A_k$ for every $j, k$. In particular, $j \neq k$ implies that $X_{jk} = 0$: $X$ is block-diagonal too. Therefore,

$$\dim \mathrm{com}(A) = \sum_{j=1}^{r} \dim \mathrm{com}(A_j).$$

(c) We are thus left with the special case where $A$ has only one eigenvalue $\lambda$, and has invariant polynomials $(X-\lambda)^{m_1}, \dots, (X-\lambda)^{m_n}$ with

$$m_1 \le \cdots \le m_n \qquad \text{and} \qquad m_1 + \cdots + m_n = n.$$

i. Show that the commutant of $A$ does not depend on $\lambda$, but only on $m_1, \dots, m_n$.

ii. Use the previous exercise to find

$$\dim \operatorname{com}(A) = \sum_{j=0}^{n-1} (2j+1) m_{n-j}.$$

(d) Deduce that for every $A \in \mathbf{M}_n(k)$, one has

$$\dim \operatorname{com}(A) \ge n.$$

239. (After C. HillarHillar & Jiawang NieNie, Jiawang.) Let $S \in \mathbf{Sym}_n(\mathbb{Q})$ be given. We assume that $S \ge 0_n$ in the sense of quadratic forms. The purpose of this exercise is to prove that $S$ is a sum of squares $(A_j)^2$ with $A_j \in \mathbf{Sym}_n(\mathbb{Q})$.


**Joseph-Louis Lagrange.**
Lagrange, *Joseph-Louis* (France)

Recall that if $n = 1$, this follows from the Lagrange'sLagrange Theorem: every positive integer is the sum of four squares of integers.

We recall the key fact that in an Abelian ring, the product of two sums of squares is a sum of squares.

We denote by $p$ the minimal polynomial of $S$, which we write

$$p(X) = X^s - a_{s-1} X^{s-1} + \cdots + (-1)^s a_s.$$

(a) Check that $p$ has simple roots. In particular, $a_0$ and $a_1$ cannot vanish both.

(b) Verify that the coefficients $a_k$ are non-negative. Deduce that $a_1 \neq 0$.

(c) We decompose $(-1)^{s-1} p(X) = q(X^2) X - r(X^2)$. Check that $T := q(S^2)$ is a sum of squares, and that it is invertible. Show then that $T^{-2} r(S^2)$ is a sum of squares. Conclude.

240. Let $A, B \in \mathbf{M}_n(\mathbb{C})$ be given. We assume that there exists a non-singular matrix $P$ such that

$$AP = PB, \qquad A^* P = P B^*.$$

Prove that $A$ and $B$ are unitarily similar: there exists a $U \in \mathbf{U}_n$ such that $AU = UB$. **Hint**: Using the polar decomposition to $P$, show that we may assume $P \in \mathbf{HPD}_n$, and then $P$ diagonal.

This statement is a part of the proof of *Specht Theorem*Specht: $A$ and $B$ are unitarily similar if, and only if, the equality $\operatorname{Tr} w(A, A^*) = \operatorname{Tr} w(B, B^*)$ holds true for every word $w$ in two letters. The rest of the proof involves representation theory and is beyond the scope of our book.

241. (From C. VillaniVillani.) Let $t \mapsto R(t) \in \mathbf{Sym}_n(\mathbb{R})$ be a continuous function over $[0, T]$. We denote $J_0(t)$ and $J_1(t)$ the matrix-valued solutions of the differential equation

$$\frac{d^2 J}{dt^2} + R(t)J = 0,$$

uniquely determined by the Cauchy data

$$J_0(0) = I_n, \quad J_0'(0) = 0_n, \qquad J_1(0) = 0_n, \quad J_1'(0) = I_n.$$

We wish to prove that $S(t) := J_1(t)^{-1} J_0(t)$ is symmetric, whenever $J_1(t)$ is non-singular.

(a) Let $u_j(t)$ $(j = 1, 2)$ be two *vector*-valued solutions of the ODE

$$u'' + R(t)u = 0. \tag{29}$$

Verify that $t \mapsto \langle u_1'(t), u_2(t)\rangle - \langle u_2'(t), u_1(t)\rangle$ is constant.

(b) For a solution $u$ of (29), show that

$$J_0(t)u(0) + J_1(t)u'(0) = u(t).$$

(c) For $t \in [0, T)$, let us define the space $\mathcal{V}_t$ of the solutions of (29) such that $u(t) = 0$. Show that it is an $n$-dimensional vector space. If $J_1(t)$ is non-singular, verify that an alternate definition of $\mathcal{V}_t$ is the equation

$$u'(0) = S(t)u(0).$$

(d) Deduce the symmetry of $S(t)$ whenever it is well-defined.

242. This is a sequel of Exercise 26, Chapter 4 (see also Exercise 153 in this list). We recall that $\Sigma$ denotes the unit sphere of $\mathbf{M}_2(\mathbb{R})$ for the induced norm $\|\cdot\|_2$. Also recall that $\Sigma$ is the union of the segments $[r, s]$ where $r \in \mathcal{R} := \mathbf{SO}_2(\mathbb{R})$ and $s \in \mathcal{S}$, the set of orthogonal symmetries. Two distinct segments may intersect only at an extremity.

We construct the *join* $J(\mathcal{R}, \mathcal{S})$ as follows: in the construction above, we replace the segments $[r, s]$ by the lines that they span. In other words, $J(\mathcal{R}, \mathcal{S})$ is the union of the (affine) lines passing through one rotation and one orthogonal symmetry. Of course,

$$\Sigma \subset J(\mathcal{R}, \mathcal{S}).$$

(a) Prove that $J(\mathcal{R}, \mathcal{S})$ is the algebraic set defined by the equation

$$\|A\|_F^2 = 1 + (\det A)^2,$$

where $\|A\|_F^2 := \operatorname{Tr}(A^T A)$ (Frobenius norm).

(b) If $A \in J(\mathcal{R},\mathcal{S}) \setminus \mathbf{O}_2(\mathbb{R})$, show that $A$ belongs to a unique line passing through a rotation and an orthogonal symmetry (these rotation and symmetry are unique).

(c) Show that for every matrix $A \in J(\mathcal{R},\mathcal{S})$, one has $\|A\|_2 \geq 1$.

(d) Find a diffeomorphism from a neighborhood of $I_2$ to a neighborhood of $0_2$, which maps the quadrics of equation $(Y+Z)^2 = 4ZT$ onto $J(\mathcal{R},\mathcal{S})$.

243. (Continuation.)

(a) Interpret the equation of $J(\mathcal{R},\mathcal{S})$ in terms of the singular values of $A$.

(b) More generally, show that the set $\mathbf{USV}_n$ of matrices $M \in \mathbf{M}_n(\mathbb{R})$, having $s = 1$ as a singular value, is an irreducible algebraic hypersurface.

(c) Show that the unit sphere of $(\mathbf{M}_n(\mathbb{R}), \|\cdot\|_2)$ is contained in $\mathbf{USV}_n$. Therefore $\mathbf{USV}_n$ is the ZariskiZariski closure of the unit sphere.

244. Let $A, B \in \mathbf{M}_2(k)$ be given, where the characteristic of $k$ is not 2. Show that $[A,B]^2$ is a scalar matrix $\lambda I_2$. Deduce the following polynomial identity in $\mathbf{M}_2(k)$:

$$\big[[A,B]^2, C\big] = 0_2. \tag{30}$$

**Remark**. Compared with $T_4(A,B,C,D) = 0_2$ (AmitsurAmitsur & LevitzkiLevitzkii Theorem, see Exercise 289), the identity (30) has one less argument, but its degree is one more.

245. In $\mathbf{M}_n(\mathbb{C})$, prove the equivalence between

$$\det M = 0,$$

and

There exists a matrix $A \in \mathbf{M}_n(\mathbb{C})$ such that $A - zM \in \mathbf{GL}_n(\mathbb{C})$ for every $z \in \mathbb{C}$.

**Hint**: Use the rank decomposition (Theorem 6.2.2); show that $M$ is equivalent to a nilpotent matrix in $\mathbf{M}_n(\mathbb{C})$.

246. Let $k$ be a field and

$$P = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be an orthogonal matrix, with $A$ and $D$ square.

Prove that

$$\det D = \pm \det A.$$

**Hint**: multiply $P$ by

$$\begin{pmatrix} A^T & C^T \\ 0 & I \end{pmatrix}.$$

Extend this result to elements $P$ of a group $\mathbf{O}(p,q)$.

247. We endow $\mathbf{M}_n(\mathbb{C})$ with the induced norm $\|\cdot\|_2$. Let $G$ a subgroup of $\mathbf{GL}_n(\mathbb{C})$ that is contained in the open ball $B(I_n; r)$ for some $r < 2$.

(a) Show that for every $M \in G$, there exists an integer $p \geq 1$ such that $M^p = I_n$. **Hint**: The eigenvalues of elements of $G$ must be of unit modulus and semi-simple (otherwise $G$ is unbounded); they may not approach $-1$.

(b) Let $A, B \in G$ be s.t. $\mathrm{Tr}(AM) = \mathrm{Tr}(BM)$ for all $M \in G$. Prove that $A = B$. **Hint**: Choose $M$ in the subgroup spanned by $B$.

(c) Deduce that $G$ is a finite group.

(d) On the contrary, find an infinite subgroup of $\mathbf{GL}_n(\mathbb{C})$, contained in $B(I_n; 2)$.

248. Let $z_1, \dots, z_n \in \mathbb{C}$ have positive real parts. Prove that the Hermitian matrix $A$ with entries

$$a_{jk} := \frac{1}{\bar{z}_j + z_k}$$

is positive definite.

**Hint**: Look for a Hilbert space $\mathcal{H}$ and elements $f_1, \dots, f_n \in \mathcal{H}$ such that

$$a_{jk} = \langle f_j, f_k \rangle.$$

249. Let $m \in \mathbb{N}^*$ be given. We denote $P_m : A \mapsto A^m$ the $m$-th power in $\mathbf{M}_n(\mathbb{C})$. Show that the differential of $P_m$ at $A$ is given by

$$\mathrm{D}P_m(A) \cdot B = \sum_{j=0}^{m-1} A^j B A^{m-1-j}.$$

Deduce the formula

$$\mathrm{D}\exp(A) \cdot B = \int_0^1 e^{(1-t)A} B e^{tA} dt.$$

250. Let $f$, an entire analytic function, be given.

If $D = \mathrm{diag}(d_1, \dots, d_n\} \in \mathbf{M}_n(\mathbb{C})$, we define a matrix $f^{[1]}(D) \in \mathbf{M}_n(\mathbb{C})$ by

$$f^{[1]}(D)_{jk} = \begin{cases} f'(d_j), & k = j, \\ \frac{f(d_j) - f(d_k)}{d_j - d_k}, & k \neq j, \end{cases}$$

where we identify

$$\frac{f(b) - f(a)}{b - a} := f'(a),$$

if $b = a$.

- If $f = P_m$ (notations of the previous exercise), check that

$$\mathrm{D}f(D)B = f^{[1]}(D) \circ B, \tag{31}$$

where $A \circ B$ denotes the HadamardHadamard product.

- Prove that (31) holds true for every analytic function $f$ (Daletskiĭ–KreinDaletskiĭKrein Formula).

**Hint**: Use polynomial approximation.

251. (B. G. ZaslavskyZaslavsky & B.-S. TamTam, Bit-Shun.) Prove the equivalence of the following properties for real $n \times n$ matrices $A$:

**Strong PerronPerron–FrobeniusFrobenius.** The spectral radius is a simple eigenvalue of $A$, the only one of this modulus ; it is associated with positive left and right eigenvectors.

**Eventually positive matrix.** There exists an integer $N \geq 1$ such that $k \geq N$ implies $A^k > 0_n$.

252. (M. GoldbergGoldberg.) In a finite dimensional associative algebra $\mathcal{A}$ with a unit, every element has a unique minimal polynomial (prove it). Actually, *associative* may be weakened into *power-associative*: the powers $a^k$ are defined in a unique way. You certainly think that if $\mathcal{B}$ is a sub-algebra and $a \in \mathcal{B}$, then the minimal polynomial of $a$ is the same in $\mathcal{A}$ and $\mathcal{B}$. So try this ....

Here $\mathcal{A} = \mathbf{M}_n(k)$. Select a matrix $M \neq I_n, 0_n$ such that $M^2 = M$. What is its minimal polynomial (it is the one in the usual, matricial, sense) ?

Then consider

$$\mathcal{B} := M\mathcal{A}M = \{MAM\,;\, A \in \mathbf{M}_n(k)\}.$$

Check that $\mathcal{B}$ is a subalgebra of $\mathcal{A}$, and that $M$ is the unit element of $\mathcal{B}$. What is its minimal polynomial in $\mathcal{B}$ ?

The explanation of this paradox lies in the notion of *subalgebra*. The equality of minimal polynomials is guarranted if the subalgebra and the algebra have the same unit, which is not the case hereabove.

253. (C. A. BergerBerger!C. A.'s theorem, proof by C. PearcyPearcy.) Recall (see Exercise 21) that the numerical radius of $A \in \mathbf{M}_n(\mathbb{C})$ is the non-negative real number

$$w(A) := \max\{|x^*Ax|\,;\, x \in \mathbb{C}^n\}.$$

The numerical radius is a norm, which is **not** submultiplicative. We show that it satisfies however the *power inequality*.

In what follows, we use the *real part* of a square matrix

$$\mathrm{Re}M := \frac{1}{2}(M + M^*),$$

which is Hermitian and satisfies

$$x^*(\mathrm{Re}M)x = \Re(x^*Mx), \qquad \forall\, x \in \mathbb{C}^n.$$

(a) Show that $w(A) \le 1$ is equivalent to the fact that $\mathrm{Re}(I_n - zA)$ is semi-definite positive for every complex number $z$ in the open unit disc.

(b) From now on, we assume that $w(A) \le 1$. If $|z| < 1$, verify that $I_n - zA$ is non-singular. **Hint**: The numerical radius dominates the spectral one.

(c) If $M \in \mathbf{GL}_n(\mathbb{C})$ has a non-negative real part, prove that $\mathrm{Re}\,(M^{-1}) \ge 0_n$. Deduce that $\mathrm{Re}(I_n - zA)^{-1} \ge 0_n$ whenever $|z| < 1$.

(d) Let $m \ge 1$ be an integer and $\omega$ be a primitive $m$-th root of unity in $\mathbb{C}$. Check that the formula

$$\frac{1}{1 - X^m} = \frac{1}{m}\sum_{k=0}^{m-1} \frac{1}{1 - \omega^k X}$$

can be recast as a polynomial identity.

Deduce that

$$(I_n - z^m A^m)^{-1} = \frac{1}{m}\sum_{k=0}^{m-1} (I_n - \omega^k z A)^{-1},$$

whenever $|z| < 1$.

(e) Deduce from above that

$$\mathrm{Re}(I_n - z^m A^m)^{-1} \ge 0_n,$$

whenever $|z| < 1$. Going backward, conclude that for every complex number $y$ in the open unit disc, $\mathrm{Re}(I_n - yA^m) \ge 0_n$ and thus $w(A^m) \le 1$.

(f) Finally, prove the power inequality

$$w(M^m) \le w(M)^m, \qquad \forall M \in \mathbf{M}_n(\mathbb{C}),\, \forall\, m \in \mathbb{N}.$$

**Nota**: A norm which satisfies the power inequality is called a *superstable* norm. It is *stable* if there exists a finite constant $C$ such that $\|A^m\| \le C\,\|A\|^m$ for every $A \in \mathbf{M}_n(k)$ and every $m \ge 1$. Induced norms are obviously superstable.

(g) Let $\nu \ge 4$ be a given constant. Prove that $N(A) := \nu w(A)$ is a submultiplicative norm over $\mathbf{M}_n(\mathbb{C})$ (GoldbergGoldberg & TadmorTadmor). Use the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

to show that this becomes false for $\nu < 4$.

254. (J. DuncanDuncan.) We denote $\langle x, y\rangle$ the usual sesquilinear product in $\mathbb{C}^n$. To begin with, let $M \in \mathbf{GL}_n(\mathbb{C})$ be given. Let us write $M = UH = KU$ the left and right polar decomposition. We thus have $H = \sqrt{M^*M}$ and $K = \sqrt{MM^*}$.

(a) Prove that $U\sqrt{H} = \sqrt{K}\,U$.

(b) Check that
$$\langle Mx, y\rangle = \langle \sqrt{H}\,x, U^*\sqrt{K}\,y\rangle, \qquad \forall\, x, y \in \mathbb{C}^n.$$
Deduce that
$$|\langle Mx, y\rangle|^2 \leq \langle Hx, x\rangle\,\langle Ky, y\rangle.$$

(c) More generally, let a rectangular matrix $A \in \mathbf{M}_{n\times m}(\mathbb{C})$ be given. Prove the generalized CauchyCauchy–SchwarzSchwarz inequality
$$|\langle Ax, y\rangle|^2 \leq \langle \sqrt{A^*A}\,x, x\rangle\,\langle \sqrt{AA^*}\,y, y\rangle, \qquad \forall\, x, y \in \mathbb{C}^n.$$

**Hint**: Use the decompositions
$$\mathbb{C}^m = \ker A \oplus^\perp R(A^*), \qquad \mathbb{C}^n = \ker A^* \oplus^\perp R(A),$$
then apply the case above to the restriction of $A$ from $R(A^*)$ to $R(A)$.

255. (a) Let $A, B \in \mathbf{M}_n(\mathbb{C})$, with $A$ normal. If $B$ commutes with $A$, prove that $B$ commutes with $A^*$. This is B. FugledeFuglede's theorem. **Hint**: Use the spectral theorem for normal operators. See also Exercise 297

(b) More generally, let $A_1, A_2$ be normal and $B$ rectangular. Assume that $A_1B = BA_2$. Prove that $A_1^*B = BA_2^*$ (Putnam'sPutnam theorem). **Hint**: Use the matrix
$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$
and apply Fuglede's theorem.

(c) Let $A, B \in \mathbf{M}_n(\mathbb{C})$ be given. Assume that the span of $A$ and $B$ is made of normal matrices. Prove that $[A, B] = 0_n$ (H. RadjaviRadjavi & P. RosenthalRosenthal). **Hint**: Use the matrices $C = A + B$ and $D = A + iB$ to prove $[A^*, B] = 0_n$, then apply Fuglede's theorem.

256. Let $k$ be a field of characteristic zero. We consider two matrices $A, B \in \mathbf{M}_n(k)$ satisfying
$$[[A, B], A] = 0_n.$$
In other words, $\Delta^2 B = 0_n$, where
$$\Delta : M \mapsto [A, M]$$
is a linear operator over $\mathbf{M}_n(k)$.

(a) Check that $\Delta$ is a *derivation*:
$$\Delta(MN) = (\Delta M)N + M(\Delta N).$$
Therefore $\Delta$ obeys a LeibnizLeibniz formula.

(b) Deduce that for every $m \geq 1$, one has

$$\Delta^m(B^m) = m!\,(\Delta B)^m. \tag{32}$$

(c) Deduce from (32) that $\Delta^m(B^j) = 0_n$ whenever $m > j$.

(d) Using the CayleyCayley–HamiltonHamilton Theorem, infer that $\Delta^n(B^n) = 0_n$.

(e) Back to (32), establish that $[A, B]$ is nilpotent (N. JacobsonJacobson).

(f) Application: let $A \in \mathbf{M}_n(\mathbb{C})$ satisfy $[[A, A^*], A] = 0_n$. Prove that $A$ is normal.


Leibniz, *Gottfried* (Romania)

**Nota**: If $k = \mathbb{C}$ or $\mathbb{R}$, one can deduce from (32) and Proposition 4.4.1 that $\rho(\Delta B) = 0$, which is the required result.

Left: **Gottfried Leibniz.**

257. (Thanks to A. GuionnetGuionnet) Let $n > m\,(\geq 1)$ be two integers. If $V \in \mathbf{U}_n$, we decompose blockwise

$$V = \begin{pmatrix} P & Q \\ R & T \end{pmatrix},$$

with $P \in \mathbf{M}_n(\mathbb{C})$. Notice that $T$ is the matrix of a contraction.

If $z \in \mathbb{C}$, of unit modulus, is not an eigenvalue of $T^*$, we define

$$W(z) := P + zQ(I_{n-m} - zT)^{-1}R \in \mathbf{M}_m(\mathbb{C}).$$

Show that $W(z)$ is unitary.

If $n = 2m$, build other such rational maps from dense open subsets of $\mathbf{U}_n$ to $\mathbf{U}_m$.

258. It seems that I have taken for granted the following fact:

If $H \in \mathbf{HPD}_n$ and $h \in \mathbf{H}_n$ are given, then the product $Hh$ is diagonalizable with real eigenvalues. The list of signs $(0, \pm)$ of the eigenvalues of $Hh$ is the same as for those of $h$.

Here is a proof:

(a) Show that $Hh$ is similar to $\sqrt{H}\, h\sqrt{H}$.

(b) Use Sylvester's inertia for the Hermitian form associated with $h$.

259. (L. MirskyMirsky.)

For a Hermitian matrix $H$ with smallest and largest eigenvalues $\lambda_\pm$, we define the *spread*

$$s(H) := \lambda_+ - \lambda_-.$$

Show that

$$s(H) = 2 \max |x^* H y|,$$

where the supremum is taken over the pairs of unit vectors $x, y \in \mathbb{C}^n$ that are orthogonal: $x^* y = 0$.

260. For a given $A \in \mathbf{GL}_n(\mathbb{C})$, we form $M := A^{-1} A^*$. Let $(\lambda, x)$ be an eigen-pair: $Mx = \lambda x$.

(a) Show that either $|\lambda| = 1$, or $x^* A x = 0$.

(b) Let us assume that $x^* A x \neq 0$. Prove that $\lambda$ is a semi-simple eigenvalue.

(c) Find an $A \in \mathbf{GL}_2(\mathbb{C})$ such that the eigenvalues of $M$ are not on the unit circle.

(d) Show that there exists matrices $A \in \mathbf{GL}_n(\mathbb{C})$ without a *bilateral polar decomposition* $A = HQH$, where as usual $Q \in \mathbf{U}_n$ and $H \in \mathbf{HPD}_n$.

261. If $A \in \mathbf{M}_n(\mathbb{C})$ is given, we denote $s(A) \in \mathbb{R}^n_+$ the vector whose components $s_1 \leq s_2 \leq \cdots \leq s_n$ are the singular values of $A$.

**Warning**. This exercise involves two norms on $\mathbf{M}_n(\mathbb{C})$, namely the operator norm $\|\cdot\|_2$ and the Schur–Frobenius norm $\|\cdot\|_F$.

(a) Using von Neumann'svonneu@von Neumann inequality (16), prove that for every matrices $A, B \in \mathbf{M}_n(\mathbb{C})$, we have

$$\|s(A) - s(B)\|_2 \leq \|A - B\|_F.$$

(b) Deduce the following property. For every semi-definite positive Hermitian matrix $H$, the projection (with respect to the distance $d(A, B) := \|A - B\|_F$) of $I_n + H$ over the unit ball of $\|\cdot\|_2$ is $I_n$.

262. Let $A \in \mathbf{GL}_n(\mathbb{C})$ be given, and $UDV^*$ be an SVD of $A$. Identify the factors $Q$ and $H$ of the polar decomposition of $A$, in terms of $U, V$ and $D$.

Let us form the sequence of matrices $X_k$ with the rule

$$X_0 = A, \qquad X_{k+1} := \frac{1}{2}(X_k + X_k^{-*}).$$

Show that $X_k$ has the form $UD_kV^*$ with $D_k$ diagonal, real and positive. Deduce that

$$\lim_{k \to +\infty} X_k = Q$$

and that the convergence is quadratic.

263. Let $H \in \mathbf{H}_n$ be positive semi-definite.

(a) Prove that $H \exp(-H)$ is Hermitian and satisfies the inequality

$$H \exp(-H) \leq \frac{1}{e} I_n.$$

(b) Deduce that the solutions of the ODE

$$\frac{dx}{dt} + Hx = 0$$

satisfy

$$\|Hx(t)\|_2 \leq \frac{1}{et} \|x(0)\|_2.$$

**Nota**. This result extends to evolution equations in Hilbert spaces. For instance, the solutions of the *heat equation* (a partial differential equation)

$$\frac{\partial u}{\partial t} = \Delta_x u, \qquad x \in \mathbb{R}^d,\, t \geq 0$$

satisfy the inequality

$$\int_{\mathbb{R}^d} |\Delta_x u(x,t)|^2 dx \leq \frac{1}{et} \int_{\mathbb{R}^d} |u(x,0)|^2 dx.$$

264. Let $A, B \in \mathbf{HPD}_n$ be given. Prove that for every vector $h \in \mathbb{C}^n$, we have

$$h^*(A \sharp B)h \leq \sqrt{h^* A h}\, \sqrt{h^* B h},$$

where $A \sharp B$ is the geometric mean of $A$ and $B$ (see Exercise 198).

**Hint**: Use the explicit formula

$$A \sharp B = A^{1/2} \left(A^{-1/2} B A^{-1/2}\right)^{1/2} A^{1/2}.$$

265. (BatailleBataille.) Let $A, B \in \mathbf{M}_n(k)$ be such that $A^2B = A$. We assume moreover that $A$ and $B$ have the same rank.

(a) Show that $\ker A = \ker B$.

(b) Prove $BAB = B$.

(c) Show that $AB$ and $A$ have the same rank and deduce $k^n = R(B) \oplus \ker A$.

(d) Finally, prove that $B^2A = B$.

266. Let us denote by $\mathcal{N}$ the set of nilpotent matrices in $\mathbf{M}_n(k)$. We also denote $G_n$ the set of polynomials $p \in k[X]$ of degree less than $n$, such that $p(0) = 0$ and $p'(0) \neq 0$. In other words, $p \in G_n$ if and only if

$$p(X) = a_1 X + \cdots + a_{n-1} X^{n-1}, \qquad a_1 \neq 0.$$

For $p, q \in G_n$, we define $p \circ q$ as the unique element $r \in G_n$ such that

$$r(X) \equiv p(q(X)), \qquad \mod X^n.$$

(a) Verify that $(G_n, \circ)$ is a group, and that $(p, N) \mapsto p(N)$ is a group action over $\mathcal{N}$.

(b) Apply this to prove that if $k$ has characteristic 0, then for every $j = 1, 2, \dots$ and every $N \in \mathcal{N}$, the matrix $I_n + N$ admits a $j$-th-root in $I_n + \mathcal{N}$, and only one in this class.

(c) Denote this $j$-th-root by $(I_n + N)^{1/j}$. If $k = \mathbb{C}$, prove that

$$\lim_{j \to +\infty} (I_n + N)^{1/j} = I_n.$$

267. (M. CavachiCavachi, Amer. Math. Monthly **191** (2009)) We consider a matrix $A \in \mathbf{GL}_n(\mathbb{Z})$ with the property that for every $k = 1, 2, \dots$, there exists a matrix $A_k \in \mathbf{M}_n(\mathbb{Z})$ such that $A = (A_k)^k$. Our target is to prove that $A = I_n$.

(a) Show that the distance of the spectrum of $A_k$ to the unit circle tends to zero as $k \to +\infty$. Deduce that the sequence of characteristics polynomials of the $A_k$'s takes finitely many values.

(b) Prove that there exists two integers $(1 \leq) j | k$ such that $j \neq k$, while $A_j$ and $A_k$ have the same characteristic polynomials. Show that their roots actually belong to the unit circle, and that they are roots of unity, of degree less than or equal to $n$.

(c) Show that in the previous question, one may choose $j$ and $k$ such that $k$ is divisible by $n!j$. Deduce that the spectrum of $A_j$ reduces to $\{1\}$.

(d) Verify that, with the terminology of Exercise 266, $A$ belongs to $I_n + \mathcal{N}$ and $A_j = A^{1/j}$.

(e) Verify that in the previous question, one may choose $j$ arbitrarily large.

(f) For $j$ as in the previous question and large enough, show that $A_j = I_n$. Conclude.

268. For a subgroup $G$ of $\mathbf{U}_n$, prove the equivalence of the three properties

**(P1)** $G$ is finite,

**(P2)** there exists a $k \geq 1$ such that $M^k = I_n$ for every $M \in G$,

**(P3)** the set $\{\operatorname{Tr} M \mid M \in G\}$ is finite.

More precisely,

(a) Prove that $(P1) \Longrightarrow (P2) \Longrightarrow (P3)$.

(b) Let us assume $(P3)$. Choose a basis $\{M_1, \dots, M_r\}$ of the subspace spanned by $G$, made of elements of $G$. Every element $M$ of $G$ writes

$$M = \sum_{j=1}^{r} \alpha_j M_j.$$

Express the vector $\vec{\alpha}$ in terms of the vector of components $\operatorname{Tr}(MM_j^*)$. Deduce that $\vec{\alpha}$ can take only finitely many values. Whence $(P1)$.

(c) The assumption that $G \subset \mathbf{U}_n$ is crucial. Find an infinite subgroup $T$ of $\mathbf{GL}_n(\mathbb{C})$, in which the trace of every element equals $n$. Thus $T$ satisfies $(P3)$ though not $(P1)$.

269. This is about the numerical range, defined in Exercise 21. We recall that the numerical range $\mathcal{H}(A)$ of a matrix $A \in \mathbf{M}_n(\mathbb{C})$ is a convex compact set, which contains the spectrum of $A$.

(a) Let $\lambda \in \mathbb{C}$ be given, such that $\rho(A) < |\lambda|$. Show that there exists a conjugate $P^{-1}AP$ such that $\lambda \notin \mathcal{H}(P^{-1}AP)$. **Hint**: Use the HouseholderHouseholder Theorem.

(b) Use the case above to show Hildebrant'sHildebrant Theorem: the intersection of $\mathcal{H}(P^{-1}AP)$, as $P$ runs over $\mathbf{GL}_n(\mathbb{C})$, is precisely the convex hull of the spectrum of $A$. **Hint**: separate this convex hull from an exterior point by a circle.

270. (Suggested by L. BergerBerger!L..) Here is a purely algebraic way to solve the problem raised in Exercise 267.

(a) Show that $\det A = 1$.

(b) Let $p$ be a prime number. Show that $(A_k)^{o(p,n)} \equiv I_n, \mod p$, where $o(p,n)$ is the order of $\mathbf{GL}_n(\mathbb{Z}/p\mathbb{Z})$.

(c) Deduce that $A \equiv I_n, \mod p$. Conclude.

271. Given $H \in \mathbf{HPD}_n(\mathbb{R})$, prove the formula

$$\frac{\pi^n}{\det H} = \int_{\mathbb{C}^n} e^{-z^* H z} dz,$$

where we identify $\mathbb{C}^n$ to $\mathbb{R}^{2n}$. **Hint**: One may split $z = x + iy$, $H = S + iA$ where $S \in \mathbf{SPD}_n$ and $A$ is skew-symmetric, and apply Exercise 75 to the integral with respect to $x$ first, then to the integral with respect to $y$.

Extend the formula above to non-Hermitian matrices such that $H+H^*$ is positive definite. **Hint**: Use holomorphy.

272. (R. BellmanBellman.) Let $A_1, \dots, A_r \in \mathbf{M}_{n\times m}(\mathbb{C})$ be strict contractions, meaning that $A_j^* A_j < I_m$. According to Exercise 221, this implies that for every pair $1 \le i, j \le r$, the matrix $I_m - A_i^* A_j$ is non-singular.

The purpose of this exercise is to prove that the Hermitian matrix $B$ whose $(i,j)$ entry is

$$\frac{1}{\det(I_m - A_i^* A_j)}$$

is positive definite. This generalizes Loo-Keng Hua'sHua, Loo-Keng inequality

(a) Let $z \in \mathbb{C}^n$ be given. Show that the matrix of entries $z^* A_i^* A_j z$ is positive semi-definite.

(b) With the help of Exercise 21, Chapter 3, prove that the matrix of entries $(z^* A_i^* A_j z)^\ell$ is positive semi-definite for every $\ell \in \mathbb{N}$. Deduce the same property for the matrix of entries $\exp(z^* A_i^* A_j z)$.

(c) Express the matrix $B$ as an integral, with the help of Exercise 271. Conclude.

273. (See also Exercise 201.) Let $A, B, \Gamma, \Delta \in \mathbf{M}_n(\mathbb{R})$ be given. We consider the transformation over $\mathbf{M}_n(\mathbb{C})$ (we warn the reader that we manipulate both fields $\mathbb{R}$ and $\mathbb{C}$)

$$T \mapsto T' := (A + T\Gamma)^{-1}(B + T\Delta).$$

In the sequel, we shall use the $2n \times 2n$ matrix

$$F := \begin{pmatrix} A & B \\ \Gamma & \Delta \end{pmatrix}.$$

In order to ensure that $T'$ is defined for every $T$ but the elements of a dense open subset of $\mathbf{M}_n(\mathbb{C})$, we assume *a priori* that $F$ is non-singular.

(a) We are interested only in those transformations that map symmetric matrices $T$ onto symmetric matrices $T'$. Show that this is equivalent to the identity

$$FJF^T = \lambda J, \qquad J := \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix},$$

for some $\lambda \in \mathbb{R}^*$.

(b) Prove the identity

$$T' - \overline{T'} = \lambda(A + T\Gamma)^{-1}(T - \overline{T})(A + T\Gamma)^{-*}.$$

Deduce that if $\lambda > 0$, and if the imaginary part of $T$ is positive definite, then the imaginary part of $T'$ is positive definite. **Nota**: Complex symmetric matrices with positive definite imaginary form the *Siegel*Siegel *domain* $\mathcal{H}_n$.

(c) Let $T \in \mathbf{M}_n(\mathbb{C})$ be given. Check that

$$X := \begin{pmatrix} I_n & T \\ I_n & \overline{T} \end{pmatrix}$$

is non-singular if, and only if, the imaginary part of $T$ is non-singular. Write in closed form its inverse.

(d) From now on, we assume that $\lambda > 0$. We are looking for fixed points ($T' = T$) of the transformation above, especially whether there exists such a fixed point in the Siegel domain. Remark that we may assume $\lambda = 1$, and therefore $F$ is a symplectic matrix.

i. Let $T \in \mathbf{Sym}_n(\mathbb{C})$ be a fixed point. We define $X$ as above. Show that

$$XFX^{-1} = \begin{pmatrix} N & 0_n \\ 0_n & \overline{N} \end{pmatrix}$$

for some $N \in \mathbf{M}_n(\mathbb{C})$.

ii. We assume now that this fixed point is in the Siegel domain. Find a matrix $K \in \mathbf{M}_n(\mathbb{C})$ such that

$$Y := \begin{pmatrix} K & 0_n \\ 0_n & \overline{K} \end{pmatrix} X$$

is symplectic. Show that $YFY^{-1}$ is symplectic and has the form

$$\begin{pmatrix} M & 0_n \\ 0_n & \overline{M} \end{pmatrix}.$$

iii. Show that $M$ is unitary. Deduce a necessary condition for the existence of a fixed point in the Siegel domain: the eigenvalues of $F$ have unit modulii.
**Nota**: Frobenius showed that this existence is equivalent to i) the eigenvalues of $F$ have unit modulii, ii) $F$ is diagonalizable. The uniqueness of the fixed point in $\mathcal{H}_n$ is much more involved and was solved completely by FrobeniusFrobenius. Let us mention at least that if $F$ has simple eigenvalues, then uniqueness holds true. This existence and uniqueness problem was posed by KroneckerKronecker. See a detailed, albeit non-technical account of this question and its solution, in T. Hawkins'Hawkins article in Arch. Hist. Exact Sci. (2008) **62**:23–57.

274. We are interested in matrices $M \in \mathbf{Sym}_3(\mathbb{R})$ with $m_{jj} = 1$ and $|m_{ij}| \leq 1$ otherwise. In particular, there exist angles $\theta_k$ such that $m_{ij} = \cos\theta_k$ whenever $\{i, j, k\} = \{1, 2, 3\}$.

(a) Prove that $\det M = 0$ if, and only if, there exists signs such that

$$\pm\theta_1 \pm \theta_2 \pm \theta_3 \in 2\pi\mathbb{Z}.$$

(b) We give ourselves a non-zero vector $x \in \mathbb{R}^3$. We ask whether there exists a matrix $M$ as above (obviously with $\det M = 0$) such that $Mx = 0$.

i. Prove the necessary condition that $|x_i| \leq |x_k| + |x_j|$ for every pairwise distinct $i, j, k$.

ii. Prove that this condition is also sufficient. **Hint**: Reduce the problem to the case where $x$ is non-negative. Then there exists a triangle whose edges have lengths $x_1, x_2, x_3$.

275. (F. HollandHolland) Let $A_1, \dots, A_r \in \mathbf{HPD}_2$ be given. The eigenvalues of $A_j$ are denoted $\lambda_1(A_j) \leq \lambda_2(A_j)$.

Let $Q_1, \dots, Q_r \in \mathbf{U}_2$ be given. Prove the inequality

$$\det\left(\sum_{j=1}^r Q_j^* A_j Q_j\right) \geq \left(\sum_{j=1}^r \lambda_1(A_j)\right)\left(\sum_{j=1}^r \lambda_2(A_j)\right).$$

**Hint**: Use the WeylWeyl Inequalities for the eigenvalues of $\sum_{j=1}^r Q_j^* A_j Q_j$.

276. (WiegmannWiegmann) Let $M$ be a complex, normal matrix.

(a) If the diagonal entries of $M$ are its eigenvalues (with equal multiplicities), show that $M$ is diagonal. **Hint**: compute the FrobeniusFrobenius norm of $M$.

(b) More generally, consider a block form of $M$, with diagonal blocks $M_{\ell\ell}$, $\ell = 1, \dots, r$. Let us assume that the union of the spectra of the diagonal blocks equal the spectrum of $M$, with the multiplicities of equal eigenvalues summing up to the multiplicity as an eigenvalue of $M$. Prove that $M$ is block-diagonal.

277. Let two matrices $A, B \in \mathbf{M}_n(k)$ be given. We say that $(A, B)$ enjoys the *property* **L** if the eigenvalues of $\lambda A + \mu B$ have the form $\lambda\alpha_j + \mu\beta_j$ $(j = 1, \dots, n)$ for some fixed scalars $\alpha_j, \beta_j$. Necessarily, these scalars are the respective eigenvalues of $A$ and $B$.

(a) If $A$ and $B$ commute, show that $(A, B)$ enjoys property **L**.

(b) Let us assume that $k$ has characteristic zero, that $A$ is diagonalizable and that $(A, B)$ enjoys property **L**. Up to a conjugation (applied simultaneously to $A$ and $B$), we may assume that $A$ is diagonal, of the form $\mathrm{diag}\{a_1 I_{m_1}, \dots, a_r I_{m_r}\}$ with $a_1, \dots, a_r$ pairwise distinct. Let us write $B$ blockwise, with the diagonal blocks $B_{\ell\ell}$ of size $m_\ell \times m_\ell$.

Prove that (MotzkinMotzkin & TausskyTaussky)

$$\det(XI_n - \lambda A - \mu B) = \prod_{\ell=1}^r \det((X - \lambda a_\ell) I_{m_\ell} - \mu B_{\ell\ell}).$$

**Hint**: Isolate one diagonal block of $XI_n - \lambda A - \mu B$. Then compute the determinant with the help of Schur'sSchur complement formula. Then look at its expansion about the point $(a_\ell, 1, 0)$. One may simplify the analysis by translating $a_\ell$ to zero.

(c) We now assume that $k = \mathbb{C}$, and $A$ and $B$ are normal matrices. If $(A, B)$ enjoys property **L**, prove that $A$ and $B$ commute (WiegmannWiegmann.) **Hint**: Use Motzkin & Taussky' result, plus Exercise 276.

278. (a) Let $\mathbf{u}_1, \dots, \mathbf{u}_{n-1} \in k^n$ and $\mathbf{x}_0, \dots, \mathbf{x}_n \in k^n$ be given. Prove that

$$\sum_{\ell=0}^{n} \det(\mathbf{u}_1, \dots, \mathbf{u}_{n-1}, \mathbf{x}_\ell) \det \widehat{X_\ell} = 0,$$

where $\widehat{X_\ell}$ denotes the matrix whose columns are $\mathbf{x}_0, \dots, \mathbf{x}_n \in k^n$, $\mathbf{x}_\ell$ being omitted.

(b) Deduce the following formula for matrices $M, N \in \mathbf{M}_n(k)$:

$$\det(MN) = \sum_{k=1}^{n} \det M_k^N \det N_M^k,$$

where $M_k^N$ denotes the matrix obtained from $M$ by replacing its last column by the $k$-th column of $N$, and $N_k^M$ denotes the matrix obtained from $N$ by replacing its $k$-th column by the last column of $M$.

(c) More generally, prove Sylvester'sSylvester Lemma: given $1 \le j_1 < \cdots < j_r \le n$, then $\det(MN)$ equals the sum of those products $\det M' \det N'$ where $M'$ is obtained by exchanging $r$ columns of $N$ by the columns of $M$ of indices $j_1, \dots, j_r$. There are $\begin{pmatrix} n \\ r \end{pmatrix}$ choices of the columns of $N$, and the exchange is made keeping the order between the columns of $M$, respectively of $N$.

279. Recall that the HadamardHadamard product of two matrices $A, B \in \mathbf{M}_{p\times q}(k)$ is the matrix $A \circ B \in \mathbf{M}_{p\times q}(k)$ of entries $a_{ij}b_{ij}$ with $1 \le i \le p$ and $1 \le j \le q$. If $A \in \mathbf{M}_n(k)$ is given blockwise

$$A = \begin{pmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

and if $a_{11}$ is invertible, then the SchurSchur complement $A_{22} - A_{21}a_{11}^{-1}A_{12}$ is denoted $A|a_{11}$ and we have the formula $\det A = a_{11}\det(A|a_{11})$.

(a) Let $A, B \in \mathbf{M}_n(k)$ be given blockwise as above, with $a_{11}, b_{11} \in k^*$ (and therefore $A_{22}, B_{22} \in \mathbf{M}_{n-1}(k)$.) Prove that

$$(A \circ B)|a_{11}b_{11} = A_{22} \circ (B|b_{11}) + (A|a_{11}) \circ E, \qquad E := \frac{1}{b_{11}} B_{21}B_{12}.$$

(b) From now on, $A$ and $B$ are positive definite Hermitian matrices. Show that

$$\det(A \circ B) \geq a_{11} b_{11} \det\left(A_{22} \circ (B|b_{11})\right).$$

Deduce Oppenheim'sOppenheim (Sir A.) Inequality:

$$\det(A \circ B) \geq \left(\prod_{i=1}^{n} a_{ii}\right) \det B.$$

**Hint**: Argue by induction over $n$.

(c) In case of equality, prove that $B$ is diagonal.

(d) Verify that Oppenheim's Inequality is valid when $A$ and $B$ are only positive semi-definite.

(e) Deduce that

$$\det(A \circ B) \geq \det A \det B.$$

**See** Exercise 285 for an improvement of Oppenheim's Inequality.

280. Let $\phi : \mathbf{Sym}_m(\mathbb{R}) \to \mathbf{Sym}_n(\mathbb{R})$ (or as well $\phi : \mathbf{H}_m \to \mathbf{H}_n$) be linear. We say that $\phi$ is *positive* if $A \geq 0_m$ implies $\phi(A) \geq 0_n$, and that it is *unital* if $\phi(I_m) = I_n$.

(a) Let $\phi$ be positive and unital. If $A$ is positive semi-definite (resp. definite), prove that

$$\begin{pmatrix} I_n & \phi(A) \\ \phi(A) & \phi(A^2) \end{pmatrix} \geq 0_n$$

or, respectively,

$$\begin{pmatrix} \phi(A^{-1}) & I_n \\ I_n & \phi(A) \end{pmatrix} \geq 0_n$$

**Hint**: Use a spectral decomposition of $A$.

(b) Deduce that $\phi(A)^2 \leq \phi(A^2)$ or, respectively, $\phi(A)^{-1} \leq \phi(A^{-1})$.

281. (Exercises 281 to 284 are taken from P. HalmosHalmos, *Linear algebra. Problem book*, MAA 1995.) Let $A, B \in \mathbf{M}_n(\mathbb{C})$ be given. We prove here that if $A$, $B$ and $AB$ are normal, then $BA$ is normal too.

(a) Let us define $C := [B, A^*A]$. Expand $C^*C$ and verify that $\operatorname{Tr} C^*C = 0$. Deduce that $B$ commutes with $A^*A$.

(b) Let $QH$ be the polar decomposition of $A$. Recall that, since $A$ is normal, $Q$ and $H$ commute. Prove that $B$ commutes with $H$. **Hint**: $H$ is a polynomial in $H^2$.

(c) Deduce the formula $Q^*(AB)Q = BA$. Conclude.

282. Let $A, B \in \mathbf{M}_n(\mathbb{R})$ be unitary similar (in $\mathbf{M}_n(\mathbb{C})$) to each other:

$$\exists U \in \mathbf{U}_n \quad \text{s.t.} \quad AU = UB.$$

(a) Show that there exists an invertible linear combination $S$ of the real and imaginary parts of $U$, such that $AS = SB$ and $A^*S = SB^*$, simultaneously.

(b) Let $QH$ be the polar decomposition of $S$. Prove that $A$ and $B$ are actually orthogonally similar:

$$AQ = QB.$$

283. Let $A_1, \dots, A_r \in \mathbf{M}_n(k)$ and $p_1, \dots, p_r \in k[X]$ be given. Prove that there exists a polynomial $p \in k[X]$ such that

$$p(A_j) = p_j(A_j), \qquad \forall j = 1, \dots, r.$$

**Hint**: This is a congruence problem in $k[X]$, similar to the *Chinese remainder lemma*.

284. Prove that for every matrix $A \in \mathbf{M}_n(\mathbb{R})$ with $n \geq 2$, there exists an invariant plane $\Pi \subset \mathbb{R}^n$ ($\dim \Pi = 2$ and $A\Pi \subset \Pi$).

285. (S. FallatFallat & C. JohnsonJohnson!Charles R..) Let $A$ and $B$ be $n \times n$, Hermitian positive semi-definite matrices. According to Exercise 279, both

$$\left(\prod_{i=1}^n a_{ii}\right) \det B \quad \text{and} \quad \left(\prod_{i=1}^n b_{ii}\right) \det A$$

admit the upper bound

$$\det A \circ B.$$

Thanks to the HadamardHadamard inequality, they also have the lower bound $\det A \det B$. In order to find a more accurate inequality than Oppenheim'sOppenheim (Sir A.), as well as a symmetric one, it is thus interesting to compare

$$\det A \circ B + \det A \det B \qquad vs \qquad \left(\prod_{i=1}^n a_{ii}\right) \det B + \left(\prod_{i=1}^n b_{ii}\right) \det A.$$

Using induction over the size $n$, we shall indeed prove that the latter is less than or equal to the former.

(a) If either $n = 2$, or $B$ is diagonal, or

$$B = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix},$$

show that actually

$$\det A \circ B + \det A \det B = \left(\prod_{i=1}^n a_{ii}\right) \det B + \left(\prod_{i=1}^n b_{ii}\right) \det A.$$

(b) We turn to the general case. To begin with, it is enough to prove the inequality for positive definite matrices $A, B$.
In the sequel, $A$ and $B$ will thus be positive definite.

(c) We decompose blockwise

$$A = \begin{pmatrix} a_{11} & x^* \\ x & A' \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & y^* \\ y & B' \end{pmatrix}.$$

Let $F := \mathbf{e}_1 \mathbf{e}_1^T$ be the matrix whose only non-zero entry is $f_{11} = 1$. Prove that

$$\tilde{A} := A - \frac{\det A}{\det A'} F$$

is positive semi-definite (actually, it is singular). **Hint**: Use Schur'sSchur complement formula.

(d) Apply the Oppenheim inequality to estimate $\det \tilde{A} \circ B$.

(e) Using Exercise 2, deduce the inequality

$$\det A \circ B \geq \left( \prod_{i=1}^{n} a_{ii} \right) \det B + \frac{\det A}{\det A'} \left( b_{11} \det A' \circ B' - \left( \prod_{i=2}^{n} a_{ii} \right) \det B \right).$$

(f) Apply the induction hypothesis. Deduce that

$$\begin{aligned} \det A \circ B \quad \geq \quad & \left( \prod_{i=1}^{n} a_{ii} \right) \det B + \left( \prod_{i=1}^{n} b_{ii} \right) \det A - \det A \det B \\ & + \frac{\det A}{\det A'} (\det B - b_{11} \det B') \left( \det A' - \prod_{i=2}^{n} a_{ii} \right). \end{aligned}$$

Conclude.

286. (S. SedovSedov.) Let $x$ be an indeterminate. For $n \geq 1$, let us define the *snail* matrix $S_n(x)$ by

$$S_n(x) := \begin{pmatrix} 1 & x & \cdots & x^{n-1} \\ x^{4n-5} & \cdots & & x^n \\ \vdots & & & \vdots \\ x^{3n-3} & \cdots & \cdots & x^{2n-2} \end{pmatrix},$$

where the powers $1, x, \ldots, x^{n^2-1}$ are arranged following an inward spiral, clockwise.

(a) Prove that for $n \geq 3$,

$$\det S_n(x) = x^{4n^2-9n+12} \left(1 - x^{4n-6}\right) \left(1 - x^{4n-10}\right) \det S_{n-2}(x).$$

**Hint**: Make a combination of the first and second rows. Develop. Then make a combination of the last two rows.

(b) Deduce the formula

$$\det S_n(x) = (-1)^{(n-1)(n-2)/2} q^{f(n)} \prod_{k=0}^{n-2} \left(1 - x^{4k+2}\right),$$

where the exponent is given by

$$f(n) = \frac{1}{3}(2n^3 - 6n^2 + 13n - 6).$$

287. Let $p \geq 2$ be a prime number. We recall that $\mathbb{F}_p$ denotes the field $\mathbb{Z}/p\mathbb{Z}$. Let $A \in \mathbf{M}_n(\mathbb{F}_p)$ be given. Prove that $A$ is diagonalizable *within* $\mathbf{M}_n(\mathbb{F}_p)$ if, and only if $A^p = A$.

**Hint**: The polynomial $X^p - X$ vanishes identically over $\mathbb{F}_p$ and its roots are simple.

288. Let $\mathbf{R}$ be a commutative ring containing the rationals (was it a field, it would be of characteristic zero), and let $A \in \mathbf{M}_n(\mathbf{R})$ be given. Let us assume that $\mathrm{Tr}(A) = \mathrm{Tr}(A^2) = \cdots = \mathrm{Tr}(A^n) = 0$. Prove that $A^n = 0_n$.


**Hint**: Begin with the case where $\mathbf{R}$ is a field and use the Newton'sNewton sums.

Left: **Isaac Newton.**

Newton, *Isaac*!(Paraguay)

289. We present the proof by S. RossetRosset!Shmuel of the Amitsur–LevitzkiAmitsurLevitzki Theorem (for users of the 2nd edition, it is the object of Section 4.4).

We need the concept of exterior algebra (see Exercise 146). Let $\{e^1, \dots, e^{2n}\}$ be a basis of $k^{2n}$. Then the monomials $e^{j_1} \wedge \cdots \wedge e^{j_r}$ with $j_1 < \cdots < j_r$ (the sequence may be empty) form a basis of the exterior algebra $\Lambda(k^{2n})$. This is an associative algebra, with the property that for two vectors $e, f \in k^{2n}$, one has $e \wedge f = -f \wedge e$.

We denote by $\mathbf{R}$ the subalgebra spanned by the 2-forms $e^i \wedge e^j$.

(a) Check that $\mathbf{R}$ is a commutative sub-algebra.

(b) If $A_1, \dots, A_{2n} \in \mathbf{M}_n(k)$, let us define

$$A := A_1 e^1 + \cdots + A_{2n} e^{2n} \in \mathbf{M}_n(\Lambda(k^{2n})).$$

i. Show that for every $\ell \geq 1$,

$$A^\ell = \sum_{i_1 < \cdots < i_\ell} \mathcal{S}_\ell(A_{i_1}, \ldots, A_{i_\ell}) e^{i_1} \wedge \cdots \wedge e^{i_\ell},$$

where the *standard polynomial* $\mathcal{S}_\ell$ in non-commutative indeterminates $X_1, \ldots, X_\ell$ is defined by

$$\mathcal{S}_\ell(X_1, \ldots, X_\ell) := \sum_\sigma \epsilon(\sigma) X_{\sigma(1)} \cdots X_{\sigma(\ell)}.$$

Hereabove, the sum runs over the permutations of $\{1, \ldots, \ell\}$, and $\epsilon(\sigma)$ denotes the signature of $\sigma$.

ii. When $\ell$ is even, show that $\operatorname{Tr} \mathcal{S}_\ell(B_1, \ldots, B_\ell) = 0$, for every $B_1, \ldots, B_\ell \in \mathbf{M}_n(k)$.

iii. If $k$ has characteristic zero, deduce that $A^{2n} = 0_n$. **Hint**: Use the previous exercise.

(c) Whence the Theorem of Amitsur & Levitzki: for every $A_1, \ldots, A_{2n} \in \mathbf{M}_n(k)$, one has

$$\mathcal{S}_{2n}(A_1, \ldots, A_{2n}) = 0_n.$$

**Hint**: First assume that $k$ has characteristic zero. Then use the fact that $\mathcal{S}_{2n}$ is a polynomial with integer coefficients.

(d) Prove that $\mathcal{S}_{2n-1}(A_1, \ldots, A_{2n-1})$ does not vanish identically over $\mathbf{M}_n(k)$. **Hint**: Specialize with the matrices $E^{11}, E^{12}, \ldots, E^{1n}, E^{21}, \ldots, E^{n1}$, where $E^{m\ell}$ is the matrix whose $(i,j)$-entry is one if $i = m$ and $j = \ell$, and zero otherwise.

290. Prove the following formula for complex matrices:

$$\log \det(I_n + zA) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k} \operatorname{Tr}(A^k) z^k.$$

**Hint**: Use an analogous formula for $\log(1 + az)$.

291. Let $A, B \in \mathbf{H}_n$ be such that

$$\det(I_n + xA + yB) \equiv \det(I_n + xA) \det(I_n + yB), \qquad \forall x, y \in \mathbb{R}.$$

(a) Show that for every $k \in \mathbb{N}$,

$$\operatorname{Tr}((xA + yB)^k) \equiv x^k \operatorname{Tr}(A^k) + y^k \operatorname{Tr}(B^k).$$

**Hint**: Use Exercise 290.

(b) Infer

$$2 \operatorname{Tr}(ABAB) + 4 \operatorname{Tr}(A^2 B^2) = 0.$$

(c) Deduce that $AB = 0$. This is the theorem of CraigCraig & SakamotoSakamoto. **Hint**: Set $X := AB$. Use the fact that $(X + X^*)^2 + 2X^*X$ is semi-positive definite.

292. We recall that the function $H \mapsto \theta(H) := -\log\det H$ is convex over $\mathbf{HPD}_n$. We extend $\theta$ to the whole of $\mathbf{H}_n$ by posing $\theta(H) = +\infty$ otherwise. This extension preserves the convexity of $\theta$. We wish to compute the LegendreLegendre transform

$$\theta^*(K) := \sup_{H\in\mathbf{H}_n} \{\mathrm{Tr}(HK) - \theta(H)\}.$$

(a) Check that $\theta^*(U^*KU) = \theta^*(K)$ for every unitary $U$.

(b) Show that

$$\theta^*(K) \geq -n - \log\det(-K).$$

In particular, $\theta^*(K)$ is infinite, unless $K$ is negative definite. **Hint**: Use diagonal matrices only.

(c) Show that, for every positive definite Hermitian matrices $H$ and $H'$, one has

$$\log\det H + \log\det H' + n \leq \mathrm{Tr}(HH').$$

**Hint**: $HH'$ is diagonalizable with positive real eigenvalues.

(d) Conclude that $\theta^*(K) \equiv -n + \theta(-K)$.

(e) Let $\chi(H) := \theta(H) - n/2$. Verify that $\chi^*(H) = \chi(-H)$. Do you know any other convex function on a real space having this property ?

293. Let $A \in \mathbf{H}_n$ be given. Show, by an explicit construction, that the set of matrices $H \in \mathbf{H}_n$ satisfying $H \geq 0_n$ and $H \geq A$ admits a least element, denoted by $A^+$:

$$\forall H \in \mathbf{H}_n, \qquad (H \geq 0_n \text{ and } H \geq A) \Longleftrightarrow (H \geq A^+).$$

Let $B$ be an other Hermitian matrix. Deduce that the set of matrices $H \in \mathbf{H}_n$ satisfying $H \geq B$ and $H \geq A$ admits a least element. We denote it by $A \vee B$ and call it the *supremum* of $A$ and $B$. We define the *infimum* by $A \wedge B := -((-A) \vee (-B))$.

Prove that

$$A \vee B + A \wedge B = A + B.$$

294. Let $J \in \mathbf{M}_{2n}(\mathbb{R})$ be the standard skew-symmetric matrix:

$$J = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}.$$

Let $S \in \mathbf{Sym}_{2n}(\mathbb{R})$ be given. We assume that $\dim\ker S = 1$.

(a) Show that 0 is an eigenvalue of $JS$, geometrically simple, but not algebraically.

(b) We assume moreover that there exists a vector $x \neq 0$ in $\mathbb{R}^{2n}$ such that the quadratic form $y \mapsto y^T Sy$, restricted to $\{x, Jx\}^\perp$ is positive definite. Prove that the eigenvalues of $JS$ are purely imaginary. **Hint**: Use Exercise 258 in an appropriate way.

(c) In the previous question, $S$ has a zero eigenvalue and may have a negative one. On the contrary, assume that $S$ has one negative eigenvalue and is invertible. Show that $JS$ has a pair of real, opposite eigenvalues. **Hint**: What is the sign of $\det(JS)$ ?

295. Let $A \in \mathbf{M}_n(\mathbb{C})$ be given. Denoting its minimal polynomial by $\pi_A$, let us define a differential operator
$$L_A := \pi_A\left(\frac{d}{dt}\right).$$
The degree of $\pi_A$ is denoted by $r$.

(a) Prove that there exists functions $f_j(t)$ for $j = 0, \dots, r-1$, such that
$$\exp(tA) = f_0(t)I_n + f_1(t)A + \cdots + f_{r-1}(t)A^{r-1}, \qquad \forall t \in \mathbb{R}.$$

(b) Prove that $t \mapsto f_j(t)$ is $\mathcal{C}^\infty$. **Hint**: This requires proving a uniqueness property.

(c) Applying $L_A$ to the above identity, show that these functions satisfy the differential equation
$$L_A f_j = 0.$$
Deduce that
$$f_j = \sum e^{t\lambda_\alpha} p_{j\alpha}(t),$$
where the $\lambda_\alpha$'s are the distinct eigenvalues and $p_{j\alpha}$ are polynomials.

(d) Determine the initial conditions for each of the $f_j$'s. **Hint**: Use the series defining the exponential.

296. (After KrishnapurKrishnapur.) Let $A \in \mathbf{M}_{n\times m}(\mathbb{C})$ be given. We assume that $m \le n$ and denote the columns of $A$ by $C_1, \dots, C_m$, that is $C_j := A\mathbf{e}^j$ with $\{\mathbf{e}^1, \dots, \mathbf{e}^m\}$ the canonical basis of $\mathbb{C}^m$. Let also $\sigma_1, \dots, \sigma_m$ be the singular values of $A$.

Besides, we define $H_i$ the subspace of $\mathbb{C}^n$ spanned the $C_j$'s for $j \ne i$. At last, we denote by $d_i$ the distance of $C_i$ to $H_i$.

(a) To begin with, we restrict to the case where $A$ has full rank: $\operatorname{rk} A = m$. Check that $A^*A$ is non-singular.

(b) Let us define the vector $V_j := (A^*A)^{-1}\mathbf{e}^j$. Show that $AV_j \cdot C_i = \delta_i^j$ for all $i, j = 1, \dots, m$.

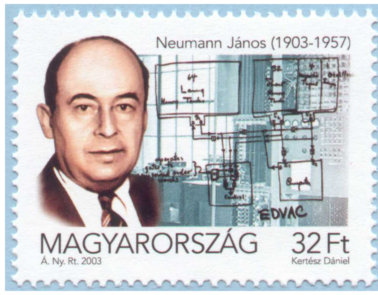
(c) Let us decompose $V_j = v_{j1}\mathbf{e}^1 + \cdots + v_{jn}\mathbf{e}^n$. Show that $AV_j$ is orthogonal to $H_j$ and that its norm equals $v_{jj}d_j$.

(d) Deduce the identity $v_{jj}d_j^2 = 1$.

(e) Prove the *Negative second moment identity*
$$\sum_j \sigma_j^{-2} = \sum_j d_j^{-2}.$$
**Hint**: Compute the trace of $(A^*A)^{-1}$.

(f) What do you think of the case where $\operatorname{rk} A < m$ ?

297. Here is another proof of Fuglede'sFuglede Theorem (see Exercise 255), due to von Neumannvonneu@von Neumann.



vonneu@von Neumann, *John*!(Hungary)

**John von Neumann.**

(a) Given $A, B, C, D \in \mathbf{M}_n(\mathbb{C})$, check
tity

$\mathrm{Tr}([A, B][C, D]+[A, C][D, B]+[A,$

(b) Deduce that if $[A, B] = 0_n$ and if
$B$ is normal, then $[A, B^*] = 0_n$.

298. (After J. von Neumannvonneu@von Neumann.) An *embedding* from $\mathbf{M}_m(\mathbb{C})$ to $\mathbf{M}_n(\mathbb{C})$ is an algebra homomorphism $f$ with the additional properties that $f(I_m) = I_n$ and $f(A^*) = f(A)^*$ for every $A \in \mathbf{M}_m(\mathbb{C})$.

(a) Prove that $f$ sends $\mathbf{GL}_m(\mathbb{C})$ into $\mathbf{GL}_n(\mathbb{C})$.

(b) Deduce that the spectrum of $f(A)$ equals that of $A$.

(c) Likewise, prove that the minimal polynomial of $f(A)$ divides that of $A$. In particular, if $A$ is semi-simple, then so is $f(A)$.

(d) If $f(A) = 0_n$ and $A$ is Hermitian, deduce that $A = 0_m$. Use this property to prove that if $f(M) = 0_n$, then $M = 0_m$ (injectivity). Deduce that $m \le n$.

(e) Let $P \in \mathbf{M}_m(\mathbb{C})$ be a unitary projector: $P^2 = P = P^*$. Prove that $f(P)$ is a unitary projector.

(f) Let $x \in \mathbb{C}^m$ be a unit vector. Thus $f(xx^*)$ is a unitary projector, whose rank is denoted by $k(x)$. Prove that $k$ is independent of $x$. **Hint**: An embedding preserves conjugacy.

(g) Show that $m$ divides $n$. **Hint**: Decompose $I_m$ as the sum of unitary projectors $xx^*$.

(h) Conversely, let us assume that $n = mp$. Prove that there exists an embedding from $\mathbf{M}_m(\mathbb{C})$ to $\mathbf{M}_n(\mathbb{C})$. **Hint**: Use the tensor product of matrices.

299. Let $A \in \mathbf{M}_n(\mathbb{C})$ be a nilpotent matrix of order two: $A^2 = 0_n$. This exercise uses the standard operator norm $\|\cdot\|_2$.

(a) Using standard properties of this norm, verify that $\|M\|_2^2 \le \|MM^* + M^*M\|_2$ for every $M \in \mathbf{M}_n(\mathbb{C})$.

(b) When $k$ is a positive integer, compute $(AA^* + A^*A)^k$ in close form. Deduce that

$$\|AA^* + A^*A\|_2 \le 2^{1/k}\|A\|_2^2.$$

(c) Passing to the limit as $k \to +\infty$, prove that

$$\|A\|_2 = \|AA^* + A^*A\|_2^{1/2}. \tag{33}$$

300. Let $T \in \mathbf{M}_n(\mathbb{C})$ be a contraction in the operator norm: $\|T\|_2 \le 1$. Let us form the matrix $S \in \mathbf{H}_{nN}$:

$$S := \begin{pmatrix} I_n & T^* & \cdots & T^{*N} \\ T & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & T^* \\ T^N & \cdots & T & I_n \end{pmatrix}.$$

We also define the matrix $R \in \mathbf{M}_{nN}(\mathbb{C})$ by

$$R := \begin{pmatrix} I_n & 0_n & \cdots & \cdots & 0_n \\ T & \ddots & & & \vdots \\ 0_n & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_n & \cdots & 0_n & T & I_n \end{pmatrix}.$$

Express $S$ in terms of $R$. If $(I_n - R)y = x$, show that

$$x^*Sx = \|y\|_2^2 - \|Ry\|_2^2.$$

Deduce that $S$ is positive semi-definite.

301. Let $H_1, \dots, H_r \in \mathbf{H}_n$ be positive semi-definite matrices such that

$$\sum_{j=1}^{r} H_j = I_n.$$

Let also $z_1, \dots, z_r \in \mathbb{C}$ be given in the unit disc: $|z_j| \le 1$.

Prove that

$$\left\| \sum_{j=1}^{r} z_j H_j \right\|_2 \le 1.$$

**Hint**: Factorize

$$\begin{pmatrix} \sum_{j=1}^{r} z_j H_j & 0_n & \cdots \\ 0_n & 0_n & \cdots \\ \vdots & \vdots & \end{pmatrix} = M^* \mathrm{diag}\{z_1 I_n, \dots, z_r I_n\} M$$

with $M$ appropriate.

302. (Sz. NagySz. Nagy.) If $M \in \mathbf{M}_n(\mathbb{C})$ is similar to a unitary matrix, show that

$$\sup_{k\in\mathbb{Z}} \|M^k\|_2 < \infty. \tag{34}$$

Conversely, let $M \in \mathbf{GL}_n(\mathbb{C})$ satisfy (34). Show that the eigenvalues of $M$ belong to the unit circle and are semi-simple. Deduce that $M$ is similar to a unitary matrix.

303. Let $M_1, \dots, M_r \in \mathbf{M}_n(\mathbb{C})$ satisfy the *anticommutation relations*

$$M_iM_j + M_jM_i = 0_n, \qquad M_i^*M_j + M_j^*M_i = \delta_i^j I_n, \qquad \forall 1 \le i, j \le r.$$

Let us define

$$M(z) = \sum_{j=1}^{r} z_j M_j, \qquad \forall z \in R^r.$$

Verify that $M(z)$ is nilpotent of order 2. Prove the identity

$$\|M(z)\|_2 = \|z\|_2.$$

**Hint**: Use Exercise 299.

304. (S. Joshi & S. BoydJoshiBoyd.) Given $A \in \mathbf{M}_n(\mathbb{C})$ and a (non trivial) linear subspace[2] $V$ of $\mathbb{C}^n$, we define

$$G(A|V) := \sup\{\|Ax\| \,|\, x \in V,\ \|x\| = 1\}, \qquad H(A|V) := \inf\{\|Ax\| \,|\, x \in V,\ \|x\| = 1\},$$

where we use the canonical Hermitian norm. Obviously, we have $H(A|V) \le G(A|V)$, thus

$$\kappa_V(A) := \frac{G(A|V)}{H(A|V)} \ge 1.$$

Notice that when $A$ is singular, $\kappa_V(A)$ may be infinite.

Given an integer $k = 1, \dots, n$, we wish to compute the number

$$\theta_k(A) := \inf\{\kappa_V(A) \,|\, \dim V = k\}.$$

(a) What is $\kappa_V(A)$ when $V = \mathbb{C}^n$ ?

(b) Show that $\theta_k$ is unitary invariant: if $B = UAV$ with $U$ and $V$ unitary, then $\theta_k(B) = \theta_k(A)$. Deduce that $\theta_k(A)$ depends only upon the singular values of $A$.

(c) From now on, we assume that $A$ is non-singular. From the previous question, we may assume that $A$ is diagonal and positive. We denote $(0 <)\sigma_n \le \cdots \le \sigma_1$ its singular values, here its diagonal entries. Show first that

$$G(A|V) \ge \sigma_{n-k+1}, \qquad H(A|V) \le \sigma_k.$$

Deduce that

$$\theta_k(A) \ge \max\left\{\frac{\sigma_{n-k+1}}{\sigma_k}, 1\right\}. \tag{35}$$

[2]This exercise is unchanged when replacing the scalar field by $\mathbb{R}$.

(d) We wish to show that (35) is actually an equality. Let $\{\vec{e}^1, \dots, \vec{e}^n\}$ be the canonical basis of $\mathbb{C}^n$. Show that given $i \le j$ and $\sigma \in [\sigma_i, \sigma_j]$, there exists a unit vector $y \in \text{Span}(\vec{e}^i, \vec{e}^j)$ such that $\|Ay\| = \sigma$.

(e) Let $N$ be least integer larger than or equal to $\frac{n+1}{2}$, that is the integral part of $1+n/2$. Show that each plane $\text{Span}(\vec{e}^{N-\ell}, \vec{e}^{N+\ell})$ with $\ell = 1, \dots, n-N$ contains a unit vector $y_\ell$ such that $\|Ay_\ell\| = \sigma_N$. Deduce that $\theta_{n-N+1}(A) = 1$. In other words, $\theta_k(A) = 1$ for every $k \le n - N + 1$.

**Nota**: This can be recast as follows. Every ellipsoid of dimension $n-1$ contains a sphere of dimension $d$, the largest integer strictly less than $n/2$.

(f) Assume on the contrary that $k > n-N+1$. Using the same trick as above, construct a linear space $W$ of dimension $k$ such that $\kappa_W(A) = \sigma_{n-k+1}/\sigma_k$. Conclude.

305. Among the class of HessenbergHessenberg matrices, we distinguish the *unit* ones, which have 1's below the diagonal:

$$M = \begin{pmatrix} * & \cdots & & \cdots & * \\ 1 & \ddots & & & \vdots \\ 0 & \ddots & & & \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & * \end{pmatrix}.$$

(a) Let $M \in \mathbf{M}_n(k)$ be a unit Hessenberg matrix. We denote by $M_k$ the submatrix obtained by retaining the first $k$ rows and columns. For instance, $M_n = M$ and $M_1 = (m_{11})$. We set $P_k$ the characteristic polynomial of $M_k$.
Show that (B. KostantKostant & N. WallachWallach, B. ParlettParlett & G. StrangStrang)

$$P_n(X) = (X - m_{nn})P_{n-1}(X) - m_{n-1,n}P_{n-2}(X) - \cdots - m_{2n}P_1(X) - m_{1n}.$$

(b) Let $Q_1, \dots, Q_n \in k[X]$ be monic polynomials, with $\deg Q_k = k$. Show that there exists one and only one unit Hessenberg matrix $M$ such that, for every $k = 1, \dots, n$, the characteristic polynomial of $M_k$ equals $Q_k$. **Hint**: Argue by induction over $n$.

**Nota**: The list of roots of the polynomials $P_1, \dots, P_n$ are called the *Ritz values*Ritz of $M$.

306. (Partial converse of Exercise 21.) Let $A$ and $B$ be $2 \times 2$ complex matrices, which have the same spectrum. We assume in addition that

$$\det[A^*, A] = \det[B^*, B].$$

Prove that $A$ and $B$ are unitarily similar. **Hint** : Prove that they both are unitarily similar to the same triangular matrix.

Deduce that two matrices in $\mathbf{M}_2(\mathbb{C})$ are unitary similar if, and only if they have the same numerical range.

307. Let $P \in \mathbf{M}_n(\mathbb{C})$ be a projection. Let us define

$$\Pi := P(P^*P + (I_n - P)^*(I_n - P))^{-1}P^*.$$

Prove that $\Pi$ vanishes over $(R(P))^\perp$. Then prove that $\Pi x = x$ for every $x \in R(P)$. In other words, $\Pi$ is the orthogonal projection onto $R(P)$. **Hint**: If $x \in R(P)$, then $x = Px$, thus you can find the solution of

$$(P^*P + (I_n - P)^*(I_n - P))y = P^*x.$$

308. Compute the conjugate of the convex function over $\mathbf{H}_n$

$$H \mapsto \begin{cases} -(\det H)^{1/n}, & \text{if } H \geq 0_n, \\ +\infty, & \text{otherwise.} \end{cases}$$

**Hint:** Use Exercise 209. Remark that the conjugate of a positively homogeneous function of degree one is the characteristic function of some convex set.

309. (a) Let $A \in \mathbf{M}_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$ be given. Show that if $\lambda$ is not in the numerical range $W(A)$, then $\lambda I_n - A$ is invertible, and its inverse verifies

$$\|(\lambda I_n - A)^{-1}\|_2 \leq \frac{1}{\operatorname{dist}(\lambda, W(A))}.$$

(b) Conversely, let us consider compact subsets $X$ of the complex plane, such that

$$\|(\lambda I_n - A)^{-1}\|_2 \leq \frac{1}{\operatorname{dist}(\lambda, X)}, \qquad \forall \lambda \notin X. \tag{36}$$

Obviously, such an $X$ is contained in the resolvant set $\mathbb{C} \setminus \operatorname{Sp}(A)$. If $\epsilon > 0$ tends to zero, show that

$$\|I_n + \epsilon A\|_2 = 1 + \epsilon \sup\{\Re z \,|\, z \in W(A)\} + O(\epsilon^2).$$

Deduce that

$$\sup\{\Re z \,|\, z \in W(A)\} \leq \sup\{\Re z \,|\, z \in X\}.$$

Finally, prove that $W(A)$ is the convex hull of $X$. In particular, if $A = J(2;0)$ is a $2 \times 2$ Jordan block, then $X$ contains the circle $C_{1/2}$. When $A$ is normal, $\operatorname{Sp}(A)$ satisfies (36).

310. Let $k$ be a field of characteristic zero. We consider a matrix $A \in \mathbf{M}_n(k)$. If $X \in \mathbf{M}_n(k)$, we define the linear form and the linear map

$$\tau_X(M) := \operatorname{Tr}(XM), \qquad \operatorname{ad}_X(M) = XM - MX.$$

(a) We assume that $A$ is nilpotent.

i. If $AM = MA$, show that $AM$ is nilpotent.
ii. Verify that $\ker \mathrm{ad}_A \subset \ker \tau_A$.
iii. Deduce that there exists a matrix $B$ such that $\tau_A = \tau_B \circ \mathrm{ad}_A$.
iv. Show that $A = BA - AB$.

(b) Conversely, we assume instead that there exists a $B \in \mathbf{M}_n(k)$ such that $A = BA - AB$.

i. Verify that for $k \in \mathbb{N}$, $BA^k - A^kB = kA^k$.
ii. Deduce that $A$ is nilpotent. **Hint**: $\mathrm{ad}_B$ has finitely many eigenvalues.

(c) If $A = J(0; n)$ is the nilpotent JordanJordan!Camille block of order $n$, find a diagonal $B$ such that $[B, J] = J$.

311. (From E. A. HermanHerman.) Let $A \in \mathbf{M}_n(\mathbb{C})$ be such that $A^2 = 0_n$. We denote its rank, kernel and range by $r$, $N$ and $R$.

(a) By computing $\|(A + A^*)x\|_2^2$, show that $\ker(A + A^*) = N \cap R^\perp$.

(b) Verify that $R \subset N$ and $N^\perp \subset R^\perp$. Deduce that $N \cap R^\perp$ is of dimension $n - 2r$ and that the rank of $\frac{1}{2}(A + A^*)$ (the *real part* of $A$) is $2r$.

(c) Show that $N$ is an isotropic subspace for the Hermitian form $x \mapsto x^*(A + A^*)x$, contained in $R(A + A^*)$. Deduce that the number of positive / negative eigenvalues of $A + A^*$ are both equal to $r$.

(d) Example: Take $n = 2r$ and

$$A := \begin{pmatrix} 0_r & B \\ 0_r & 0_r \end{pmatrix}, \qquad B \in \mathbf{GL}_r(\mathbb{C}).$$

Find the equations of the stable and the unstable subspaces of $A + A^*$. **Hint**: the formula involves $\sqrt{BB^*}$.

312. Let $E$ be a hyperplane in $\mathbf{H}_n$. We use the scalar product $\langle A, B \rangle := \mathrm{Tr}(AB)$. Prove the equivalence of the following properties.

- Every nonzero matrix $K \in E$ has at least one positive and one negative eigenvalues.
- $E^\perp$ is spanned by a positive definite matrix.

313. (After von Neumannvonneu@von Neumann, HalperinHalperin, AronszajnAronszajn, KayalarKayalar & WeinertWeinert.) We equip $\mathbb{C}^n$ with the standard scalar product, and $M_n(\mathbb{C})$ with the induced norm. Let $M_1$ and $M_2$ be two linear subspaces, and $P_1$, $P_2$ the corresponding orthogonal projections. We recall that $P_j^* = P_j = P_j^2$. We denote $M := M_1 \cap M_2$ and $P$ the orthogonal projection onto $M$. Finally, we set $Q := I_n - P$ and $\Omega_j := P_j Q$. Our goal is two prove

$$\lim_{m \to +\infty} (P_2 P_1)^m = P$$

(von Neumann–Halperin Theorem), and to give a precise error bound.

(a) Show that $P_j P = P$ and $PP_j = P$.

(b) Deduce that $\Omega_j = P_j - P$. Verify that $\Omega_j$ is an orthogonal projection.

(c) Show that $(\Omega_2\Omega_1)^m = (P_2P_1)^m - P$.

(d) Deduce that

$$\|(P_2P_1)^m - P\|^2 = \|(\Omega_1\Omega_2\Omega_1)^{2n-1}\| = \|\Omega_1\Omega_2\Omega_1\|^{2n-1}.$$

(e) Let $x \in \mathbb{C}^n$ be given. If $\Omega_1 x = \Omega_2 x = x$, show that $x = 0$. Deduce that $\|\Omega_1\Omega_2\Omega_1\| < 1$ and establish the von Neumann–Halperin Theorem.

314.


Hadamard,

*Jacques* (Rép. de Guinée)

**Jacques Hadamard.**

A HadamardHadamard matrix is a matrix $M \in \mathbf{M}_n(\mathbb{Z})$ whose entries are $\pm 1$s and such that $M^T M = nI_n$. The latter means that $\frac{1}{\sqrt{n}}M$ is orthogonal.

We construct inductively a sequence of matrices $H_m \in \mathbf{M}_{2^m}(\mathbb{Z})$ by

$$H_0 = (1), \qquad H_{m+1} = \begin{pmatrix} H_m & -H_m \\ H_m & H_m \end{pmatrix}.$$

Verify that each $H_m$ is a Hadamard matrix.

**Nota**: It is unknown whether there exists or not a Hadamard whose size is not a power of 2.

315. (M. RossetRosset!Myriam and S. RossetRosset!Shmuel.) Let $A$ be a Principal Ideal Domain. We define the vector product $X \times Y$ in $A^3$ with the same formulæ as in $\mathbb{R}^3$. For instance, the first coordinate is $x_2y_3 - x_3y_2$.

(a) We admit for a minute that the map $\phi : (X, Y) \mapsto X \times Y$ is surjective from $A^3 \times A^3$ into $A^3$. Prove that every matrix $M \in \mathbf{M}_2(A)$ with zero trace is a commutator $BC - CB$ with $B, C \in \mathbf{M}_2(A)$.

(b) We now prove the surjectivity of $\phi$. Let $Z = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in A^3$ be given. Show that there exists $X = \begin{pmatrix} x \\ y \\ c \end{pmatrix}$ such that $ax + by + cz = 0$ and $\gcd\{x, y, z\} = 1$. Then Bézout gives a vector $U = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ such that $ux + vy + wz = 1$. Set $Y = Z \times U$. Then $Z = X \times Y$.

316. Let $k$ be a field, $UT_n(k)$ be the set of upper triangular matrices with 1s along the diagonal. It is called the *unipotent group*.

(a) Prove that $UT_n(k)$ is a subgroup of $\mathbf{GL}_n(k)$.

(b) If $G$ is a group, $D(G)$ is the group generated by the commutators $xyx^{-1}y^{-1}$. It is a normal subgroup of $G$. Show that $D(UT_n(k))$ consists of the matrices such that $m_{i,i+1} = 0$ for every $i = 1, \dots, n-1$.

(c) Let $G_0 = U_n(k)$ and $G_1 = D(UT_n(k))$. We define $G_r$ by induction; $G_{r+1}$ is the group generated by the commutators where $x \in G_0$ and $y \in G_r$. Describe $G_r$ and verify that $G_n = \{I_n\}$. One says that the group $UT_n(k)$ is a *nilpotent*.

317. (Follow-up of the previous exercise. Thanks to W. ThurstonThurston.) We take $k = \mathbb{Z}/3\mathbb{Z}$ and $n = 3$.

(a) Show that every element $M \neq I_3$ of $UT_3(\mathbb{Z}/3\mathbb{Z})$ is of order 3.

(b) What is the order of $UT_3(\mathbb{Z}/3\mathbb{Z})$ ?

(c) Find an abelian group of order 27, with 26 elements of order 3.

(d) Deduce that the number of elements of each order does not characterizes a group in a unique manner.

318. Let $p \geq 3$ be a prime number. $M \in \mathbf{GL}_n(\mathbb{Z})$ be of finite order ($M^r = I_n$ for some $r \geq 1$), and such that $M \equiv I_n \mod p$.

(a) Prove that $M = I_n$. **Hint**: If $M \neq I_n$, write $M = I_n + p^\alpha A$ with $A$ not divisible b $p$. Likewise, $r = p^\beta \ell$ with $\beta \geq 0$ and $\ell$ not divisble by $p$. Verify that for each $k \geq 2$,

$$p^{k\alpha}\binom{r}{k}$$

is divisible by $p^{\alpha+\beta+1}$. Deduce that $M^r \equiv I_n + \ell p^{\alpha+\beta} A \mod p^{\alpha+\beta+1}$. Conclude.

(b) Let $G$ be a finite subgroup of $\mathbf{GL}_n(\mathbb{Z})$. Deduce that the reduction $\mod p$ is injective over $G$. Yet, this reduction is not injective over $\mathbf{GL}_n(\mathbb{Z})$.

(c) The result is false if $p = 2$: find a matrix $I_n + 2A$ of finite order, with $A \neq 0_n$.

319. (After C. S. BallantineBallantine.) We prove that every Hermitian matrix $H$ with strictly positive trace can be written as $H = AB + BA$ with $A, B \in \mathbf{HPD}_n$.

(a) We first treat the case where the diagonal entries $h_{jj}$ are strictly positive. Prove that such a pair $(A, B)$ exists with $A$ diagonal. **Hint**: Induction over $n$. Choose $a_n > 0$ small enough.

(b) Conclude, with the help of Exercise 131.

(c) Conversely, if $A, B \in \mathbf{HPD}_n$ are given, prove that $\mathrm{Tr}(AB + BA) > 0$.

320. Let $S$ be a finite set of cardinal $n$. Let $E_1, \dots, E_m$ be subsets of $S$ with the properties that

- every $E_j$ has an odd cardinal,
- for every $i \neq j$, $E_i \cap E_j$ has an even cardinal.

(a) Let us form the matrix $A$ with $n$ columns (indexed by the elements of $S$) and $m$ rows, whose entry $a_{jx}$ equals 1 if $x \in E_j$, and 0 otherwise. Prove that $AA^T = I_m$. **Hint**: Yes, $I_m$ ! Leave some room to your imagination.

(b) Deduce $m \leq n$.

321. Let $A, B \in \mathbf{M}_n(\mathbb{C})$ satisfy the relation $[A, B] = A$. Let us define the following matrix-valued functions of $t \in \mathbb{R}$:

$$X(t) := Be^{-tA}, \qquad Y(t) = Ae^{-tB}, \qquad Z(t) = e^{t(A+B)}e^{-tA}e^{-tB}.$$

(a) Find a differential equation for $X$. Deduce that

$$[B, e^{-tA}] = tAe^{-tA}.$$

(b) Find a differential equation for $Y$. Deduce that

$$Ae^{-tB} = e^{-t}e^{-tB}A.$$

(c) Find a differential equation for $Z$. Deduce that

$$e^{t(A+B)} = e^{\tau(t)A}e^{tB}e^{tA}, \qquad \tau(t) := 1 - (t+1)e^{-t}.$$

**Remarks**.

- From Exercise 256, we know that $A$ is nilpotent.
- This result can be used in order to establish an explicit formula for the semi-group generated by the Fokker–PlanckFokkerPlanck equation $\partial_t f = \Delta_v f + v \cdot \nabla_v f$.

322. Let $A \in \mathbf{M}_n(\mathbb{R})$. We denote $\mathbf{1}$ the vector whose coordinates are ones. Prove the equivalence of the following properties.

- The semi-group $(M_t := e^{tA})_{t \geq 0}$ is MarkovianMarkov, meaning that $M_t$ is stochastic ($M_t \geq 0_n$ and $M_t\mathbf{1} = \mathbf{1}$).
- The off-diagonal entries of $A$ are non-negative, and $A\mathbf{1} = 0$.
- $A\mathbf{1} = 0$, and for every $X \in \mathbb{R}^n$, we have

$$A(X \circ X) \geq 2X \circ (AX),$$

where we use the HadamardHadamard product, $(X \circ Y)_j = x_j y_j$.

323. (S. BoydBoyd & L. VandenbergheVandenberghe). The spectral radius of a non-negative matrix $A \in \mathbf{M}_n(\mathbb{R})$ is an eigenvalue (Perron–FrobeniusPerronFrobenius Theorem), which we denote $\lambda_{\mathrm{pf}}(A)$.

(a) When $A > 0_n$, prove that

$$\log \lambda_{\rm pf}(A) = \lim_{k\to+\infty} \frac{1}{k} \log \mathbf{1}^T A^k \mathbf{1},$$

where $\mathbf{1}$ is the vector whose entries are all ones.

(b) Let $A, B$ be positive matrices and let us define $C$ by $c_{ij} := \sqrt{a_{ij} b_{ij}}$ (the square root, in the sense of Hadamard product).

i. Show that for every $m \geq 1$,

$$(C^m)_{ij} \leq \sqrt{(A^m)_{ij} (B^m)_{ij}}\,.$$

ii. Deduce that $\lambda_{\rm pf}(C) \leq \sqrt{\lambda_{\rm pf}(A)\lambda_{\rm pf}(B)}\,$.

(c) More generally, show that $\log \lambda_{\rm pf}(A)$ is a convex function of the variables $\log a_{ij}$. **Nota**: This is not what is usually called *log-convexity*.

324. Given $B \in \mathbf{M}_n(\mathbb{C})$, assume that the only subspaces $F$ invariant under both $B$ and $B^*$ (that is $BF \subset F$ and $B^* F \subset F$) are $\mathbb{C}^n$ and $\{0\}$. Let us denote by $L$ the sub-algebra of $\mathbf{M}_n(\mathbb{C})$ spanned by $B$ and $B^*$, *i.e.* the smallest algebra containing $B$ and $B^*$.

(a) Verify that $\ker B \cap \ker B^* = \{0\}$.

(b) Construct a matrix $H \in L$ that is Hermitian positive definite.

(c) Deduce that $I_n \in L$ : $L$ is *unital*.

(d) We show now that $L$ does not admit a proper two-sided ideal (we say that the unital algebra $L$ is *simple*). So let $J$ be a two-sided ideal of $L$.

i. We define

$$K = \bigcap_{M\in J} (\ker M \cap \ker M^*).$$

Show that there exist finitely many elements $M_j \in J$ such that

$$K = \bigcap_j (\ker M_j \cap \ker M_j^*) = \ker \sum_j (M_j M_j^* + M_j^* M_j).$$

ii. Verify that $BK \subset K$ and $B^* K \subset K$.

iii. Deduce that either $J = (0_n)$, or $J$ contains an invertible element.

iv. Conclude.

**Comment**. This can be used to prove the following statement, which interpolates the unitary diagonalization of normal matrices and the AmitsurAmitsur–LevitskiLevitski theorem that the standard noncommutative polynomial $\mathcal{S}_{2n}$ vanishes over $\mathbf{M}_n(k)$.

Let us say that a matrix $A \in \mathbf{M}_n(\mathbb{C})$ is *$r$-normal* if the standard polynomial in $2r$ non-commuting variables $\mathcal{S}_{2r}$ vanishes identically over the sub-algebra spanned by $A$ and $A^*$. In particular, every $A$ is $n$-normal, whereas $A$ is 1-normal if and only if it is normal.

Then $A$ is $r$-normal if and only if there exists a unitary matrix $U$ such that $U^*AU$ is block-diagonal with diagonal blocks of size $m \times m$ with $m \leq r$.

The reader will prove easily that a matrix that is unitarily similar to such a block-diagonal matrix is $r$-normal.

325. We consider here the linear equation $AM = MB$ in $M \in \mathbf{M}_{n\times m}(k)$, where $A \in \mathbf{M}_n(k)$ and $B \in \mathbf{M}_m(k)$ are given. The solution set is a vector space denoted $\mathcal{S}(A, B)$.

(a) If $R \in k[X]$, verify that $R(A)M = MR(B)$. If the spectra of $A$ and $B$ are disjoint, deduce that $M = 0$.

(b) When $A = J(0; n)$ and $B = J(0; m)$, compute the solutions, and verify that the dimension of $\mathcal{S}(A, B)$ is $\min\{m, n\}$.

(c) If $A$ is conjugate to $A'$ and $B$ conjugate to $B'$, prove that $\mathcal{S}(A', B')$ is obtained from $\mathcal{S}(A, B)$ by applying an equivalence. In particular, their dimensions are equal.

(d) In this question, we assume that $k$ is algebraically closed

i. Let $\{\lambda_1, \dots, \lambda_\ell\}$ be the union of the spectra of $A$ and $B$. If $i \leq \ell$, we denote $(X - \lambda_i)^{\alpha_{ij}}$ the elementary divisors of $A$, and $(X - \lambda_i)^{\beta_{ik}}$ those of $B$. Using a canonical form $A'$ and $B'$, prove that the dimension of $\mathcal{S}(A, B)$ equals the sum of the numbers

$$N_i := \sum_{j,k} \min\{\alpha_{ij}, \beta_{ik}\}.$$

ii. Deduce that the dimension of the solution set of the matrix equation $AM = MB$ equals

$$\sum \deg[\text{g.c.d.}(p_i, q_j)],$$

where $p_1, \dots, p_n$ are the invariant factors of $A$ and $q_1, \dots, q_m$ are those of $B$.

(e) Show that the result above persists when $k$ is an arbitrary field, not necessarily algebraically closed. This is the Cecioni–FrobeniusCecioniFrobenius Theorem. **Hint**: $\mathcal{S}(A, B)$ is defined by a linear system in $k^{nm}$. Its dimension remains the same when one replaces $k$ by a field $K$ containing $k$.

326. A symmetric matrix $S \in \mathbf{Sym}_n(\mathbb{R})$ is said *compatible* if it is of the form $ab^T + ba^T$ with $a, b \in R^n$. Prove that $S \in \mathbf{Sym}_n(\mathbb{R})$ is compatible if and only if

- either $S = 0_n$,
- or $S$ is rank-one and non-negative,
- or $S$ has rank two and its non-zero eigenvalues have opposite signs.

327. (After R. A. HornHorn!Roger and C. R. JohnsonJohnson!Charles R. (I).) Let $A \in \mathbf{M}_n(\mathbb{C})$ be given.

   (a) Suppose that $A$ is similar to a matrix $B \in \mathbf{M}_n(\mathbb{R})$. Prove that $A$ is similar to $A^*$. **Hint**: $B^T$ is similar to $B$.

   (b) Conversely, we assume that $A$ is similar to $A^*$.

      i. Verify that $A$ is similar to $\overline{A}$. Deduce that the spectrum of $A$ is invariant under conjugation, and that when $\lambda$ is a non-real eigenvalue, the JordanJordan!Camille blocks corresponding to $\bar{\lambda}$ have the same sizes as those corresponding to $\lambda$.

      ii. Deduce that $A$ is similar to a matrix $B \in \mathbf{M}_n(\mathbb{R})$.

328. (After R. A. HornHorn!Roger and C. R. JohnsonJohnson!Charles R. (II).) Let $A \in \mathbf{M}_n(\mathbb{C})$ be given. We assume that $A$ is similar to $A^*$: $A = T^{-1}A^*T$.

   (a) Show that there exists $\theta \in \mathbb{R}$ such that $e^{i\theta}T + e^{-i\theta}T^*$ is non-singular.

   (b) Deduce that there exists $S \in \mathbf{H}_n \cap \mathbf{GL}_n(\mathbb{C})$ such that $A = S^{-1}A^*S$.

329. (After R. A. HornHorn!Roger and C. R. JohnsonJohnson!Charles R. (III).) Let $A \in \mathbf{M}_n(\mathbb{C})$ be given.

   (a) We assume that $A$ is similar to $A^*$ *via* a Hermitian transformation: $A = S^{-1}A^*S$ with $S \in \mathbf{H}_n$. Verify that $A^*S$ is Hermitian. Deduce that $A$ has the form $HK$ where $H$ and $K$ are Hermitian, one of them being non-singular.

   (b) Conversely, assume that $A$ has the form $HK$ where $H$ and $K$ are Hermitian, one of them (say $H$ for definiteness) being non-singular. Verify that $A$ is similar to $A^*$.

   So far, we have shown that $A$ is similar to a matrix $B \in \mathbf{M}_n(\mathbb{R})$ if and only if it is of the form $HK$ where $H$ and $K$ are Hermitian, one of them being non-singular.

330. (After R. A. HornHorn!Roger and C. R. JohnsonJohnson!Charles R. (IV).) We now assume that $A = HK$ where $H$ and $K$ are Hermitian. We warn the reader that we allow both $H$ and $K$ to be singular.

   (a) To begin with, we assume that $H = \begin{pmatrix} H' & 0 \\ 0 & 0 \end{pmatrix}$ where $H' \in \mathbf{H}_p$ is non-singular. Let $\begin{pmatrix} K' & * \\ * & * \end{pmatrix}$ be the block form of $K$ matching that of $H$ so that
$$A = \begin{pmatrix} H'K' & * \\ 0 & 0 \end{pmatrix}.$$

      i. Let $\lambda$ be a non-real eigenvalue of $A$. Show that $\lambda$ is an eigenvalue of $A' := H'K'$ and that the Jordan blocks corresponding to $\lambda$ in $A$ or in $A'$ are the same.

ii. Deduce that the Jordan blocks of $A$ corresponding to $\bar{\lambda}$ have the same sizes as those corresponding to $\lambda$. **Hint**: According to the previous exercises, $A'$ is similar to a matrix $B' \in \mathbf{M}_p(\mathbb{R})$.

iii. Deduce that $A$ is similar to a matrix $B \in \mathbf{M}_n(\mathbb{R})$.

(b) Prove the same result in the general case. **Hint**: Diagonalize $H$.

**Summary**: The following properties are equivalent to each other for every $A \in \mathbf{M}_n(\mathbb{C})$.

- $A$ is similar to a matrix $B \in \mathbf{M}_n(\mathbb{R})$,
- $A$ is similar to $A^*$,
- $A$ is similar to $A^*$ *via* a Hermitian transformation,
- There exist $H, K \in \mathbf{H}_n$ such that $A = HK$ and one of them is non singular,
- There exist $H, K \in \mathbf{H}_n$ such that $A = HK$.

331. If $A \in \mathbf{M}_n(\mathbb{R})$ and $x \in (0, +\infty)$, verify that $\det(xI_n + A^2) \geq 0$. Deduce that if $n$ is odd, then $-I_n$ cannot be written as $A^2 + B^2$ with $A, B \in \mathbf{M}_n(\mathbb{R})$. **Note**: On the contrary, if $n$ is even, then every matrix $M \in \mathbf{M}_n(\mathbb{R})$ can be written as $A^2 + B^2$ with $A, B \in \mathbf{M}_n(\mathbb{R})$.

332. If $\sigma \in \mathfrak{S}_n$, we denote by $P_\sigma$ the permutation matrix associated with $\sigma$. A finite sum of permutation matrices is obviously a matrix $M \in \mathbf{M}_n(\mathbb{N})$, whose sums of rows and columns are equal. We shall prove the converse statement: If $M \in \mathbf{M}_n(\mathbb{N})$ has equal sum $S$ for rows and columns, then $M$ is a finite sum of permutation matrices.

Let $I$, $J$ be two sets of indices $1 \leq i, j \leq n$. If the bloc $M_{IJ}$ is identically 0, prove that the sum of the entries of the opposite bloc $M_{I^cJ^c}$ equals $(n - p - q)S$. If $S \geq 1$, deduce that $p + q \leq n$.

Let us recall (Exercise 9) that this property implies that there exists a permutation $\sigma$ such that $m_{i\sigma(i)} \neq 0$ for every $1 \leq i \leq n$. Then argue by induction over $S$.

333. Let $M \in \mathbf{M}_n(\mathbb{R})$ be a given non-negative matrix. If $\sigma \in \mathfrak{S}_n$, let us denote

$$m^\sigma := \sum_{i=1}^{n} m_{i\sigma(i)}.$$

Finally, we define

$$S := \max_{\sigma \in \mathfrak{S}_n} m^\sigma.$$

We assume that for every entry $m_{ij}$, there exist $\sigma \in \mathfrak{S}_n$ such that $\sigma(i) = j$ and $m^\sigma = S$.

(a) Find a positive linear form $f$ over $\mathbf{M}_n(\mathbb{R})$ such that $m^\sigma = f(P_\sigma \circ M)$ for every $\sigma \in \mathfrak{S}_n$, where $A \circ B$ is the Hadamard product.

(b) Rephrase the assumption in the following way: There exists a subset $X$ of $\mathfrak{S}_n$ such that $m^\sigma = S$ for every $\sigma \in X$, and

$$Q := \sum_{\sigma \in X} P_\sigma > 0_n.$$

(c) Let $\theta \in \mathfrak{S}_n$ be given.

i. Verify that $Q - P_\theta$ is a sum of $k-1$ permutation matrices, with $k := |X|$. **Hint**: Use exercise 332.

ii. Deduce that $m^\theta \geq S$, and therefore $= S$.

334. Recall that a numerical function $f : (a,b) \to \mathbb{R}$ is *operator monotone* if whenever $n \geq 1$, $A, B \in \mathbf{H}_n$ are given with the spectra of $A$ and $B$ included in $(a,b)$, we have

$$(A \leq B) \Longrightarrow (f(A) \leq f(B)),$$

where $H \leq K$ is understood in the sense of Hermitian forms. More generally, $f$ is operator monotone *of grade* $n$ if this holds true at fixed size $n \times n$. We prove here the Loewner'sLoewner Theorem, under some regularity assumption.

(a) Check that if $m \leq n$, then operator monotonicity of grade $n$ implies operator monotonicity of grade $m$.

(b) We already know that if both $H$ and $K$ are $\geq 0_n$, their HadamardHadamard product $H \circ K$ is $\geq 0_n$ too. Here is a converse: Let $K \in \mathbf{H}_n$ be given. If $K \circ H \geq 0_n$ whenever $H \in \mathbf{H}_n$ is $\geq 0_n$, prove that $K \geq 0_n$.

(c) We assume that $f \in \mathcal{C}(a,b)$ and recall the assumptions of Exercise 250: If $D = \mathrm{diag}(d_1, \dots, d_n\} \in \mathbf{M}_n(\mathbb{C})$, we define a matrix $f^{[1]}(D) \in \mathbf{M}_n(\mathbb{C})$ by

$$f^{[1]}(D)_{jk} = \frac{f(d_j) - f(d_k)}{d_j - d_k},$$

where we identify

$$\frac{f(b) - f(a)}{b - a} := f'(a),$$

if $b = a$.

If $f$ is operotor monotone of grade $n$, prove that $f^{[1]}(D) \geq 0_n$ whenever $d_1, \dots, d_n \in (a,b)$. **Hint**: Use Daletskiĭ–KreinDaletskiĭKrein formula.

(d) If $n \geq 2$, deduce that either $f$ is constant, or it is strictly increasing.

(e) We assume that $f \in \mathcal{C}^3(a,b)$ and $n \geq 2$. We may take $n = 2$. Compute $P^T f^{[1]}(D) P$ when

$$P = \begin{pmatrix} 1 & -\frac{1}{d_2 - d_1} \\ 0 & \frac{1}{d_2 - d_1} \end{pmatrix}.$$

Deduce that $2f'f''' \geq 3f''^2$. In other words, either $f$ is constant or $(f')^{-1/2}$ is concave.

335. The *relative gain array* of a square matrix $A \in \mathbf{GL}_n(k)$ is defined as $\Phi(A) := A \circ A^{-T}$, where the product is that of HadamardHadamard (entrywise). It was studied by C. R. JohnsonJohnson!Charles R. & H. ShapiroShapiro). Many questions about it remain open, including that of the range of $\Phi$.

(a) If $A$ is triangular, show that $\Phi(A) = I_n$.

(b) Verify that 1 is an eigenvalue of $\Phi(A)$, associated with the eigenvector $\vec{e}$ whose components are all ones. **Open problem**: Is this the only constraint ? In other words, if $M\vec{e} = \vec{e}$, does there exist a matrix $A$ such that $\Phi(A) = M$ ?

(c) If $A$ is not necessarily invertible, we may define $\Psi(A) := A \circ \widehat{A}$, with $\widehat{A}$ the cofactor matrix. Therefore we have $\Phi(A) = \frac{1}{\det A}\Psi(A)$ when $A$ is non-singular.

Show that there exists a homogeneous polynomial $\Delta$ in the entries of $A$, such that $\det \Psi(A) \equiv \Delta(A) \det A$.

(d) If $n = 3$ and $A$ is symmetric,

$$A = \begin{pmatrix} a & z & y \\ z & b & x \\ y & x & c \end{pmatrix},$$

verify that

$$\Delta(A) = (abc - xyz)^2 + abx^2y^2 + bcy^2z^2 + caz^2x^2 - (ax^2)^2 - (by^2)^2 - (cz^2)^2.$$

Deduce that $\Delta(A) = 0$ when $A$ is rank-one, or when a row of $A$ vanishes. However $\det A$ does not divide $\Delta(A)$.

336. (The symmetric positive definite case.) We continue with the study of the relative gain array. We now restrict to matrices $S \in \mathbf{SPD}_n$.

(a) Show that $\Phi(S) \geq I_n$ (FiedlerFiedler), with equality only if $S = \mu I_n$ for some $\mu$. **Hint**: Write $S = \sum \lambda_j v_j v_j^T$ in an orthonormal basis. Compute $S^{-T}$ and $\Phi(S)$. Then use the inequality $\frac{a}{b} + \frac{b}{a} \geq 2$.

(b) Following the same strategy, show that

$$\Phi(S) \leq \frac{1}{2}\left(\kappa(S) + \frac{1}{\kappa(S)}\right) I_n,$$

where $\kappa(S)$ is the condition number of $S$.

(c) Deduce the inequality

$$\kappa(\Phi(S)) \leq \frac{1}{2}\left(\kappa(S) + \frac{1}{\kappa(S)}\right).$$

(d) Draw the conclusion: if $S \in \mathbf{SPD}_n$, then

$$\lim_{m \to +\infty} \Phi^{(m)}(S) = I_n.$$

Verify that this convergence has order 2 at least, like in a Newton'sNewton method.

337. (S. MaciejMaciej, R. StanleyStanley.)

Let us recall that the *permanent* of $A \in \mathbf{M}_n(k)$ is defined by the formula

$$\mathrm{per} A := \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} a_{i\sigma(i)}.$$

It is the same formula as for the determinant, with the exception that the coefficient $\epsilon(\sigma)$ has been replaced by $+1$. We consider matrices with real entries $a_{ij} \in [0,1]$. We assume that there are $m$ zeroes among the entries, with $m \leq n$. We wish to bound the permanent of $A$.

(a) Verify that the maximum of the permanent is achieved, at some matrix whose $n^2 - m$ other entries are 1's.

(b) Prove the formula

$$\mathrm{per} A = \sum_{1 \leq j < k \leq n} \mathrm{per} A^{jk} \cdot \mathrm{per} B^{jk},$$

where $A^{jk}$ denotes the block obtained by retaining only the first two rows and the $j$-th and $k$-th colums, whereas $B^{jk}$ is the block obtained by deleting the first two rows and the $j$-th and $k$-th colums.

(c) Let us assume that $A$ has $m$ zeroes and $n^2 - m$ ones, and that two zeroes of $A$ belong to the same row.

i. Show that $A$ has a row of ones.

ii. Wlog, we may assume that $a_{11} = a_{12} = 0$ and the second row is made of ones. We define $A'$ form $A$ by switching $a_{11}$ and $a_{21}$. Thus the upper-left block of $A'$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using the formula above, show that $\mathrm{per} A < \mathrm{per} A'$.

(d) If $\mathrm{per} A$ is maximal among the admissible matrices, deduce that the zeroes of $A$ are on $m$ distinct rows and $m$ distinct columns.

(e) We may therefore assume that $a_{ii} = 0$ for $i = 1, \dots, m$ and $a_{ij} = 0$ otherwise. Prove that

$$\mathrm{per} A = \sum_{\ell=0}^{m} (-1)^{\ell} \binom{m}{\ell} (n-\ell)!\,.$$

Deduce that

$$\mathrm{per} A \leq n! \left(1 - \frac{m}{2n}\right).$$

338. Let $\delta \in \mathbb{Z}$ be given. We assume that $\delta$ is not the square of an integer. We consider the set $E_\delta$ of matrices $A \in \mathbf{M}_n(\mathbb{Z})$ whose characteristic polynomial is $X^2 - \delta$. If $a, b \in \mathbb{Z}$ and $b | \delta - a^2$, we denote

$$M_{(a,b)} := \begin{pmatrix} a & c \\ b & -a \end{pmatrix} \in E_\delta, \qquad c := \frac{\delta - a^2}{b}.$$

Finally, we say that two matrices $A, B \in \mathbf{M}_n(\mathbb{Z})$ are similar in $\mathbb{Z}$ if there exists $P \in \mathbf{GL}_2(\mathbb{Z})$ such that $PA = BP$.

(a) If $(a, b)$ is above and $\lambda \in \mathbb{Z}$, verify that $M_{(a,b)}$, $M_{(a,-b)}$, $M_{(a+\lambda b,b)}$, $M_{(-a,(\delta-a^2)/b)}$ are similar in $\mathbb{Z}$.

(b) Let $M \in E_\delta$ be given. We define $\beta(M)$ as the minimal $b > 0$ such that $M$ is similar to $M_{(a,b)}$. Prove that this definition makes sense.

(c) Show that there exists an $a \in \mathbb{Z}$ such that $|a| \le \frac{1}{2}\beta(M)$, such that $M$ is similar to $M_{(a,b)}$.

(d) Compare $|\delta - a^2|$ with $\beta(M)^2$. Deduce that

$$\beta(M) \le \begin{cases} \sqrt{\delta}, & \text{if } \delta > 0, \\ \sqrt{4|\delta|/3}, & \text{if } \delta < 0. \end{cases}$$

(e) Finally, show that $E_\delta$ is the union of finitely many conjugation classes in $\mathbb{Z}$.

339. Let $k$ be a field and $P \in k[X]$ be a monic polynomial of degree $n$.

(a) When is the companion matrix $B_P$ diagonalizable ?

(b) Show that the Euclidian algorithm can be used to split $P$ into factors having simple roots, in finitely many elementary operations.

(c) Deduce an explicit construction of a diagonalizable matrix $A_P \in \mathbf{M}_n(k)$ whose characteristic polynomial is $P$ (*diagonalizable companion*).

**Nota**: When $k \in \mathbb{R}$ and the roots of $P$ are real, Exercise 92 gives an alternate construction.

340. Elaborate a test which tells you in finite time whether a matrix $A \in \mathbf{M}_n(k)$ is diagonalizable or not. **Hints**: – $A$ is diagonalizable if and only if $P(A) = 0_n$ for some polynomial with simple roots, – One may construct explicitly the factor $P$ of the characteristic polynomial $P_A$, whose roots are simple and are those of $P_A$ (see Exercise 339).

341. Let $A \in \mathbf{M}_n(\mathbb{C})$ be a strictly diagonally dominant matrix. We denote

$$r_i := \frac{|a_{ii}|}{\sum_{j\ne i} |a_{ij}|} < 1, \qquad i = 1, \dots n.$$

Let us recall that $A$ is non-singular, and denote $B := A^{-1}$.

(a) Let $i \ne k$ be indices. Show that

$$|b_{ik}| \le r_i \max\{|b_{jk}| \,;\, j \ne i\}.$$

(b) Deduce that for every $i \ne k$, we have $|b_{ik}| \le r_i |b_{kk}|$. **Hint**: Fix $k$ and consider the index $j$ that maximizes $|b_{jk}|$.

Remark that $A^{-1}$ is not necessarily strictly diagonally dominant.

342. We come back to the *relative gain array* defined in Exercise 335:

$$\Phi(A) := A \circ A^{-T}.$$

We consider the strictly diagonally dominant case and use the notations of Exercise 341.

(a) Verify that
$$\sum_{j \neq i} |\Phi(A)_{ij}| \leq r_i (\max_{j \neq i} r_j) |\Phi(A)_{ii}|.$$

(b) Deduce that $\Phi(A)$ is strictly diagonally dominant too. Denoting $r(A) := \max_i r_i$, we have $r(\Phi(A)) \leq r(A)^2$.

(c) We now consider the iterates $A^{(k)} := \Phi^{(k)}(A)$. Show that $A^{(k)}$ is strictly diagonally dominant and that $r(A^{(k)}) \to 0$. Deduce that $A^{(k)} = D^{(k)}(I_n + E^{(k)})$, where $D^{(k)}$ is diagonal and $E^{(k)} \to 0_n$.

(d) Show that $A^{(k+1)} = D^{(k)} M^{(k)} \left(D^{(k)}\right)^{-1}$ where $M^{(k)} \to I_n$. Deduce that $D^{(k+1)} \to I_n$.

(e) Finally prove (JohnsonJohnson!Charles R. & ShapiroShapiro)
$$\lim_{k \to +\infty} \Phi^{(k)}(A) = I_n$$
for every strictly diagonally dominant matrix.

343. We continue the analysis of the *relative gain array* defined in Exercise 335, following JohnsonJohnson!Charles R. & ShapiroShapiro. We consider permutation matrices and linear combinations of two of them. We recall that permutation matrices are orthogonal: $P^{-T} = P$.

(a) Assume that $C$ is the permutation matrix associated with an $n$-cycle. If $z \in \mathbb{C}$ is such that $z^n \neq 1$, verify that
$$(I_n - zC)^{-1} = \frac{1}{1 - z^n}(I_n + zC + \cdots + z^{n-1}C^{n-1}).$$
Deduce that
$$\Phi(I_n - zC) = \frac{1}{1 - z^n}(I_n - z^n C).$$

(b) Use the formula above to prove that
$$\Phi^{(k)}(I_n - zC) = \frac{1}{1 - z^{n^k}}(I_n - z^{n^k} C).$$
Deduce that if $|z| < 1$ (respectively $|z| > 1$) then $\Phi^{(k)}(I_n - zC)$ converges towards $I_n$ (resp $C$) as $k \to +\infty$.

(c) If $z^n = z \neq 1$, deduce that $\frac{1}{1-z}(I_n - zC)$ is a fixed point of $\Phi$.

(d) If $A \in \mathbf{GL}_n(\mathbb{C})$ and $P$ is a permutation matrix, verify that $\Phi(PA) = P\Phi(A)$.

(e) Show that if $P$ and $Q$ are permutation matrices and $P+Q$ is non-singular, then $\frac{1}{2}(P+Q)$ is a fixed point of $\Phi$. **Hint**: reduce to the case where $P=I_n$, then work blockwise to deal only with cycles.

344. The QR method for the calculation of the eigenvalues of a complex matrix was designed in 1961–62 by J. FrancisFrancis. Not only he proved the convergence when the eigenvalues have distinct moduli, but he recognized the necessity of shifts:

- shifts help to enhance the convergence by reducing the ratio $\lambda_n/\lambda_{n-1}$, where $\lambda_n$ is the smallest eigenvalue,
- complex shifts help to discriminate pairs of distinct eigenvalues that have the same modulus. This problem is likely to happen for matrices with real entries, because of complex conjugate pairs.

We describe below a few basic facts about shifts. The algorithm works as follows. We start with $A$, presumably a Hessenberg matrix. We choose a $\rho_0 \in \mathbb{C}$ and make the $QR$ factorization

$$A-\rho_0 I_n = Q_0 R_0.$$

Then we re-combine $A_1 := R_0 Q_0 + \rho_0 I_n$. More generally, if $A_j$ is an iterate, we choose $\rho_j \in \mathbb{C}$, decompose $A_j - \rho_j I_n = Q_j R_j$ and recompose $A_{j+1} := R_j Q_j + \rho_j I_n$. Thus the standard QR algorithm (without shift) corresponds to choices $\rho_j = 0$.

We still denote $P_j := Q_0 \cdots Q_{j-1}$ and $U_j := R_{j-1} \cdots R_0$.

(a) Verify that $A_{j+1} = Q_j^* A_j Q_j$ and then $A_k = P_k^* A P_k$.

(b) Show that $P_k U_k$ is the QR factorization of the product $(A-\rho_{k-1} I_n)\cdots(A-\rho_0 I_n)$.

(c) We consider the case where $A \in \mathbf{M}_n(\mathbb{R})$. We choose $\rho_1 = \bar{\rho}_0$. Show that $P_2 \in \mathbf{O}_n(\mathbb{R})$ and deduce that $A_2 \in \mathbf{M}_n(\mathbb{R})$.

**Nota**: J. FrancisFrancis found a way to perform the two first iterations at once, by using only calculations within real numbers. Therefore the shifted QR method does not need to dive into the complex numbers when $A$ has real entries. See D. Watkins, *American Math. Monthly*, May 2011, pp 387–401.

345. A square matrix $A$ is said to be *non-derogatory* if for every eigenvalue $\lambda$, one has $\dim\ker(A-\lambda I_n) = 1$.

(a) Let $J = J(0;r)$ be the basic Jordan block. If $B \in \mathbf{M}_r(k)$ commutes with $J$, show that $B$ is a polynomial in $J$.

(b) More generally, if $B$ commutes with a non-derogatory matrix $A$, show that there exists $p \in k[X]$ such that $B = p(A)$. **Hint**: Jordanization, plus polynomial interpolation.

346. (BezerraBezerra, R. HornHorn!Roger.) This is a follow-up of the previous exercise. Let $A \in \mathbf{M}_n(\mathbb{C})$ be non-derogatory, and suppose that $B \in \mathbf{M}_n(\mathbb{C})$ commutes with both $A$

and $A^*$. Show that $B$ is normal. **Hint**: Use Schur's Theorem that a matrix is unitarily trigonalizable.

347. If $v_1, \dots, v_r \in \mathbb{R}^n$ have non-negative entries, then the matrix $v_1 v_1^T + \cdots + v_r v_r^T$ is symmetric positive semidefinite and has non-negative entries. A natural question is whether the converse holds true: given a symmetric matrix $S$, positive semidefinite with non-negative entries, do there exist vectors $v_j \geq 0$ such that $S = v_1 v_1^T + \cdots + v_r v_r^T$ ? According to P. H. DianandaDiananda, and to M. HallHall!M. & M. NewmanNewman, this is true for $n \leq 4$. The following example, due to Hall, shows that it is false when $n \geq 5$:

$$S = \begin{pmatrix} 4 & 0 & 0 & 2 & 2 \\ 0 & 4 & 3 & 0 & 2 \\ 0 & 3 & 4 & 2 & 0 \\ 2 & 0 & 2 & 4 & 0 \\ 2 & 2 & 0 & 0 & 4 \end{pmatrix}.$$

(a) Verify that $S$ is positive semidefinite. In particular, $\det S = 0$, thus $S$ has a non-trivial kernel. Compute a generator of the kernel

(b) Suppose that $S$ was a $v_1 v_1^T + \cdots + v_r v_r^T$ for some non-negative vectors $v_j$.

i. Show that $S$ can be written as a sum $S_1 + S_2$ with

$$S_1 = \begin{pmatrix} 4 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & a & 0 \\ 2 & 0 & 0 & 0 & b \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 3 & 0 & 2 \\ 0 & 3 & 4 & 2 & 0 \\ 0 & 0 & 2 & x & 0 \\ 0 & 2 & 0 & 0 & y \end{pmatrix},$$

and $a, b, x, y \geq 0$ and both matrices are positive semidefinite.

ii. Show that necessarily, $a = b = 0$ and $x = y = 4$. **Hint**: Use the kernel of $S$.

iii. Deduce a contradiction.

(c) Let $P_n$ denote the cone of $n \times n$ symmetric matrices with non-negative entries. Let $\mathbf{Sym}_n^+$ denote the cone of positive semidefinite matrices. Finally $C_n$ denotes the cone of symetric $n \times n$ matrices $S$ having the property that for $v \in \mathbb{R}^n$,

$$(v \geq 0) \Longrightarrow (v^T S v \geq 0).$$

Show that if $n \geq 5$, then $\mathbf{Sym}_n^+ + P_n \subsetneq C_n$. **Hint**: argue by duality.

348. (a) Parametrization: Given $(a, b, c, d)$ such that $a^2 + b^2 + c^2 + d^2 = 1$, verify that the matrix

$$\begin{pmatrix} a^2 + b^2 - c^2 - d^2 & 2bc - 2ad & 2bd + 2ac \\ 2bc + 2ad & a^2 - b^2 + c^2 - d^2 & 2cd - 2ab \\ 2bd - 2ac & 2cd + 2ab & a^2 - b^2 - c^2 + d^2 \end{pmatrix}$$

is orthogonal.

(b) Interpretation: Let $\mathbb{H}$ be the skew field of quaternions, whose basis is $(1, \mathbf{i}, \mathbf{j}, \mathbf{k})$. If $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H}$, we denote $\bar{q} := a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$ its *conjugate*. We identify the Euclidian space $\mathbb{R}^3$ with the *imaginary* quaternions $q = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. In other words, $v \in \mathbb{R}^3$ iff $\bar{v} = -v$. We have $\overline{qr} = \bar{r}\bar{q}$. Finally, the norm over $\mathbb{H}$ is

$$\|q\| := \sqrt{q\bar{q}} = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

One has $\|qq'\| = \|q\| \, \|q'\|$.

Suppose that $q \in \mathbb{H}$ has unit norm. Verify that $q^{-1} = \bar{q}$. We consider the linear map $r \mapsto L_q r := qrq^{-1}$. Show that $L_q(\mathbb{R}^3) \subset \mathbb{R}^3$. Verify that the restriction $R_q$ of $L_q$ to $\mathbb{R}^3$ is an isometry.

Prove that $q \mapsto R_q$ is a continuous group homomorphism from the unit sphere of $\mathbb{H}$ into $\mathbf{O}_3(\mathbb{R})$. Deduce that its image is included into $\mathbf{SO}_3(\mathbb{R})$.

(c) Conversely, let $R$ be a rotation of $\mathbb{R}^3$ with axis $\vec{u}$ and angle $\alpha$. Show that $R = R_q$ for $q := \cos\frac{\alpha}{2} + \vec{u}\sin\frac{\alpha}{2}$. The morphism $q \mapsto R_q$ is thus onto.

349. Let $M \in \mathbf{M}_n(k)$ be given. We denote $M^{(j)}$ the $j$-th principal minor,

$$M^{(j)} := M\begin{pmatrix} 1 & \dots & j \\ 1 & \dots & j \end{pmatrix}.$$

We assume that these principal minors are all nonzero. Recall that this ensures a factorization $M = LU$.

(a) For any fixed pair $(i, j)$, use the Desnanot–JacobiDesnanotJacobi formula (DodgsonDodgson (see Lewis C.) condensation formula, exercise 24) to establish the identity

$$\frac{M\begin{pmatrix} 1 & \dots & k-1 & i \\ 1 & \dots & \dots & k \end{pmatrix} M\begin{pmatrix} 1 & \dots & \dots & k \\ 1 & \dots & k-1 & j \end{pmatrix}}{M^{(k)}M^{(k-1)}} = A_{k-1} - A_k,$$

where

$$A_k := \frac{M\begin{pmatrix} 1 & \dots & k & i \\ 1 & \dots & k & j \end{pmatrix}}{M^{(k)}}.$$

(b) Deduce the formula

$$\sum_{k=1}^{n} \frac{M\begin{pmatrix} 1 & \dots & k-1 & i \\ 1 & \dots & \dots & k \end{pmatrix} M\begin{pmatrix} 1 & \dots & \dots & k \\ 1 & \dots & k-1 & j \end{pmatrix}}{M^{(k)}M^{(k-1)}} = m_{ij}.$$

(c) Deduce that in the factorization $M = LU$, we have

$$\ell_{ij} = \frac{M\begin{pmatrix} 1 & \dots & j-1 & i \\ 1 & \dots & \dots & j \end{pmatrix}}{M^{(j)}}, \qquad u_{ij} = \frac{M\begin{pmatrix} 1 & \dots & \dots & i \\ 1 & \dots & i-1 & j \end{pmatrix}}{M^{(i-1)}}.$$

(d) In particular, if $M$ is *totally positive*, meaning that all minors $M\begin{pmatrix} i_1 & \cdots & i_r \\ j_1 & \cdots & j_r \end{pmatrix}$ are strictly positive whenever $i_1 < \cdots < i_r$ and $j_1 < \cdots < j_r$, then the non-trivial entries of $L$ and $U$ are strictly positive. **Comment**: Actually, all the minors

$$L\begin{pmatrix} i_1 & \cdots & i_r \\ j_1 & \cdots & j_r \end{pmatrix}, \qquad \text{respectively } U\begin{pmatrix} i_1 & \cdots & i_r \\ j_1 & \cdots & j_r \end{pmatrix},$$

with $j_1 < \cdots < j_r \leq i_1 < \cdots < i_r$ (resp. $i_1 < \cdots < i_r \leq j_1 < \cdots < j_r$) are strictly positive.

350. Let $\mathcal{T}_n \subset \mathbf{M}_n(\mathbb{R})$ be the set of $n \times n$ tridiagonal bi-stochastic matrices.

(a) Verify that every $M \in \mathcal{T}_n$ is of the form

$$\begin{pmatrix} 1-c_1 & c_1 & 0 & \cdots & & 0 \\ c_1 & 1-c_1-c_2 & c_2 & \ddots & & \\ 0 & c_2 & 1-c_2-c_3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & c_{n-1} & 0 \\ & & \ddots & c_{n-1} & 1-c_{n-1}-c_n & c_n \\ 0 & & \cdots & 0 & c_n & 1-c_n \end{pmatrix}.$$

In particular, $M$ is symmetric.

(b) Verify that $\mathcal{T}_n$ is a convex compact subset of $\mathbf{M}_n(\mathbb{R})$, defined by the inequalities

$$0 \leq c_i \leq 1, \qquad c_i + c_{i+1} \leq 1.$$

(c) Prove that the extremal points of $\mathcal{T}_n$ are those matrices $M$ for which $(c_1, \ldots, c_n)$ is a sequence of 0s and 1s, in which two 1s are always separated by one or several 0s. We denote $\mathcal{F}_n$ the set of those sequences.

(d) Find a bijection between $\mathcal{F}_n \cup \mathcal{F}_{n-1}$ and $\mathcal{F}_{n+1}$. Deduce that the cardinal of $\mathcal{F}_n$ is the $n$-th Fibonacci number.

351. (After Y. BenoistBenoist and B. JohnsonJohnson!Bill.) In Exercise 330, we proved that every real matrix $M \in \mathbf{M}_n(\mathbb{R})$ can be written as the product $HK$ of (possibly complex) Hermitian matrices. Of course, one has $\|M\| \leq \|H\| \cdot \|K\|$. We show here that if $n = 3$ (and therefore also if $n > 3$), there does not exist a finite number $c_n$ such that $(H, K)$ can always be chosen so that $\|H\| \cdot \|K\| \leq c_n \|M\|$. Thus this factorization is *unstable*.

So let us assume that for some finite $c_3$, the following property holds true: for every $M \in \mathbf{M}_3(\mathbb{R})$, there exist $H, K \in \mathbf{H}_3$ such that $M = HK$ and $\|H\| \cdot \|K\| \leq c_3 \|M\|$.

We recall that the condition number of a non-singular matrix $P$ is

$$\kappa(P) := \|P\| \cdot \|P^{-1}\| \geq 1.$$

(a) Let $M \in \mathbf{GL}_n(\mathbb{R})$ be given. Show that $\chi(M) \le c_3\kappa(M)$, where $\chi(M)$ is defined as the infimum of $\kappa(P)$ where $P^{-1}MP = M^T$.

(b) Let $\mathcal{B} := (v_1, v_2, v_3)$ be a basis of $\mathbb{R}^3$ and let $(w_1, w_2, w_3)$ be the dual basis. If $Mv_i = \lambda_i v_i$ for all $i$, verify that $M^T w_i = \lambda_i w_i$.

(c) Chosing pairwise distinct eigenvalues $\lambda_i$, deduce that $\Phi(\mathcal{B}) \le c_3\kappa(M)$, where $\Phi(\mathcal{B})$ is the infimum of $\kappa(P)$ for which $Pv_i \parallel w_i$.

(d) Deduce that for every basis $\mathcal{B}$, one has $\Phi(\mathcal{B}) \le c_3$.

(e) We choose the basis $\mathcal{B}_\epsilon$ given by

$$v_{1,\epsilon} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad v_{2,\epsilon} = \begin{pmatrix} 1 \\ \epsilon \\ 0 \end{pmatrix}, \qquad v_{3,\epsilon} = \begin{pmatrix} 1 \\ 0 \\ \epsilon \end{pmatrix}.$$

Compute the dual basis. If the inequality above is true, prove that there exists a subsequence $(P_\epsilon)_{\epsilon_m \to 0}$ that converges to some $P_0 \in \mathbf{GL}_3(\mathbb{C})$, such that $P_\epsilon v_{i,\epsilon} \parallel w_{i,\epsilon}$. Conclude.

352. Let $H \in \mathbf{H}_n$ be a Hermitan matrix, with indices of inertia $(n_-, 0, n_+)$. Hence $H$ is non-degenerate.

(a) Let us write $H$ blockwise as

$$\begin{pmatrix} H_- & X \\ X^* & H_+ \end{pmatrix},$$

where $H_-$ has size $n_- \times n_-$. If $H_-$ is definite negative, prove that its Schur complement $H_+ - X^* H_-^{-1} X$ is positive definite.

(b) Deduce that if $E \subset \mathbb{C}^n$ is a subspace of dimension $n_-$ on which the form $x \mapsto x^* H x$ is negative definite ($E$ is a *maximal negative* subspace for $H$), then the form $y \mapsto y^* H^{-1} y$ is positive definite on $E^\perp$ ($E^\perp$ is a *maximal positive* subspace for $H^{-1}$).

353. We consider the method of JacobiJacobi for the approximate calculation of the spectrum of a Hermitian matrix $H$. We take the notations of Section 13.4 of the second edition.

(a) Recall that the equation $t^2 + 2t\sigma - 1 = 0$ admits two roots $t, t'$, with $t = \tan\theta$ and $\theta \in [-\frac{\pi}{4}, \frac{\pi}{4})$. Verify that the other root corresponds to $t' = \tan\theta'$ with $\theta' = \theta + \frac{\pi}{2}$.

(b) We call $\theta$ the *inner* angle and $\theta'$ the *outer* angle. Show that if the choice of angle $\theta$ or $\theta'$ leads to an iterate $K$ or $K'$, then $K'$ is conjugated to $K$ by a rotation of angle $\frac{\pi}{2}$ in the $(p, q)$-plane.

(c) Deduce that if we fix the list of positions $(p_k, q_k)$ to be set to zero at step $k$, the choice of the angles $\theta_k$ is irrelevant because, if $A^{(k)}$ and $B^{(k)}$ are two possible iterates at step $k$, then they are conjugated by a *sign-permutation* matrix. Such a matrix is a permutation matrix in which some 1s have been replaced by $-1$s.

354. (After X. TuniTuni.) We show here that it is not possible to define a continuous square root map over $\mathbf{GL}_2(\mathbb{C})$.

(a) Let $A \in \mathbf{GL}_2(\mathbb{C})$ be given. If $A$ has two distinct eigenvalues, prove that there are exactly four matrices $X$ such that $X^2 = A$.

(b) If instead $A$ is not semi-simple, verify that there are exactly two matrices $X$ such that $X^2 = A$.

(c) We suppose that there exists a square root map $A \mapsto A^{1/2}$ over $\mathbf{GL}_2(\mathbb{C})$, which is continuous. Without loss of generality, me may assume that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{1/2} = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}.$$

For $x \in [0, \pi)$, prove the formula

$$\begin{pmatrix} e^{2ix} & 1 \\ 0 & 1 \end{pmatrix}^{1/2} = \begin{pmatrix} e^{ix} & (1+e^{ix})^{-1} \\ 0 & 1 \end{pmatrix}.$$

**Hint**: Use a continuity argument.

(d) By letting $x \to \pi$, conclude.

355. Let $A \in \mathbf{Sym}_n$ be a positive semi-definite matrix. We assume that its diagonal part $D$ is positive definite. If $b \in R(A)$, we consider the Gauss–SeidelGaussSeidel method to solve the system $Ax = b$:

$$(D - E)x^{(m+1)} = E^T x^{(m)} + b.$$

Remark that $D - E$ is non-singular from the assumption. The iteration matrix is $G = (D - E)^{-1}E^T$.


Gauß,

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(a) Define $y^{(m)} = x^{(m)} - \bar{x}$, where $\bar{x}$ is some solution of $A\bar{x} = b$. Verify that $y^{(m+1)} = Gy^{(m)}$. In order that $y^{(m)}$ converges for every choice of the initial data $y^{(0)}$, prove that it is necessary and sufficient that

- if $\lambda = 1$ is an eigenvalue of $G$, then it is semi-simple,
- the rest of the spectrum of $G$ is of modulus $< 1$.

(b) Verify that $\ker(G - I_n) = \ker A$.

(c) Show that $G$ commutes with $(D - E)^{-1}A$. Deduce that $(D - E)^{-1}R(A)$ is a $G$-invariant subspace.

(d) Prove that $\mathbb{R}^n = \ker A \oplus (D-E)^{-1}R(A)$. **Hint**: if $v^T(D-E)v = 0$, then $v^T(D+A)v = 0$, hence $v = 0$.

(e) Prove that the spectrum of the restriction of $G$ to $(D-E)^{-1}R(A)$ has modulus $< 1$. Conclude. **Hint**: follow the proof of Lemma 20 in Chapter 12 of the 2nd edition.

356. (After C. HillarHillar.) A matrix $A \in \mathbf{M}_n(\mathbb{C})$ is *completely invertible* if its numerical range $W(A)$ does not contain 0. In particular, it is invertible, because $W(A)$ contains the spectrum of $A$.

We suppose that $A$ is completely invertible.

(a) Show that there exists $\theta \in \mathbb{R}$ and $\epsilon > 0$ such that $W(e^{i\theta}A)$ is contained in the half-plane $\Re z \geq \epsilon$.

(b) Let us denote $B := e^{i\theta}A$. Verify that $B + B^* \geq 2\epsilon I_n$. Deduce that $B^{-*} + B^{-1}$ is positive definite.

(c) Prove that there exists $\alpha > 0$ such that $W(B^{-1})$ is contained in the half-plane $\Re z \geq \alpha$.

(d) Conclude that $A^{-1}$ is completely invertible. Therefore a matrix is completely invertible if and only if its inverse is so.

357. I shall not comment the title of this exercise, but it is related to the following fact: if $M \in \mathbf{SO}_n(\mathbb{R})$ is given in block form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A$ and $D$ are square matrices, not necessarily of the same size, prove that $\det A = \det D$. **Hint**: Find block-triangular matrices $L$ and $U$ such that $ML = U$.

The same identity holds true if $M \in \mathbf{SO}(p,q)$.

358. Let us say that a lattice

$$\mathcal{L} = \oplus_{j=1}^{d} \mathbb{Z}\mathbf{v}_j$$

of $\mathbb{R}^d$ has a *five-fold symmetry* if there exists a matrix $A \in \mathbf{M}_d(\mathbb{R})$ such that $A^5 = I_d$, $A \neq I_d$ and $A\mathcal{L} = \mathcal{L}$.

(a) Verify that the lattice $\mathbb{Z}^5$ has a five-fold symmetry.

(b) If $A\mathcal{L} = \mathcal{L}$, verify that $A$ is similar to a matrix $B \in \mathbf{M}_d(\mathbb{Z})$. Deduce that its eigenvalues are algebraic integers of degree $\leq d$.

(c) Suppose $d = 3$ and $\mathcal{L}$ has a five-fold symmetry $A$. Show that $A$ is diagonalizable in $\mathbf{M}_3(\mathbb{C})$, with eigenvalues 1, $\omega$ and $\bar{\omega}$, where $\omega$ is a primitive root of unity of order 5. Deduce that there does not exist a 3-dimensional lattice with a five-fold symmetry.

(d) In the same vein, if $\mathcal{L}$ is a 4-dimensional lattice with a five-fold symmetry $A$, show that $A^4 = -I_4 - A - A^2 - A^3$.

(e) Let $\alpha \neq \beta$ be the roots of $X^2 - X - 1$. Let us form the matrix

$$A := \frac{1}{2}\begin{pmatrix} -1 & -\alpha & -\beta & 0 \\ 1 & 0 & \alpha & -\beta \\ 1 & \beta & 0 & -\alpha \\ -1 & 1 & 1 & -1 \end{pmatrix}.$$

Define

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_2 = A\mathbf{v}_1, \qquad \mathbf{v}_3 = A\mathbf{v}_2, \qquad \mathbf{v}_4 = A\mathbf{v}_3.$$

Show that the $\mathbf{v}_j$'s span a lattice that has a five-fold symmetry.

359. (After O. Taussky and H. ZassenhausTausskyZassenhaus.) We show here that if $A \in \mathbf{M}_n(k)$, then there exists a *symmetric* matrix $S \in \mathbf{GL}_n(k)$ such that $A^T = SAS^{-1}$ (compare with Exercises 327–330). Of course, we already know that there exists a (possibly non-symmetric) matrix $R \in \mathbf{GL}_n(k)$ such that $A^T = RAR^{-1}$.

(a) Let us begin with the case where the characteristic and minimal polynomials of $A$ coincide. We recall that then, the subspace $\mathrm{Com}(A)$ of matrices commuting with $A$ equals

$$\{P(A) \mid P \in k[X]\},$$

and that its dimension equals $n$.

i. Define the subspace $\mathrm{CT}(A) \subset \mathbf{M}_n(k)$ by the equation $MA = A^TM$. Define also the subspace $\mathrm{ST}(A) \subset \mathbf{M}_n(k)$ by the equations

$$SA = A^TS \quad \text{and} \quad S = S^T.$$

Verify that $\mathrm{ST}(A) \subset \mathrm{CT}(A) \subset R \cdot \mathrm{Com}(A)$.

ii. Show that $\dim \mathrm{ST}(A) \geq n$.

iii. Deduce that every matrix $M$ such that $MA = A^TM$ is symmetric.

(b) We now drop the condition about the characteristic and minimal polynomials of $A$. Using the case above, show that there exists a symmetric and non-singular $S$ such that $SA = A^TS$. **Hint**: Apply the case above to the diagonal blocks of the FrobeniusFrobenius form of $A$.

**Remark**. This situation is interesting in infinite dimension too. For instance, let us take a differential operator $L = D^2 + D \circ a$, where $a$ is a $\mathcal{C}^1$-bounded function, that is $Lu = u'' + (au)'$. As an unbounded operator over $L^2(\mathbb{R})$, $L$ has an adjoint $L^* = D^2 - aD$, that is $Lv = v'' - av'$. There are a lot of self-adjoint operators $S$ satisfying $SL = L^*S$. For instance,

$$Sz = (\alpha z')' + \gamma z$$

works whenever $\alpha(x)$ and $\gamma(x)$ solve the linear differential equations

$$\alpha' = a\alpha \quad \text{ and } \quad \gamma = \alpha a' + \text{cst}\,\alpha.$$

More generally, the analysis above suggests that for every $P \in \mathbb{R}[X]$, $S := \alpha P(L)$ is a self-adjoint operator satisfying $SL = L^* S$.

However, the situation can be very different from the finite-dimensional one, just because it may happen that an operator $M$ is not conjugated to $M^*$. This happens to the derivation $D = \frac{d}{dx}$ within the algebra of differential operators. We recover conjugacy by leaving the realm of differential operators: the symmetry

$$S_0 : (x \mapsto f(x)) \longmapsto (x \mapsto f(-x))$$

satisfies $S_0 D = D^* S_0 = -DS_0$. Then, as above, every $S = S_0 P(D)$ with $P \in \mathbb{R}[X]$ is self-adjoint and satisfies $SD = D^* S$.

360. (Continuation.) We investigate now the real case: $k = \mathbb{R}$. We ask under which condition it is possible to choose $S$ symmetric positive definite such that $SA = A^T S$.

(a) Show that a necessary condition is that $A$ has a real spectrum.

(b) Prove that if $A$ is similar to $B$, and if there exists $\Sigma \in \mathbf{SPD}_n$ such that $\Sigma B = B^T \Sigma$, then there exists $S \in \mathbf{SPD}_n$ such that $SA = A^T S$.

(c) If $A$ is a Jordan block, describe explicitly the solutions of $SA = A^T S$. Verify that some of them are positive definite.

(d) Deduce that a necessary and sufficient condition is that $A$ has a real spectrum.

361. Let $n \geq 2$ be a given integer. We identify below the convex cone $K$ spanned by matrices of the form $-A^2$, with $A \in \mathbf{M}_n(\mathbb{R})$ running over skew-symmetric matrices.

(a) If $A$ is skew-symmetric with real entries, verify that $-A^2$ is symmetric, positive semi-definite, and that its non-zero eigenvalues have even multiplicities.

(b) Show that $K$ is contained in the set

$$\left\{ S \in \mathbf{Sym}_n \,|\, 0_n \leq S \leq \frac{\operatorname{Tr} S}{2} I_n \right\}.$$

(c) Let us define

$$C_1 := \left\{ a \in \mathbb{R}^n \,|\, \sum_j a_j = 1 \quad \text{and} \quad 0 \leq a_j \leq \frac{1}{2}, \forall j \right\}.$$

i. Show that $C_1$ is a compact, convex subset of $\mathbb{R}^n$, and that its extremal points have the form $\frac{1}{2}(\mathbf{e}_i + \mathbf{e}_j)$ for some $i < j$.

ii. Deduce that $C_1$ is the convex hull of

$$C_1 := \left\{ \frac{1}{2}(\mathbf{e}_i + \mathbf{e}_j) \,|\, 1 \le i < j \le n \right\}.$$

iii. Finally, show that

$$\left\{ a \in \mathbb{R}^n \,|\, 0 \le a_j \le \frac{1}{2} \sum_k a_k, \, \forall j \right\} = \operatorname{conv}\left( \{ \lambda(\mathbf{e}_i + \mathbf{e}_j) \,|\, \lambda \ge 0, \, 1 \le i < j \le n \} \right).$$

(d) Using this last result, show that actually,

$$K = \left\{ S \in \mathbf{Sym}_n \,|\, 0_n \le S \le \frac{\operatorname{Tr} S}{2} I_n \right\}.$$

362. (After FeitFeit & HigmanHigman.) Let $M \in \mathbf{M}_n(k)$ be given, and $p \in k[X]$ a non-zero polynomial such that $p(M) = 0$ (that is, a multiple of the minimum polynomial of $M$). Let $m$ be the multiplicity of $\lambda \in k$ as a root of $p$, and define the polynomial $q(X) = p(X)/(X - \lambda)^m$. Prove that the multiplicity of $\lambda$ as an eigenvalue of $M$ equals

$$\frac{\operatorname{Tr} q(M)}{q(\lambda)}.$$

**Hint**: decompose $k^n$ into $R((M - \lambda)^m)$ and $\ker(M - \lambda)^m$.

363. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous convex function. If $x, y \in \mathbb{R}^n$ are such that $x \prec y$, prove that

$$\sum_{j=1}^n \phi(y_j) \le \sum_{j=1}^n \phi(x_j).$$

**Hint**: Use Proposition 6.4 in the 2nd edition.

**Application**: Let $M \in \mathbf{M}_n(\mathbb{C})$ be given, with eigenvalues $\lambda_j$ and singular values $\sigma_j$. If $s$ is a positive real number, deduce from above and from Exercise 69 the inequality

$$\sum_{j=1}^n |\lambda_j|^s \le \sum_{j=1}^n \sigma_j^s.$$

364. (Golden, Wasin So, ThompsonGoldenSo, WasinThompson.)

(a) Let $H, K \in \mathbf{HPD}_n$ be given. Using the previous exercise, prove that for every integer $m \ge 1$,

$$\operatorname{Tr}\left( (HK)^{2m} \right) \le \operatorname{Tr}\left( (KH^2K)^m \right).$$

(b) If $m = 2$, prove that the equality holds above if, and only if $[H, K] = 0_n$.

(c) Let $A, B$ be Hermitian matrices. Show that the sequence $(u_k)_{k\geq 1}$ defined by

$$u_k = \operatorname{Tr}\left((e^{A/2^k}e^{B/2^k})^{2^k}\right)$$

is non-increasing.

(d) Deduce that

$$\operatorname{Tr}(e^A e^B) \geq \operatorname{Tr} e^{A+B}.$$

**Hint**: UseTrotter's formula

$$\lim_{m\to+\infty}(e^{A/m}e^{B/m})^m = e^{A+B}.$$

(e) In the equality case, that is if $\operatorname{Tr}(e^A e^B) = \operatorname{Tr} e^{A+B}$, show that $\operatorname{Tr}((e^A e^B)^2) = \operatorname{Tr}(e^A e^{2B} e^A)$. Then deduce that $[e^A, e^B] = 0_n$, and actually that $[A, B] = 0_n$.

365. Let $t \mapsto A(t)$ be a continuous map with values in the cone of real $n \times n$ matrices with non-negative entries. We denote by $E$ the vector space of solutions of the differential equation

$$\frac{dx}{dt} = -A(t)x.$$

We also define $\mathbf{e} = (\mathbf{1}, \ldots, \mathbf{1})^{\mathbf{T}}$.

(a) Verify that the map $x \mapsto \|x(0)\|_1$ is a norm over $E$.

(b) If $\tau > 0$ is given, we denote $x^\tau \in E$ the solution satisfying $x^\tau(\tau) = \mathbf{e}$. Verify that $x^\tau(t) \geq 0$ for every $t \in [0, \tau]$.

(c) Let us define $y^\tau = x^\tau / \|x^\tau(0)\|_1$. Show that the family $(y^\tau)_{\tau>0}$ is relatively compact in $E$, and that it has a cluster point as $\tau \to +\infty$.

(d) Deduce Hartman–Wintner'sHartmanWintner Theorem: There exist a non-zero solution $y(t)$ of the ODE such that $y(t) \geq 0$ and $y'(t) \leq 0$ for every $t \geq 0$. In particular, $y(t)$ admits a limit as $t \to +\infty$.

(e) When $A$ is constant, give such a solution in close form.

366. Let $K$ be a non-void compact convex subset in finite dimension. If $x \in \partial K$, the Hahn–BanachHahnBanach Theorem ensures that there exists at least one convex cone (actually a half-space) with apex $x$, containing $K$. The set of all such convex cones admits a smaller one, namely the intersection of all of them. We call it the *supporting cone* of $K$ at $x$, and denote it $C_K(x)$. If there is no ambiguity, we just write $C(x)$.

We admit the following properties, which are classical in convex analysis:

- $K$ is the intersection of its supporting cones $C(x)$ when $x$ runs over the extremal points of $K$,
- If $x \in \partial K$, $C(x)$ is the smallest cone with apex at $x$, containing all the extremal points of $K$.

In what follows, we determine $C_K(I_n)$ when $K$ is the convex hull of $\mathbf{SO}_n(\mathbb{R})$ in $\mathbf{M}_n(\mathbb{R})$. We assume that $n \geq 2$.

(a) We look for those matrices $Q \in \mathbf{M}_n(\mathbb{R})$, such that $\operatorname{Tr} Q(R - I_n) \leq 0$ for every $R \in \mathbf{SO}_n(\mathbb{R})$.

- Using the one-parameters subgroups of $\mathbf{SO}_n(\mathbb{R})$, show that for every skew-symmetric matrix $A$, one has
$$\operatorname{Tr}(QA) = 0, \qquad \operatorname{Tr}(QA^2) \leq 0.$$
- Show that $Q$ is symmetric.
- We denote $q_1 \leq q_2 \leq \cdots \leq q_n$ the eigenvalues of $Q$. Using Exercise 361, show that $q_1 + q_2 \geq 0$. In other words,
$$|q_1| \leq q_2 \leq \cdots \leq q_n.$$
**Hint**: Use the extremal points of the cone $C_1$.

(b) Conversely, let $Q \in \mathbf{Sym}_n(\mathbb{R})$ be such that $q_1+q_2 \geq 0$. Using Dacorogna–MaréchalDacorognaMaréc Inequality (17), prove that $\operatorname{Tr}(QR) \leq \operatorname{Tr} Q$ for every $R \in \mathbf{SO}_n(\mathbb{R})$.

(c) Deduce that $C(I_n)$ is the set of all matrices such that $\operatorname{Tr}(QM) \leq \operatorname{Tr} Q$ for every symmetric $Q$ whose least eigenvalues satisfy $q_1 + q_2 \geq 0$.

(d) Show that $C(I_n)$ is the set of matrices $M$ whose symmetric part $S = \frac{1}{2}(M + M^T)$ satisfies
$$\left(1 + \frac{\operatorname{Tr} S - n}{2}\right) I_n \leq S \leq I_n.$$
**Hint**: Use again Exercise 361.

367. (From Zhiqin LuLu, Zhiqin.) In this exercise and in the next one, one can replace the scalar field $\mathbb{R}$ by $\mathbb{C}$, to the price that $X^T$ be replaced by $X^*$.

Let $X \in \mathbf{M}_n(\mathbb{R})$ be given. We form the matrix $P = \begin{pmatrix} 0_n & X \\ 0_n & 0_n \end{pmatrix} \in \mathbf{M}_{2n}(\mathbb{R})$. We define the linear map
$$T_X : V \in \mathbf{M}_n(\mathbb{R}) \mapsto [P^T, [P, V]].$$

(a) Show that $T_X$ is self-adjoint over $\mathbf{M}_{2n}(\mathbb{R})$, for the standard Euclidian product $\langle V, W\rangle = \operatorname{Tr}(W^T V)$.

(b) Verify that the set of block-diagonal matrices $\begin{pmatrix} B & 0_n \\ 0_n & C \end{pmatrix}$ is an invariant subspace for $T_X$.

The restriction of $T_X$ to this subspace will be denoted $U_X$.

(c) Let $s_1 \geq \cdots \geq s_n$ be the singular values of $X$. Show that the spectrum of $U_X$ consists in the numbers $\mu = s_i^2 + s_j^2$, with multiplicities equal to the number of pairs $(i,j)$ such that this equality holds.

For instance, if all the numbers $s_i^2$ and $s_j^2 + s_k^2$ are pairwise distinct (except for the trivial $s_j^2 + s_k^2 = s_k^2 + s_j^2$) then $s_i^2$ is simple and $s_j^2 + s_k^2$ has multiplicity two for $j \neq k$.

368. (Continuation of the previous one.) We present below the proof by Zhiqin LuLu, Zhiqin of the Böttcher–Wenzelbottcher@BöttcherWenzel Inequality

$$\|[X,Y]\|_F^2 \leq 2\|X\|_F^2 \|Y\|_F^2, \qquad \forall X, Y \in \mathbf{M}_n(\mathbb{R}).$$

For $X \in \mathbf{M}_n(\mathbb{R})$, one defines $S_X : M \mapsto [X^T, [X,Y]]$.

(a) Show that $S_X$ is self-adjoint, positive semi-definite.

(b) Verify that the ratio $\frac{\|[X,Y]\|_F^2}{\|Y\|_F^2}$ is maximal if and only if $Y \neq 0$ belongs to the eigenspace $E$ associated with the largest eigenvalue of $S_X$.

(c) If $Y \in E$, show that $[X^T, Y^T] \in E$. Show also that $[X^T, Y^T]$ is not colinear to $Y$, unless $X = 0$ or $Y = 0$. Deduce that $\dim E \geq 2$.

(d) Show that there exists $Z \neq 0$ in $E$ such that $Z_1 = \begin{pmatrix} Z & 0_n \\ 0_n & Z \end{pmatrix}$ is orthogonal to the the main eigenvector of $U_X$ (that associated with $s_1^2$).

(e) Show that

$$\sup_{Y \neq 0} \frac{\|[X,Y]\|_F^2}{\|Y\|_F^2} = \langle U_X Z_1, Z_1 \rangle$$

and deduce that

$$\sup_{Y \neq 0} \frac{\|[X,Y]\|_F^2}{\|Y\|_F^2} \leq (s_1^2 + s_2^2)\|Y\|_F^2.$$

Then conclude.

**Remark**: This proof gives a little more when $n \geq 3$, because we now that $X \mapsto \phi(X) = \sqrt{s_1^2 + s_2^2}$ is a norm (use Exercise 162), with $\phi \leq \|\cdot\|_F$. We do have $\|[X,Y]\|_F \leq \sqrt{2}\,\phi(X)\|Y\|_F$.

369. (With the help of P. MigdolMigdol.) Let $n \geq 2$ be given. If $M \in \mathbf{M}_n(\mathbb{C})$, we denote $r(M)$ the *numerical radius*

$$r(M) = \sup_{\|x\|_2 = 1} |x^* M x|.$$

Recall that $r$ is norm over $\mathbf{M}_n(\mathbb{C})$. We also define the *real* and *imaginary parts* of $M$ by

$$\Re M = \frac{1}{2}(M + M^*) \in \mathbb{H}_n \qquad \Im M = \frac{1}{2i}(M - M^*).$$

(a) Prove that

$$r(M) = \sup_{\theta \in \mathbb{R}/2\pi\mathbb{Z}} \|\Re(e^{-i\theta} M)\|_2.$$

(b) Let $A, B \in \mathbf{M}_n(\mathbb{C})$ be given. We apply the previous question to $M = [A, B]$. Let $\theta$ be such that $r(M) = \|\Re(e^{-i\theta}M)\|_2$.

i. Let us denote $X = \Re(e^{-i\theta}A)$, $Y = \Im(e^{-i\theta}A)$, $Z = \Re B$ and $T = \Im B$. Check that $r([A, B]) = \|[X, T] + [Y, Z]\|_2$ and deduce

$$r([A, B]) \leq 2(\|X\|_2 \cdot \|T\|_2 + \|Y\|_2 \cdot \|Z\|_2).$$

ii. Conclude that

$$r([A, B]) \leq 4r(A)r(B), \qquad \forall A, B \in \mathbf{M}_n(\mathbb{C}). \tag{37}$$

(c) By picking up a convenient pair $A, B \in \mathbf{M}_n(\mathbb{C})$, show that the constant 4 in (37) is the best possible. In other words,

$$\sup_{A,B \neq 0} \frac{r([A, B])}{r(A)r(B)} = 4.$$

370. Let $\Sigma$ be Hermitian positive definite, with CholeskyCholesky factorization $LL^*$.

(a) Show that in the polar factorization $L = HU$, one has $H = \sqrt{\Sigma}$.

(b) Show that in the QR-factorization $\sqrt{\Sigma} = QR$, one has $R = L^*$.

371. We consider real symmetric matrices; the Hermitian case could be treated the same way. We recall that the eigenvalues and eigenvectors of a matrix are smooth functions of its entries so long as the eigenvalues are simple (see Theorem 5.3 of the second edition). We also recall that a functional calculus is available over Hermitian matrices with continuous functions: if $f : I \to \mathbb{R}$ is continuous and $H = U^*DU$ where $U$ is unitary and $D = \mathrm{diag}(a, b)$, then $f(H) = U^*f(D)U$ with $f(D) = \mathrm{diag}(f(a), f(b))$, whenever the spectrum of $H$ is contained in $I$. This construction does not depend upon the way we diagonalize $H$.

Loewner'sLoewner theory is the study of *operator monotone* functions. A numerical function $f$ over an interval $I$ is operator monotone if whenever the real symmetric matrices $A, B$ have their spectrum included in $I$, then $A \leq B$ implies $f(A) \leq f(B)$. This notion depends upon the size $n$ of the matrices under consideration, the class of operator monotone functions getting narrower as $n$ increases.

Finally, we recall that if $M \in \mathbf{GL}_n(\mathbb{R})$, then $A \leq B$ if and only if $MAM^T \leq MBM^T$.

(a) If $f$ is operator monotone, show that $f$ is monotone in the classical sense. **Hint**: Take $A = \lambda I_2$.

(b) Let $\theta \mapsto S(\theta)$ be a smooth curve such that $S(\theta)$ has simple eigenvalues. Up to a unitary conjugation, we may assume that $S(\theta_0) = \mathrm{diag}(a_1, \ldots, a_n)$. Writing $S(\theta) = P(\theta)\mathrm{diag}(\lambda_1(\theta), \lambda_2(\theta))P(\theta)^T$ with $\lambda_j(\theta_0) = a_j$, $P(\theta_0) = I_n$ and $P(\theta)$ orthogonal, compute the derivatives of $\lambda_j$ and of $P$ at $\theta_0$. Deduce that

$$\left.\frac{d}{d\theta}\right|_{\theta_0} f(S(\theta)) = H \circ S'(\theta_0),$$

is the Hadamard product of $S'(\theta_0)$ with the matrix $H = H(\vec{a})$ whose entries are

$$h_{ij} = \begin{cases} f'(a_j), & \text{if} \quad i = j, \\ \frac{f(a_j)-f(a_i)}{a_j-a_i}, & \text{if} \quad i \neq j. \end{cases}$$

(c) If $f$ is operator monotone over $n \times n$ matrices whose spectrum belong to $I$, deduce from above that $H(\vec{a}) \geq 0_n$ for every $a_1, \dots, a_n \in I$.

(d) Conversely, we suppose that the matrices $H(\vec{a})$ above are non-negative. Let $A, B \in \mathbf{Sym}_n(\mathbb{R})$ have their spectra in $I$, with $B - A$ positive definite. We admit that there exists a smooth curve $\theta \mapsto S(\theta)$ such that $S(0) = A$, $S(A) = B$, $S(\theta)$ has simple eigenvalues for every $\theta \in (0,1)$ and $S'(\theta) \geq 0_n$. This follows from the fact that the set of symmetric matrices $C$ such that $A \leq C \leq B$ is open, and the subset of matrices with a multiple eigenvalue has codimension 2 in $\mathbf{Sym}_n(\mathbb{R})$.
Prove that $\frac{d}{d\theta} f(S(\theta)) \geq 0_2$ for all $\theta \in (0,1)$. Deduce that $f(A) \leq f(B)$.

(e) By continuity, extends this result to the case where $B - A \geq 0_n$.
Hence a $C^1$-function $f$ is operator monotone over $n \times n$ matrices (property (OM$n$)) if and only if the matrices $H(\vec{a})$ are non-negative for every $a_1, \dots, a_n$ in the domain of $f$.

(f) If $k \leq n$ verify that (OM$n$) implies (OM$k$).

(g) Show that (OM$n$) amounts to saying that for all $1 \leq k \leq n$, and for every $a_1, \dots, a_k \in I$, then

$$\det H(\vec{a}) \geq 0.$$

(h) Show that (OM2) amounts to saying that $f' \geq 0$ and $\frac{1}{\sqrt{f'}}$ is concave.

(i) If $I = \mathbb{R}$ and $f$ is operator monotone, show that $f$ is affine.

(j) Consider the case $I = (0, +\infty)$ and $f(t) = t^\alpha$. Show that $f$ is operator monotone if and only if $\alpha \in [-1, 1]$.

372. Let $k$ be a field and $E$ a linear subspace of $\mathbf{M}_n(k)$. We assume that every element of $E$ is singular. We wish to prove that there exist $P, Q \in \mathbf{GL}_n(k)$ such that $PEQ =: E'$ has the property that every $M \in E'$ has a zero entry $m_{nn}$.

(a) This is true if $n = 1$.

(b) Suppose that $M \neq \{0_n\}$. Show that there exist non-singular $P_1, Q_1$ and $1 \leq r < n$ such that

$$J_r = \begin{pmatrix} I_r & 0 \\ 0 & 0_{n-r} \end{pmatrix} \in P_1 E Q_1 =: E_1.$$

(c) Let $F \subset \mathbf{M}_{n-r}(k)$ be the subpace of matrices $N$ such that there exists a matrix

$$A = \begin{pmatrix} \cdot & \cdot \\ \cdot & N \end{pmatrix} \in E_1.$$

Prove that every element of $F$ is singular. **Hint**: Apply Schur'sSchur complement formula to $tA + J_r$.

(d) Then argue by induction over $n$.

373. (From SmyrlisSmyrlis.) We address the following 'practical' problems:

- Let $n = 2p + 1$ be an odd integer, and let $n$ coins be given. The facial value of each coin is an integer (in cents, say). Suppose that, whenever we remove one coin, the $n - 1$ remaining ones can be arranged into two sets of $p$ coins, the total value of the sets being equal to each other. Prove that all coins have the same value.
- Let $n = 2p + 1$ be an odd integer, and let $n$ complex numbers $z_j$ be given. Suppose that, whenever we remove one number, the $n-1$ remaining ones can be arranged into two sets of $p$ numbers, which share the same isobarycenter. Prove that all numbers coincide.

Of course the first problem is a special case of the second, a natural integer being a complex number. Yet, we shall prove the general case after proving the special case.

(a) We begin with the first problem. Show that there exists a matrix $A \in \mathbf{M}_n(\mathbb{Z})$ and $x \in \mathbb{Z}^n$ with $x > 0$ such that $Ax = 0$, the diagonal entries of $A$ vanish, and the other entries of any $i$th line are $\pm 1$, summing up to 0.

(b) Prove that the coordinates of $x$ all have the same parity.

(c) Let us define $y = x - x_1(1, \ldots, 1)^T$. Verify that $Ay = 0$ and $y_1 = 0$. Prove that the coordinates of $y$ have to be even. Finally, prove $y = 0$ and conclude.

(d) One turns towards the second problem and denote $z = (z_1, \ldots, z_n)^T$. Then there exists a matrix $A$ with the same properties as above, and $Az = 0$. In particular, 0 is an eigenvalue of $A$.

(e) Using the answer to the first problem, prove that the kernel of $A$, when acting over $\mathbb{Q}^n$, is one-dimensional. Deduce that its kernel, when acting over $\mathbb{C}^n$, is one-dimensional. Conclude.

374. This gives an alternate argument about the rank of $A$ in the previous exercise.

Let $A \in \mathbf{M}_n(\mathbb{Z})$ ($n$ needs not be odd) be such that its diagonal entries are even, whereas its off-diagonal entries are odd. Prove that $\operatorname{rk} A \geq n - 1$. **Hint**: After deleting the first row and the last column, compute the determinant *modulo* 2.

375. (After A. BezerraBezerra & H.-J. WernerWerner.) Recall that an $n \times n$ matrix $M$ is idempotent if $M^2 = M$. In particular, it satisfies

$$k^n = R(M) \oplus \ker M, \qquad \ker(I_n - M) = R(M), \qquad \ker M = R(I_n - M).$$

Let $A$ and $B$ be $n \times n$ idempotent matrices

(a) Show that
$$\ker(I_n - AB) = R(A) \cap (R(B) \oplus (\ker A \cap \ker B)).$$

(b) Deduce that

$$\dim\ker(I_n - AB) = \dim(R(A) \cap R(B)) + \dim((R(A) + R(B)) \cap \ker A \cap \ker B).$$

**Hint**: Use twice the identity $\dim(X + Y) + \dim(X \cap Y) = \dim X + \dim Y$.

376. This is to show that when $H = A + A^*$, then the spectrum of $A$ does not tell us much about that of $H$, apart from $\mathrm{Tr}H = 2\Re(\mathrm{Tr}A)$, and the obvious fact that the spectrum of $H$ is real.

Thus, let $H$ be a Hermitian $n \times n$ matrix. Show that there exists a matrix $A = \left(\frac{1}{n}\mathrm{Tr}H\right) I_n + N$ with $N$ nilpotent, such that $A + A^* = H$. Therefore the spectrum of $A$ is a singleton. **Hint**: Use Exercise 131 above, which is Exercise 9 of Chapter 5 in the 2nd edition.

377. Let $A$ and $B \in \mathbf{M}_n(\mathbb{C})$ be given. We assume that they span a two-dimensional subspace. We wish to prove that there exist non-trivial factors $s_jA + t_jB$ $(1 \le j \le N = 2^n - 1)$ such that

$$\prod_{j=1}^{N}(s_jA + t_jB) = 0_n.$$

Here *non-trivial* means that $(s_j, t_j) \neq (0, 0)$.

(a) Let $M, P \in \mathbf{M}_n(k)$ be given, with $r = \mathrm{rk}R$. Using the rank decomposition, we write

$$R = \sum_{i=1}^{r} x_i a_i^T.$$

Show that $\mathrm{rk}(RMR) \le r$, and that $\mathrm{rk}(RMR) < r$ if and only if the $r \times r$ matrix $P$ defined by $p_{ij} := a_i^T R x_j$ is singular.

(b) Show, by induction, that for every $1 \le r \le n$, there exists a product of $2^{n-r} - 1$ factors whose rank is at most $r$.

378. (Thanks to F. Brunault.)Brunault We prove here that if $n \ge 2$ and $P \in \mathbb{C}[X]$ is such that $P(A)$ is diagonalizable for every $A \in \mathbf{M}_n(\mathbb{C})$, then $P$ is a constant polynomial.

(a) If $P \in \mathbb{C}[X]$, $z \in \mathbb{C}$ and $B \in \mathbf{M}_n(\mathbb{C})$, verify

$$P(zI_n + B) = P(z)I_n + P'(z)B + \frac{1}{2!}P''(z)B^2 + \cdots,$$

where the sum is finite. **Hint**: this is Taylor expansion for a polynomial in a commutative algebra.

(b) If $w \in \mathbb{C}$, $M$ is nilpotent and $wI_n + M$ is diagonalizable, prove that $M = 0_n$.

(c) Given $P \in \mathbb{C}[X]$, suppose that $P(A)$ is diagonalizable for every $A \in \mathbf{M}_n(\mathbb{C})$ for some $n \ge 2$.

i. If $N$ is nilpotent, show that for every $z \in \mathbb{C}$,

$$P'(z)N + \frac{1}{2!}P''(z)N^2 + \cdots = 0_n.$$

ii. Chosing $N \neq 0_n$ above, deduce that $\det(P'(z)I_n + N') = 0$ for some nilpotent $N'$.

iii. Conclude.

*Remark: In this analysis,* $\mathbb{C}$ *can be replaced by any field of characteristic* 0.

379. (Continuing.) On the contrary, consider some finite field $k = \mathbb{F}_{p^m}$.

(a) Given $n \geq 2$, prove that there exists a $P_n \in k[X]$ that is divisible by every polynomial $p \in k[X]$ of degree $n$.

(b) Show that for all $A \in \mathbf{M}_n(k)$, one has $P_n(A) = 0_n$ ; hence the matrix $P_n(A)$ is diagonalizable!

(c) We consider the case $m = 1$, that is $k = \mathbb{F}_p$. Show that we may take

$$P_n(X) = \prod_{m=1}^{n}(X^{p^m} - X).$$

Its degree is $p\frac{p^n-1}{p-1}$.

380. We begin with some geometry over closed convex cones in $\mathbb{R}^n$. Let $K$ be such a cone. We assume in addition that $K \cap (-K) = \{0\}$. By Hahn–Banach Theorem, it is not hard to see that there exists a compact convex section $K_0$ of $K$ such that $K = \mathbb{R}^+ \cdot K_0$. For instance, if $K = (\mathbb{R}^+)^n$, then the unit simplex works.

If $x, y \in \mathbb{R}^n$, we write $x \leq y$ when $y - x \in K$.

(a) If $x, y \in K$, we define

$$\alpha(x,y) = \sup\{\lambda \geq 0 \,|\, \lambda x \leq y\}, \qquad \beta(x,y) = \inf\{\mu \geq 0 \,|\, \mu x \geq y\}.$$

We may have $\alpha = 0$ or $\beta = +\infty$. We set

$$\theta(x,y) = \log \frac{\beta(x,y)}{\alpha(x,y)}\,.$$

i. Verify that $\theta(x,y) \in [0,+\infty]$, $\theta(x,y) = \theta(y,x)$ and $\theta(x,z) \leq \theta(x,y) + \theta(y,z)$.

ii. Suppose that the interior $U$ of $K$ is non-void. Show that $\theta$ is a distance over the (projective) quotient of $U$ by the following relation: $y \sim x$ if there exists $t \in (0,+\infty)$ such that $y = tx$. This is called the HilbertHilbert distance.

(b) Suppose now that a matrix $A \in \mathbf{M}_n(\mathbb{R})$ is given, such that $AK \subset K$. Let us define

$$\Delta := \sup\{\theta(Ax, Ay) \,|\, x, y \in K\}.$$

i. Let $x, y \in K$ be given, and $\alpha = \alpha(x, y)$, $\beta = \beta(x, y)$. Show that $\alpha \leq \alpha(Ax, Ay)$ and $\beta \geq \beta(Ax, Ay)$, and therefore $\theta(Ax, Ay) \leq \theta(x, y)$. In summary, $A$ induces a non-expansive map over $U/\sim$.

ii. Define (remark that $\mu \geq \lambda \geq 0$

$$\lambda = \alpha(A(y - \alpha x), A(\beta x - y)), \qquad \mu = \beta(A(y - \alpha x), A(\beta x - y)).$$

Show that
$$\frac{\mu\alpha + \beta}{\mu + 1} \leq \alpha(Ax, Ay), \qquad \frac{\lambda\alpha + \beta}{\lambda + 1} \geq \beta(Ax, Ay).$$

Deduce that

$$\theta(Ax, Ay) \leq \log \frac{\lambda\alpha + \beta}{\lambda + 1}\,\frac{\mu + 1}{\mu\alpha + \beta} = \int_0^{\theta(x,y)} f'(t)\,dt, \qquad f(t) := \log \frac{\lambda + e^t}{\mu + e^t}.$$

iii. Verify that
$$s \in (0, +\infty) \mapsto \log \frac{\lambda + s}{\mu + s}$$
is maximal at $s = \sqrt{\lambda\mu}$.

iv. Deduce that
$$\theta(Ax, Ay) \leq k \cdot \theta(x, y), \qquad k := \tanh e^{\Delta/4}.$$
In particular, if $\Delta < +\infty$, then $A$ induces a **contraction** over $U/\sim$.

(c) Deduce an other proof of a part of the Perron–Frobenius Theorem: if $A$ is strictly positive, then it has one and only one positive eigenvector.

It is possible to recover the full Perron–Frobenius Theorem with arguments in the same vein. This proof is due to G. BirkhoffBirkhoff.

381. Let $R$ be an abelian ring and $A \in \mathbf{M}_n(R)$ be given. The *left annihilator* $\mathrm{Ann}^\ell(A)$ is the set of $B \in \mathbf{M}_n(R)$ such that $BA = 0_n$ ; it is a left-submodule. Likewise, the *right annihilator* $\mathrm{Ann}^r(A)$ is the set of $B \in \mathbf{M}_n(R)$ such that $AB = 0_n$.

(a) If $R$ is a principal ideal domain, show that there exists a non-singular matrix $Q$ (depending on $A$) such that

$$\mathrm{Ann}^\ell(A)^T = Q \cdot \mathrm{Ann}^r(A).$$

Deduce that $\mathrm{Ann}^\ell(A)$ and $\mathrm{Ann}^r(A)$ have the same cardinality.

(b) (After K. ArdakovArdakov.) We choose instead $R = k[X, Y]/(X^2, XY, Y^2)$, where $k$ is a finite field.

i. Verify that $R = k \oplus \mathfrak{m}$, where $\mathfrak{m} = kx \oplus ky$ is a maximal ideal (the unique one) satisfies $\mathfrak{m}^2 = (0)$. We have $|R| = |k|^3$ and $|\mathfrak{m}| = |k|^2$.

ii. Let $a, b \in R$ be such that $ax + by = 0$. Show that $a, b \in \mathfrak{m}$.

iii. Let us define

$$A = \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}.$$

Describe its left- and right-annihilators. Verify that

$$|k|^8 = \mathrm{Ann}^\ell(A) \neq \mathrm{Ann}^r(A) = |k|^{10}.$$

382. Let $\mathrm{Sym}_d$ be the space of $d \times d$ real symmetric matrices, where $d \geq 2$ is given. When endowed with the scalar product $\langle S, T\rangle = \mathrm{Tr}(ST)$, this becomes a Euclidian space of dimension $N = \frac{d(d+1)}{2}$, isomorphic to $\mathbb{R}^N$. The space $\mathcal{Q}$ of quadratic forms over $\mathrm{Sym}_d$ is therefore isomorphic to $\mathrm{Sym}_N$ (!!). If $q \in \mathcal{Q}$ and $V$ is a subspace of $\mathrm{Sym}_d$, the trace of $q$ over $V$ is well-defined, and it equals

$$\sum_i q(S_i),$$

where $S_1, \ldots,$ is any orthonormal basis of $V$.

By duality, the space of linear forms over $\mathcal{Q}$ is isomorphic to $\mathcal{Q}$ itself, through $L(q) = \mathrm{Tr}(\Sigma\sigma_q)$, where $\Sigma \in \mathbf{Sym}_N$ and $\sigma_q \in \mathbf{Sym}_n$ is the matrix associated with $q$.

(a) Let $L$ be a linear form over $\mathcal{Q}$. Show that there exist a unitary basis $S_1, \ldots, S_N$ and numbers $\alpha_i$ such that

$$L(q) = \sum_i \alpha_i q(S_i), \qquad \forall q \in \mathcal{Q}.$$

In particular, the numbers $\alpha_i$ are unique up to a permutation, and given a number $\alpha$, the subspace $V_\alpha$ of $\mathbf{Sym}_d$ spanned by the $S_i$'s such that $\alpha_i = \alpha$ is unique. **Hint**: Diagonalize the matrix $\Sigma$ associated with $L$.

Finally, show that

$$L(q) = \sum_\alpha \alpha \,\mathrm{Tr}\left(q|_{V_\alpha}\right), \qquad \forall q \in \mathcal{Q}.$$

(b) We consider the linear representation of $\mathbf{O}_d$ over $\mathrm{Sym}_d$ by

$$(U, S) \mapsto U^T SU.$$

It induces a representation of $\mathbf{O}_d$ over $\mathcal{Q}$ by

$$(U, q) \mapsto q^U, \qquad q^U(S) = q(U^T SU).$$

We suppose that a linear form $L$ over $\mathcal{Q}$ is invariant under this action:

$$L(q) = L(q^U), \qquad \forall q \in \mathcal{Q},\, U \in \mathbf{O}_d.$$

Show that there exist numbers $\alpha, \beta$ such that

$$L(q) = \alpha q(I_d) + \beta \,\mathrm{Tr}\, q, \qquad \forall q \in \mathcal{Q}.$$

(c) Let $K$ be the (convex) cone of semi-positive quadratic forms, $K \subset \mathcal{Q}$. Show that the extremal rays of $K$ are spanned by the forms $q_H : S \mapsto (\mathrm{Tr}(HS))^2$.

(d) Let $L(q) = 0$ be the equation of a supporting hyperplane of $K$ at $q_{I_d}$. By convention, $L \geq 0$ over $K$. We thus have $L(q_{I_d}) = 0$. Let us define

$$\mathcal{L}(q) := \int L(q^U)\, \mathrm{d}\mu(U),$$

where $\mu$ is the HaarHaar measure over $\mathbf{O}_d$. Verify that $\mathcal{L}(q) = 0$ is the equation of a supporting hyperplane to $K$ at $q_{I_d}$, and also that $\mathcal{L}$ is invariant under the action of $\mathbf{O}_d$.

(e) Deduce that

$$\mathcal{L}(q) = \gamma(d \,\mathrm{Tr}\, q - q(I_d)), \qquad \forall q \in \mathcal{Q},$$

for some positive constant $\gamma$.

383. Let $P_n$ be the set of $n \times n$ symmetric real matrices that can be written as a sum $\sum_\alpha v^\alpha \otimes v^\alpha$ where the vectors $v^\alpha \in \mathbb{R}^n$ are non-negative.

(a) Show that $P_n$ is a convex cone, stable under the HadamardHadamard product.

(b) As an example, let $\vec{a} = (a_1, \dots, a_n)$ and $\vec{b} = (a_1, \dots, a_n)$ be two sequences of real numbers, with $0 < a_1 < \cdots < a_n$ and $0 < b_n < \cdots < b_1$. We form the symmetric matrix $S(\vec{a}, \vec{b})$ whose entries $s_{ij}$ are given by $a_{\min(i,j)} b_{\max(i,j)}$. Show that $S(\vec{a}, \vec{b}) \in P_n$

384. Let us order $\mathfrak{S}_n$, for instance by lexicographic order. Given a matrix $A \in \mathbf{M}_n(k)$, we form a matrix $P \in \mathbf{M}_{n!}(k)$, whose rows and columns are indexed by permutations, in the following way:

$$p_{\sigma\rho} := \prod_{i=1}^{n} a_{\sigma(i)\rho(i)}.$$

Notice that $p_{\sigma\rho}$ depends only upon $\sigma^{-1}\rho$. Thus $P$ is a kind of *circulant* matrix.

(a) Show that $\det A$ and

$$\mathrm{per} A := \sum_{\sigma \in \mathfrak{S}_n} a_{i\sigma(i)}$$

are eigenvalues of $P$, and exhibit the corresponding eigenvectors.

(b) If $k = \mathbb{R}$ and $A$ is entrywise non-negative, prove that $\mathrm{per}A$ is the Perron eigenvalue of $P$, that is $P \geq 0_n$ and $\mathrm{per}A$ is the spectral radius of $P$.

385. This exercise is at the foundation by FrobeniusFrobenius of the representation theory. Let $G$ be a finite group, with $n = |G|$. If $(X_g)_{g \in G}$ are indeterminates, DedekindDedekind formed the matrix $A \in \mathbf{M}_n(\mathbb{C}[X_e, \cdots])$ whose entries are

$$a_{gh} = X_{gh^{-1}}, \qquad g, h \in G.$$

Remark that if $G$ is cyclic, then $A$ is a circulant matrix.

Let $V := \mathbb{C}[G]$ and $\rho$ be the regular representation, where a basis of $V$ as a linear space is $G$, and we have $\rho(g)h = gh$ ($G$ acts on itself by left-multiplications).

(a) Show that

$$L := \sum_{g \in G} X_g \rho(g)$$

acts linearly on $\mathbb{C}[X_e, \cdots] \otimes V =: W$, and its matrix in the basis $G$ is $A$.

(b) By decomposing $V$ into irreducible representations $V_1, \cdots$, show that $W$ decomposes as the direct sum of $W_1, \cdots$, in which each factor is stable under $L$.

(c) Deduce that the polynomial $P(X_e, \cdots) := \det A$ splits as

$$P = \prod_i P_i^{n_i}$$

where each $P_i$ is a homogeneous polynomial of degree $n_i = \dim V_i$.

**Remark**: One may prove that each $P_i$ is irreducible, and that they are pairwise prime.

386. Let $H \in \mathbf{HPD}_n$ be given. Using CholeskyCholesky factorization, find an other proof of HadamardHadamard Inequality

$$\det H \leq \prod_{i=1}^{n} h_{ii}.$$

This gives also the equality case ($H$ must be diagonal).

387. We show here that if $a_1, \dots, a_n$ are integers, then

$$\prod_{1 \leq i < j \leq n} \frac{a_j - a_i}{j - i}$$

is an integer.

(a) Define $p_j(X) := X(X-1)\cdots(X-j+1) \in \mathbb{Z}[X]$, where $p_0(X) = 1$ (empty product) and $p_1(X) = X$. Show that there exists an infinite triangular matrix $T = (t_{ij})_{0 \leq i,j}$ with 1's on the diagonal, such that the basis $\{p_0, p_1, p_2, \dots\}$ is the image of the basis $\{1, X, X^2, \dots\}$ under $T$.

(b) Deduce that

$$\det \begin{pmatrix} 1 & 1 & \cdots \\ p_1(x_1) & p_1(x_2) & \cdots \\ p_2(x_1) & p_2(x_2) & \cdots \\ \vdots & \vdots & \end{pmatrix} = \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

**Hint**: There is a VandermondeVandermonde determinant.

(c) Let us denote
$$\binom{a}{k} = \frac{a(a-1)\cdots(a-k+1)}{k!} = \frac{p_k(a)}{k!}.$$
Prove that
$$\det\begin{pmatrix} 1 & 1 & \cdots \\ \binom{x_1}{1} & \binom{x_2}{1} & \cdots \\ \binom{x_1}{2} & \binom{x_2}{2} & \\ \vdots & \vdots & \end{pmatrix}$$
is an integer. Conclude.

388. (After E. StarlingStarling.) Let $k$ be a field whose characteristic is not 2. Let $A, B \in \mathbf{M}_n(k)$ be such that $A^2 = B^2 = I_n$. If $A+B$ is non-singular, define
$$A \star B = (A+B)^{-1}(A - B + 2I_n).$$

(a) Show that
$$k^n = E_+(A) \oplus E_-(A) = E_+(B) \oplus E_-(B),$$
where $E_\pm(M)$ denotes the eigenspace of $M$ associated with the eigenvalue $\pm 1$.

(b) If $A+B$ is non-singular, show that $\dim E_\pm(A) = \dim E_\pm(B)$. In other words, $A$ and $B$ have the same spectrum.

(c) Show that
$$E_+(A \star B) = E_+(B), \qquad E_-(A \star B) = E_-(A).$$

(d) Deduce that $(A \star B)^2 = I_n$.

389. Let $S \in \mathbf{M}_n(\mathbb{R})$ be symmetric, with non-negative entries. Concerning the diagonal entries, we even assume $s_{ii} > 0$ for every $i = 1, \dots, n$.

(a) Let us define the subset of $\mathbb{R}^n$
$$K := \{x \geq 0 \mid \prod_{i=1}^{n} x_i = 1\}.$$
We consider a minimizing sequence $(x^k)_{k\in\mathbb{N}}$ of $x \mapsto \frac{1}{2}x^T S x$ over $K$. Show that each coordinate sequence $(x_j^k)_{k\in\mathbb{N}}$ is bounded below by some $\epsilon > 0$. Deduce that the sequence is bounded, and therefore it has a cluster point in $K$.

(b) Deduce that $x \mapsto \frac{1}{2}x^T S x$ achieves its infimum over $K$, at some point $x^* \in K$.

(c) Using $x^*$, show that there exists a vector $X \in \mathbb{R}^n$ such that $X \circ (SX) = \mathbf{1}$, where $\circ$ is the Hadamard product and $\mathbf{1} = (\mathbf{1}, \dots, \mathbf{1})^{\mathbf{T}}$.

(d) Deduce that there exists a diagonal matrix $D > 0_n$ such that $DSD$ is bi-stochastic.

390. Recall that an idempotent matrix $M \in \mathbf{M}_n(k)$ represents a projector: $M^2 = M$. Let $A_1, \dots, A_r$ be idempotent matrices and define $A = A_1 + \cdots + A_r$.

(a) If $A_iA_j = 0_n$ for every pair $i \neq j$, verify that $A$ is idempotent.

(b) Conversely, we suppose that $A$ is idempotent. Using the trace, show that $R(A) = \oplus_{i=1}^r R(A_i)$. Deduce that $A_iA_j = 0_n$ for every $i \neq j$.

391. This a third proof (due to M. RosenblumRosenblum) of the Putnam–Fuglede'sPutnamFuglede Theorem that if $M, N \in \mathbf{M}_n(\mathbb{C})$ are normal matrices and if $BN = MB$, then $BN^* = M^*B$. The first proof is given in Exercise 255 and the second one in Exercise 297.

(a) Verify that if $f : \mathbb{C} \to \mathbb{C}$ is an entire function, then $Bf(N) = f(M)B$.

(b) Deduce that the holomorphic function

$$F(z) := e^{izM^*}Be^{-izN^*} : \mathbb{C} \to \mathbf{M}_n(\mathbb{C})$$

is bounded. **Hint**: Remark that $e^{izM^*}e^{i\bar{z}M}$ is unitary, hence bounded. This is where the assumption of normality enter in the play.

(c) Deduce that $F$ is constant. Conclude by calculating $F'(0)$.

392. This exercise gives a version of Lemma 20, Section 13.3 (2nd edition), where the assumption that matrices are "Hermitian positive definite" is replaced by "entrywise positive". We borrow it from *Nonnegative matrices in the mathematical sciences*, by A. BermanBerman & R. J. PlemmonsPlemmons, chapter 5.

Let $A, M \in \mathbf{M}_n(\mathbb{R})$ be non-singular. We form $N = M - A$ and $H = M^{-1}N$, so that $H$ is the iteration matrix of the scheme

$$Mx^{k+1} = Nx^k + b$$

in the resolution of $Ax = b$.

We assume that $H \geq 0$ (which could be verified by a discrete maximum principle). Show that the iteration is convergent, that is $\rho(H) < 1$, if and only if $A^{-1}N \geq 0_n$. **Hint**: If $\rho(H) < 1$, prove that $A^{-1}N = \sum_{k\geq 1} H^k$. Conversely, if $A^{-1}N$ is non-negative, apply Perron–FrobeniusPerronFrobenius and give a relation between the eigenvalues of $A^{-1}N$ and those of $H$.

393. (After K. CostelloCostello & B. YoungYoung.)

Let $F$ be a finite set of cardinal $n \geq 2$ and denote $\mathcal{P}(F)$ its Boolean algebra, made of all subsets of $F$. Let us define a square matrix $B$, whose rows and columns are indexed by the *non-void* subsets of $F$:

$$b_{ij} = \begin{cases} 1 & \text{if} \quad I \cap J = \emptyset, \\ 0 & \text{if} \quad I \cap J \neq \emptyset. \end{cases}$$

We shall prove that $B$ is invertible and compute its inverse matrix.

(a) Show that there exists only one permutation of $\mathcal{P}(F)$ satisfying $\sigma(I) \cap I = \emptyset$ for every $I \subset F$.

(b) Let $M$ be the square matrix, whose rows and columns are indexed by *all* the subsets of $F$, defined by

$$m_{ij} = \begin{cases} 0 & \text{if} \quad I \cap J = \emptyset, \\ 1 & \text{if} \quad I \cap J \neq \emptyset. \end{cases}$$

Deduce that $\det M = 1$.

(c) Let $A$ be the square matrix, whose rows and columns are indexed by *all* the subsets of $F$, defined by

$$a_{ij} = \begin{cases} 1 & \text{if} \quad I \cap J = \emptyset, \\ 0 & \text{if} \quad I \cap J \neq \emptyset. \end{cases}$$

Show that $A$ has rank $\geq 2^n - 1$. Deduce that $B$ is invertible. **Hint**: $M = \mathbf{e}\mathbf{e}^T - A$ where $\mathbf{e}$ is defined below.

(d) We chose an order in $\mathcal{P}(F)$ such that the last element is $F$ itself. Denote

$$v = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{e} = \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}.$$

Verify $Bv = \mathbf{e}$.

(e) We identify $B^{-1}$ through an induction over $F$. If $G = F \cap \{a\}$ with $a \notin F$, and if an order has been given in $\mathcal{P}(F)$, then we order $\mathcal{P}(G)$ as follows: first list the non-void subsets of $F$, then $\{a\}$, then the non-void subsets of $F$ augmented of $a$. If $B$ and $C$ denote the matrices associated with $F$ and $G$ respectively, verify that

$$C = \begin{pmatrix} B & 0 & B \\ 0^T & 1 & \mathbf{e}^T \\ B & \mathbf{e} & \mathbf{e}\mathbf{e}^T \end{pmatrix}.$$

(f) If $B$ is invertible, show that $C$ is too, with inverse

$$C^{-1} = \begin{pmatrix} 0 & -v & B{-}1 \\ -v^T & 0 & v^T \\ B^{-1} & v & -B^{-1} \end{pmatrix}.$$

Conclude.

394. Let $M \in \mathbf{M}_n(\mathbb{R})$ be bistochastic, with singular values $\sigma_1 \geq \cdots \geq \sigma_n$.

(a) Show that $\sigma_1 = 1$.

(b) If $J$ denotes the matrix whose all entries equal 1, what are the singular values of $M - \frac{1}{n}J$ ?

395. Let $T \in \mathbf{M}_n(\mathbb{C})$ be given. Prove that $T^*T$ and $TT^*$ are unitarily similar. **Hint**: Use the singular value decomposition.

More generally, if $T \in \mathbf{M}_{n\times m}(\mathbb{C})$ with $n \geq m$, prove that there exists an isometry $W \in \mathbf{M}_{n\times m}(\mathbb{C})$ (that is $W^*W = I_m$) such that $T^*T = W^*(TT^*)W$.

396. (After J.-C. BourinBourin & E.-Y. LeeLee, Eun-Yung.)

Let $H \in \mathbf{H}^+_{n+m}$ a Hermitian matrix, positive semi-definite, given blockwise

$$H = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}, \qquad A \in \mathbf{H}^+_n, B \in \mathbf{H}^+_m.$$

(a) Show that there exists a decomposition

$$H = T^*T + S^*S, \qquad T = \begin{pmatrix} C & Y \\ 0 & 0 \end{pmatrix}, S = \begin{pmatrix} 0 & 0 \\ Y^* & D \end{pmatrix}.$$

**Hint**: Use the square root of $H$.

(b) Deduce that there exist unitary matrices $U, V$, such that

$$H = U \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} V^*.$$

**Hint**: Use Exercise 395.

**Remark**: This decomposition implies a lot of inequalities between sums of eigenvalues of $A, B$ and $H$, respectively, following A. HornHorn!Alfred.

(c) Arguing by induction, show that there exist vectors $x_1, \ldots, x_n \in \mathbf{S}^{n-1}$ (the unit sphere) such that

$$H = \sum_{j=1}^{n} h_{jj} x_j x_j^*.$$

397. (After J.-C. BourinBourin.)

Usually, one uses the convexity of the numerical range to prove that for a given matrix $M \in \mathbf{M}_n(\mathbb{C})$, there exists a unitarily similar $U^*MU$ that has a constant diagonal. However, one may prove directly the latter property.

(a) Show that $M$ is unitarily similar to a matrix $R$ such that $i \neq j$ implies $r_{ji} = -\overline{r_{ij}}$. **Hint**: Consider the so-called real part $\frac{1}{2}(M + M^*)$.

(b) If $x \in \mathbb{C}^n$ is such that $|x_j| = n^{-1/2}$, show that $x^*Rx = \frac{1}{n} \operatorname{Tr} M + i\Im\phi(x, x)$ where $\phi$ is a sesquilinear form to be determined.

(c) Deduce that there exists a unit vector $y$ such that $y^*My = \frac{1}{n} \operatorname{Tr} M$.

(d) Show that $M$ is unitarily similar to a matrix of constant diagonal.

398. (After D. R. RichmanRichman.)

Let $k$ be a field and $n \geq 2$ an integer.

(a) Consider a matrix $M \in \mathbf{M}_n(k)$ of Hessenberg form

$$M = \begin{pmatrix} \cdot & 1 & & & & \\ \vdots & \ddots & 1 & & O & \\ & & \ddots & \ddots & & \\ & & & \ddots & 1 & \\ \vdots & & & & \ddots & 1 \\ a_1 & \cdots & & & \cdots & a_n \end{pmatrix}.$$

If $X$ is an indeterminate, we form $M + XJ \in \mathbf{M}_n(k[X])$, where $J = \operatorname{diag}(1, \dots, 1, 0)$. Prove that $\det(M + XJ) = 0$ if and only if $a_1 = \cdots = a_n = 0$. **Hint**: Induction over $n$.

(b) Let $p \in k[X]$ be a given monic polynomial. Let $M$ be given as above. Deduce that there exists a vector $z \in k^n$ such that the characteristic polynomial of $M + \vec{e_n} z^T$ equal $p$. **Hint**: Linearity of the determinant as a function of a row.

399. (After D. R. RichmanRichman.)

(a) Let $S$ be an integral ring and $p$ a prime number such that $pS = \{0\}$. If $B \in \mathbf{M}_n(S)$, prove that $\operatorname{Tr}(B^p) = (\operatorname{Tr} B)^p$. **Hint**: Work in a splitting field of the characteristic polynomial.

(b) Let $R$ be an integral ring and $n, k \geq 2$ integers. If $M \in \mathbf{M}_n(R)$ is the sum of $k$-th powers of matrices $B_j \in \mathbf{M}_n(R)$, prove that for every prime factor $p$ of $k$, there exists an $x \in R$ such that $\operatorname{Tr} M \equiv x^p$ *mod* $pR$.

**Nota**: RichmanRichman gives also a sufficient condition for the Waring problem to have a solution. For instance, when $p$ is prime and $p \leq n$, the fact that $\operatorname{Tr} M$ is a $k$-th power *mod* $pR$ implies that $M$ is a sum of $p$-th powers. As an example, every $M \in \mathbf{M}_n(\mathbb{Z})$ with $n \geq 2$ is a sum of squares of integral matrices.

400. Here is another proof of the concavity of $f : S \mapsto \log \det S$ and $g : S \mapsto (\det S)^{1/n}$ over $\mathbf{SDP}_n$. Of course, it works in the Hermitian case too.

(a) Show that the differential of $f$ at $S$ is $T \mapsto \operatorname{Tr}(S^{-1}T)$.

(b) Verify that the Hessian of $f$ at $S$ is $T \mapsto -\operatorname{Tr}\left((S^{-1}T)^2\right)$.

(c) Conclude that $f$ is concave. **Hint**: Use the fact that if $\Sigma \in \mathbf{SPD}_n$ and $T$ is symmetric, then the spectrum of $\Sigma T$ is real.

(d) Follow the same strategy to prove that $g$ is concave. **Hint**: At the end, you have to apply Cauchy–Schwarz inequality to the vector of all ones and the vector of eigenvalues of $S^{-1}T$.

401. Here is an iterative method for the calculation of the factors in the polar decomposition $UH$ of a given matrix $A \in \mathbf{GL}_n(\mathbb{C})$. Define a sequence of matrices by

$$A_0 = A, \qquad A_{k+1} = \frac{1}{2}(A_k + A_k^{-*}).$$

If $U_k H_k$ is the polar decomposition of $A_k$, prove that $U_k = U$ and

$$H_{k+1} = \frac{1}{2}(H_k + H_k^{-1}).$$

Deduce that the sequence is defined for every $k \geq 0$, and

$$\lim_{k \to +\infty} A_k = U.$$

Verify that the convergence is quadratic.

DOUBLON avec l'Exercice 262 !!!

402. Let $A \in \mathbf{M}_n(k)$ and $w, z \in k^n$ be given.

(a) If $\det A = 0$, show that

$$z^T \widehat{A} w \widehat{A} - \widehat{A} w z^T \widehat{A} = 0_n,$$

where $\widehat{A}$ is the cofactor matrix. **Hint**: $\widehat{A}$ is a rank-one matrix (see Exercise 56).

(b) Suppose that $A$ is diagonalisable, that is

$$A = \sum_j \lambda_j x_j y_j^T, \qquad \text{where } y_i^T x_j = \delta_i^j,\ \forall i, j.$$

Prove the formula

$$\frac{1}{\det A}(z^T \widehat{A} w \widehat{A} - \widehat{A} w z^T \widehat{A}) = \sum_{j<k} \widehat{\lambda_{j,k}} [(z \cdot y_j) y_k - (z \cdot y_k) y_j]\, [(w \cdot x_j) x_k - (w \cdot x_k) x_j)]^T,$$

where

$$\widehat{\lambda_{j,k}} = \prod_{m \neq j,k} \lambda_m.$$

(c) Show actually that there is a polynomial mapping $P_{w,z} : \mathbf{M}_n(k) \sim k^{n^2} \to \mathbf{M}_n(k)$, such that

$$\frac{1}{\det A}(z^T \widehat{A} w \widehat{A} - \widehat{A} w z^T \widehat{A}) = P_{w,z}(A).$$

**Hint**: Apply Desnanot–JacobiDesnanotJacobi formula (see Exercise 24).

Remark that $P_{w,z}(A) w = 0$ and $z^T P_{w,z}(A) = 0$.

(d) If $n = 2$, verify that $P_{w,z}(A) \equiv: z' w'^T$, where $z' := \begin{pmatrix} z_2 \\ -z_1 \end{pmatrix}$.

403. (After R. MarsliMarsli & F. J. HallHall!F. J..)

(a) Let $E$ be a subspace of $\mathbb{C}^n$, of dimension $r$. Show that there exists a basis $\{v^1, \dots, v^r\}$ of $E$ and pairwise distinct indices $i_1, \dots, i_r$ such that $\|v^j\|_\infty = |v^j_{i_j}|$ for every $j = 1, \dots, r$. **Hint**: Argue by induction over the dimension $r$.

(b) Let $A \in \mathbf{M}_n(\mathbb{C})$ be given. Deduce that, if $\lambda$ is an eigenvalue of $A$ of geometric multiplicity $r$, then $\lambda$ belongs to at least $r$ GershgorinGershgorin disks $D(a_{ii}; r_i)$, where we recall that
$$r_i = \sum_{j \neq i} |a_{ij}|.$$

404. Let $A, B \in \mathbf{M}_{n\times p}(\mathbb{C})$ be given.

(a) Show that
$$\begin{pmatrix} AA^* & A \\ A^* & I_p \end{pmatrix} \geq 0_{n+p}.$$

(b) Deduce that
$$\begin{pmatrix} (AA^*) \circ (BB^*) & A \circ B \\ A^* \circ B^* & I_p \end{pmatrix} \geq 0_{n+p}.$$

(c) Finally, show that
$$(AA^*) \circ (BB^*) \geq (A \circ B)(A \circ B)^*.$$

(d) As an application, let $H, K$ be Hermitian non-negative, of same size. Show that
$$(H \circ K)^{1/2} \geq H^{1/2} \circ K^{1/2}.$$

405. Let $A, B$ be Hermitian matrices of same size, with $B$ positive definite and $A$ positive semi-definite. We already know that $A \circ B$ is positive semidefinite.

(a) If $a_{ii} > 0$ for all $A$, show that $A \circ B$ is positive semi-definite (SchurSchur). **Hint**: go back to the proof that it is positive semidefinite.

(b) In general, show that the rank of $A \circ B$ equals the number of positive diagonal entries of $A$ (BallantineBallantine).

(c) Finally, prove that a positive semidefinite Hermitian matrix $C$ can be factorized $A \circ B$ with $A, B$ Hermitian, $B$ positive definite and $A$ positive semi-definite, if an only if the rank of $C$ equals its number of positive diagonal entries.

(d) The positive definite case is easier and more explicit. Let $J$ be the matrix with all entries equal to 1, and denote $K_\alpha = (1-\alpha)I_n + \alpha K$. If $C$ is Hermitian positive definite, verify that $C \circ K_\alpha$ is positive definite for some $\alpha > 1$. Check that $C = (C \circ K_\alpha) \circ K_{1/\alpha}$ and conclude (DjokovicDjokovic).

406. If $f : \mathbb{R}^N \to \mathbb{R} \cap \{+\infty\}$ is a proper ($f \not\equiv +\infty$) convex function, and $h$ is a positive parameter, the *Yosida approximation*Yosida of $f$ is

$$f_h(u) = \inf_{v \in \mathbb{R}^N} \left( \frac{h}{2}|u-v|^2 + f(v) \right).$$

On another hand, we know that $f : \mathbf{Sym}_n(\mathbb{R}) \to \mathbb{R} \cap \{+\infty\}$, defined by

$$f(S) = \begin{cases} -\log\det S & \text{if } S > 0_n, \\ +\infty & \text{otherwise,} \end{cases}$$

is convex.

Compute its Yosida approximation $f_h$.

407. Let $A \in \mathbf{GL}_n(\mathbb{R})$ be such that $a_{ij} \le 0$ for every pair $i \ne j$, and $A^{-1} \ge 0_n$ (entrywise). Prove that $a_{ii} > 0$ for every $i = 0, \dots, n$. Such a matrix is called an $M$-matrix.

(a) If $M \ge 0_n$ entrywise, show that $\rho(M) < 1$ if and only if $I_n - M$ is non-singular with $(I_n - M)^{-1} \ge 0_n$. **Hint**: Sufficiency comes from Perron–FrobeniusPerronFrobenius Theorem, while necessity involves a series.

(b) Let $M \in \mathbf{M}_n(\mathbb{R})$ be such that $m_{ii} > 0$, while $m_{ij} \le 0$ otherwise. Show that $M$ is an $M$-matrix if and only if $\rho(I_n - D^{-1}M) < 1$, where $D = \operatorname{diag}\{m_{11}, \dots, m_{nn}\}$.

(c) Let $A$ be an $M$-matrix, and $B \in \mathbf{M}_n(\mathbb{R})$ be such that $A \le B$ entrywise, and $b_{ij} \le 0$ for every pair $i \ne j$. We denote $D_A$, $D_B$ their diagonals. Verify that $D_A^{-1}A \le D_B^{-1}B$ and deduce that $B$ is an $M$-matrix.

408. We recall that the function $H \mapsto \phi(H) := -(\det H)^{1/n}$ is convex over $\mathbf{HPD}_n$. We extend $\phi$ to the whole of $\mathbf{H}_n$ by posing $\phi(H) = +\infty$ otherwise. This extension preserves the convexity of $\phi$. We define as usual the LegendreLegendre transform

$$\phi^*(K) := \sup_{H \in \mathbf{H}_n} \{\operatorname{Tr}(HK) - \phi(H)\}.$$

Show that

$$\phi^*(K) = \begin{cases} 0 & \text{if} \quad K \in E, \\ +\infty & \text{otherwise,} \end{cases}$$

where $E$ denotes the set of matrices $K \in \mathbf{H}_n$ that are non-positive and satisfy

$$(\det(-K))^{1/n} \ge \frac{1}{n}.$$

409. (After W. MascarenhasMascarenhas.) We continue our analysis of the JacobiJacobi algorithm.

(a) In an iteration, compute $(k_{pp} - k_{qq})^2 - (h_{pp} - h_{qq})^2$. Deduce that
$$|k_{pp} - k_{qq}| \geq |h_{pp} - h_{qq}|.$$
This means that an iteration tends to separate the relevant diagonal terms. This is reminiscent to the well-known repulsion phenomenon in quantum mechanics between two energy levels.

(b) We define
$$\Sigma := \sum_{i,j} |h_{jj} - h_{ii}|, \qquad \Sigma' := \sum_{i,j} |k_{jj} - k_{ii}|.$$

i. If $x \leq y \leq w \leq z$ are such that $x + z = w + y$, verify that the function $a \mapsto |a-x| - |a-y| - |a-w| + |a-z|$ is non-negative. Deduce that
$$|h_{ii} - k_{pp}| + |h_{ii} - k_{qq}| - |h_{ii} - h_{pp}| - |h_{ii} - h_{qq}| \geq 0.$$

ii. Show that
$$\Sigma' - \Sigma \geq 2|k_{pp} - h_{pp}|.$$

(c) Let $\vec{\delta}^k$ be the diagonal of $A^{(k)}$. We define also $\Delta^k$ to be $|k_{pp} - h_{pp}|$ if $H = A^{(k)}$, $K = A^{(k+1)}$ and $(p,q) = (p_k, q_k)$. Finally, $\Sigma$ and $\Sigma'$ above are denoted $\Sigma^k$ and $\Sigma^{k+1}$.

i. Verify that $\|\vec{\delta}^{k+1} - \vec{\delta}^k\|_\infty \leq \Delta^k$.

ii. Deduce that $\sum_k \|\vec{\delta}^{k+1} - \vec{\delta}^k\|_\infty$ is finite, and that the diagonal of $A^{(k)}$ converges as $k \to +\infty$.

**Remark**. This result is independent of the choice of the sequence $(p_k, q_k)$. It does not say that $A^{(k)}$ converges to a diagonal matrix. Therefore we don't claim that the limit of the diagonal is the spectrum of the initial matrix $A$.

410. Let $A \in \mathbf{Sym}_n(\mathbb{Z}/2\mathbb{Z})$ be such that $a_{ii} = 1$ for all diagonal entries. Show that the vector
$$\mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$
belongs to the range of $A$. **Hint**: using $x^T A x$, show that $\ker A \perp \mathbf{1}$.

*Interpretation.* $A$ is the adjacency matrix of a graph, an electric network whose vertices are light bulbs. At the beginning, all bulbs are turned off. If you switch (off or on) a bulb, its neighbours are switched simultaneously (but their resulting states depend on their original states). The problem is to act so that all bulbs are switched on at the end. This is equivalent to finding an $x \in (\mathbb{Z}/2\mathbb{Z})^n$ such that $Ax = \mathbf{1}$.

411. Recall that a circulant matrix has the form
$$A = \begin{pmatrix} a_1 & a_2 & \cdots & \cdots & a_n \\ a_n & a_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & a_2 \\ a_2 & \cdots & \cdots & a_n & a_1 \end{pmatrix} = P(J), \qquad J = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

where $P$ is a polynomial. We consider complex circulant matrices.

(a) Find a permutation matrix $P$ such that every circulant matrix satisfies $M^T = P^{-1}MP$.

(b) Deduce that if $p \in [1, \infty]$ and $p'$ is the conjugate exponent, then circulant matrices satisfy $\|M\|_{p'} = \|M\|_p$.

(c) Show that the map $p \mapsto \|M\|_p$ is non-increasing over $[1, 2]$, non-decreasing over $[2, \infty]$. **Hint**: remember that the map

$$\frac{1}{p} \mapsto \log \|M\|_p$$

enjoys a nice property.

(d) Compute $\|M\|_p$ for $p = 1, 2, \infty$.

(e) For $p \in [1, \infty)$, deduce the inequality

$$\sum_{k=1}^{n} \left| \sum_{j=1}^{n} a_{j+k} x_j \right|^p \leq \left( \sum_{j=1}^{n} |a_j| \right)^p \sum_{j=1}^{p} |x_j|^p.$$

Examine the equality case.

412. Given $S^1, \dots, S^r \in \mathbf{Sym}_n(\mathbb{R})$, prove that the following statements are equivalent to each other:

- None of the matrices $x_1 S^1 + \cdots + x_r S^r$ is positive definite, when $x$ runs over $\mathbb{R}^r$,
- There exists a matrix $\Sigma \in \mathbf{Sym}_n(\mathbb{R})$ such that $\Sigma \geq 0_n$ and $\mathrm{Tr}(\Sigma S^j) = 0$ for every $j = 1, \dots, r$.

**Hint**: Apply Hahn-Banach.

413. This is graph theory. A graph is a pair $(V, E)$ where $V$ is a finite set (the *vertices*) and $E \subset V \times V$ (the *edges*) is symmetric: $(s, t) \in E$ implies $(t, s) \in E$. The *adjacency matrix* $A$ is indexed by $V \times V$, with $a_{st} = 1$ if $(s, t) \in E$ and $a_{st} = 0$ otherwise ; this is a symmetric matrix. The group of the graph is the subgroup $G$ of $\mathbf{Bij}(V)$ formed by elements $g$ such that $(gs, gt) \in E$ if and only if $(s, t) \in E$.

A *unit distance representation* of the graph is a map from $V$ in some Euclidian space $\mathbb{R}^d$, where (after denoting $u_1, \dots, u_n$ the images of the vertices) $|u_j - u_i| = 1$ whenever $(u_i, u_j)$ is an edge. For instance, the complete graph ($E = V \times V$) with $n$ vertices has a UDR in dimension $d = n - 1$, a regular simplex. A graph admits a UDR if and only if it does not contains an edge $(s, s)$.

Given a graph $(V, E)$ without edges $(s, s)$ (the diagonal of $A$ is zero), we look for the smallest $R \geq 0$ such that there is a UDR contained in a ball of radius $R$. We denote $n$ the number of vertices.

(a) Show that $R \leq 1$ and that the minimal UDR can be taken in a space of dimension $d \leq n-1$.

(b) Show that $(V, E)$ admits a UDR in a ball of radius $\rho$ if and only if there exists a matrix $S \in \mathbf{Sym}_n^+$ such that

$$\begin{aligned} S_{pp} &\leq \rho^2 \qquad \forall p \in V, \\ S_{pp} - 2S_{pq} + S_{qq} &= 1 \qquad \forall (p,q) \in E. \end{aligned}$$

Show that such a UDR exists in a space whose dimension equals the rank of $S$. **Hint**: Consider a GramGram matrix.

(c) Let $R$ be the infimum of those $\rho \geq 0$ for which the graph has a UDR in a ball of radius $\rho$. Show that a UDR exists in a ball of radius $R$ (*minimal* UDR).

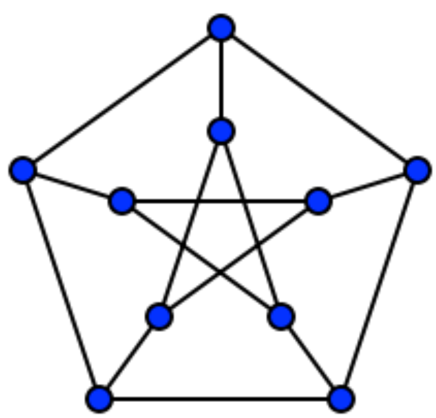
(d) Show that among the MUDR's, there is at least one that is invariant under the group of the graph:

$$P_g^T S P_g = S, \qquad \forall g \in G,$$

where $P_g$ denotes the permutation matrix associated with $g$. We call it a symmetric minimal UDR.

(e) Consider the complete graph with $n$ vertices. For a SMUDR, show that the matrix $S$ above is of the form $\frac{1}{2} I_n + y J_n$, where $J_n$ is the matrix of entries 1 everywhere. Prove that $R^2 = \frac{n-1}{2n}$. Prove that it can be realized in dimension $n-1$ but not in smaller dimension.

(f) Consider PetersenPetersen graph:



The invariance group of Petersen graph has the property that it is transitive on edges: if $(st)$ and $(uv)$ are two edges, there exists a $g \in G$ such that $gs = u$ and $gt = v$. It is also transitive on anti-edges (!): if both $(st)$ and $(uv)$ are *not* edges, there exists a $g \in G$ such that $gs = u$ and $gt = v$. Remark that if two vertices are not neighbours, there exists a unique path of length two from one to the other.

i. Let $S$ be the matrix associated with an SMUDR of Petersen graph. Show that it has the form

$$S = S(x, y, z) = x I_{10} + y J_{10} + z A, \qquad x + y = R^2.$$

ii. Verify that 3 is a simple eigenvalue of $A$. For which eigenvector?

iii. We admit that the other eigenvalues of $A$ are 1 and $-2$. Compute the eigenvalues of $S(x, y, z)$. Determine the triples for which $S(x, y, z)$ is positive semi-definite.

iv. Finally, show that $R^2 = \frac{3}{10}$ and that the SMUDR of Petersen graph can be realized in dimension 4.

(g) Consider the graph of an octahedron (the dual of a cube). Show that its SMUDR has radius $\sqrt{1/3}$ and dimension 2. Of course, this dimension is pathologic: the map $V \to \mathbb{R}^2$ is not injective !

(h) What is the SMUDR of a square, of a pentagon, more generally of an $n$-agon ?

414. Consider the product
$$P(z_1, \dots, z_n) := \prod_{1 \le j < k \le n} |z_k - z_j|.$$
Prove that the maximum of $P$, as $z_1, \dots, z_n$ run over the unit disk $\mathbb{D}$, equals $n^{n/2}$, and that it is achieved when the points form a regular $n$-agon.

**Hint**: Consider a VandermondeVandermonde determinant. Apply the HadamardHadamard inequality and use the equality case.

415. Denote $\mathcal{A}_n(F)$ the space of alternate matrices over a field $F$. Recall that $\mathcal{S}_r$ is the *standard non commutative polynomial* in $r$ variables (see exercise 289).

(a) Prove that $\mathcal{S}_4 \equiv 0$ over $\mathcal{A}_3(F)$. **Hint**: Invoque the dimension of $\mathcal{A}_3$.

(b) With a similar argument, prove that there exists a matrix $J \in \mathcal{A}_4(F)$, and an alternate 6-form $\phi$ over $\mathcal{A}_4(F)$ such that
$$\mathcal{S}_6(A^1, \dots, A^6) = \phi(A^1, \dots, A^6) \cdot J, \qquad \forall\, A^1, \dots, A^6 \in \mathcal{A}_4(F).$$

(c) Let $\Omega^{ij} = e_i e_j^T - e_j e_i^T$ be the elements standard basis of $\mathcal{A}_4$. Verify that the products of all the $\Omega^{ij}$'s in any order, is trivial. Deduce $J = 0_4$. Hence $\mathcal{S}_6 \equiv 0$ over $\mathcal{A}_4$.

*Comment*: B. KostantKostant and L. H. RowenRowen proved that $\mathcal{S}_{2n-2} \equiv 0$ over $\mathcal{A}_n$; the case $n = 2$ being trivial.

(d) Likewise, show that
$$\mathcal{S}_3(A, B, C) = \phi(A, B, C) \cdot I_3, \qquad \forall\, A, B, C \in \mathcal{A}_3(F),$$
where $\phi$ is a non-zero alternate 3-form over $\mathcal{A}_3(F)$.

416. Let $k$ be a field, $n = 2m$ and $A \in \mathbf{M}_n(k)$ be alternate. Define the polynomial
$$P_A^f(X) := \mathrm{Pf}(XJ_n + A), \qquad J_n = \begin{pmatrix} 0_m & I_m \\ -I_m & 0_m \end{pmatrix}.$$
The matrix $XJ_n + A \in \mathbf{M}_n(k[X])$ is alternate. Its alternate adjoint $\widehat{XJ_n + A}$ is defined in Exercise 11.

(a) Show that the entries of $\widehat{XJ_n + A}$ have degrees less or equal to $m - 1$.

(b) Mimicking the proof of Cayley–HamiltonCayleyHamilton Theorem, deduce that

$$P_A^f(JA) = 0_n.$$

**Comment**. It is remarquable that if $A$ is alternate, then the minimal polynomial of $JA$ has degree less than or equal to $\frac{n}{2} = m$.

(c) Let $K, B \in \mathbf{M}_n(k)$ be alternate matrices, with $K$ invertible. Define $Q(X) = \mathrm{Pf}(XK - B)$. Deduce from above the identity

$$Q(K^{-1}B) = 0_n.$$

417. We consider elements $M$ of the orthogonal group $\mathbf{O}(p, q; \mathbb{R})$. *Wlog*, we assume $p \geq q$. We use the block decomposition

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad A \in \mathbf{M}_p(\mathbb{R}).$$

(a) Write the equations satisfied by the blocks. Deduce that $A$ and $D$ are non-singular.

(b) Show the identity

$$A^T A = A^T B(I_q + B^T B)^{-1} B^T A + I_p = A^T f(BB^T) A + I_p, \qquad f(t) = \frac{t}{1+t},$$

equivalently

$$A^{-T} A^{-1} + f(BB^T) = I_p.$$

(c) Let us denote $a_1 \geq \cdots \geq a_p (> 0)$ the singular values of $A$. Likewise, $b_j, c_j, d_j$ are those of $B, C, D$, in decreasing order (with $1 \leq j \leq q$). Prove the relations

$$a_i^2 = 1 + b_i^2, \quad a_i = d_i, \quad b_i = c_i, \qquad i = 1, \ldots, q$$

and $a_i = 1$ for $q < i \leq p$.

In particular, one has

$$\begin{pmatrix} \|A\|_2 & \|B\|_2 \\ \|C\|_2 & \|D\|_2 \end{pmatrix} \in \mathbf{O}(1, 1).$$

(d) Prove that the image of $\mathbf{O}(p, q; \mathbb{R})$ under the projection $M \mapsto A$ is precisely the set of matrices $A \in \mathbf{GL}_p(\mathbb{R})$ satisfying $a_p \geq 1$ (that is $\|A^{-1}\|_2 \leq 1$) and

$$(q < i \leq p) \Longrightarrow (a_i = 1).$$

**Hint**: First, construct a matrix $B$, then $C$ and $D$.

418. (Thanks to P.-L. LionsLions, P.-L..) Let us say that a matrix $A \in \mathbf{M}_n(\mathbb{C})$ is *monotone* if for every vector $x \in \mathbb{C}^n$, one has $\Re x^* A x \geq 0$. It is *strictly* monotone if $\Re x^* A x > 0$ for every non-zero vector. This amounts to saying that the numerical range of $A$ is contained in the right half-space (closed or open, respectively) of the complex plane.

(a) Verify that $A$ is monotone if and only if $A^*$ is so. If $P \in \mathbf{M}_n(\mathbb{C})$, then $P^*AP$ is monotone too.

(b) Show that $A$ is strictly monotone if and only if $A^{-1}$ is so.

(c) Let $A$ be monotone and $B$ strictly monotone. Prove that the spectrum of $AB$ (or $BA$ as well) is contained in $\mathbb{C} \setminus (-\infty, 0)$. If both $A, B$ are strictly monotone, then the spectrum of $AB$ avoids $(-\infty, 0]$.

(d) Conversely, let $M \in \mathbf{M}_n(\mathbb{C})$ be given, its spectrum being contained in $\mathbb{C} \setminus (-\infty, 0]$. We define $D$ a diagonal matrix with the same spectrum.

  i. Obviously, $D = D_0^2$ with $D_0$ strictly monotone. Prove that there exists a neighbourhood $\mathcal{V}$ of $D$ such that if $N \in \mathcal{V}$, then $N = A^2$ with $A$ strictly monotone. **Hint**: The square root of $N$ makes sense thanks to DunfordDunford calculus.

  ii. Prove that $M$ is the product of two strictly monotone matrices. **Hint**: $M$ is similar to an $N \in \mathcal{V}$.

(e) Adapt the previous proof to the real case: if the spectrum of $M \in \mathbf{M}_n(\mathbb{R})$ avoids $(-\infty, 0]$, then $M$ is the product of two *real* strictly monotone matrices. **Hint**: Consider first the block diagonal case, where the diagonal blocks are either scalar or rotation matrices.

(f) Show that the solution of the ODE

$$\frac{dX}{dt} + X^2 = 0_n, \qquad X(0) = M$$

has a global solution over $\mathbb{R}^+$ if and only if the spectrum of $M$ avoids $(-\infty, 0)$.

419. (After R. BhatiaBhatia & R. SharmaSharma.) Let $A \in \mathbf{M}_n(\mathbb{C})$ be normal, with eigenvalues $\lambda_j$ for $j = 1, \dots, n$. Prove

$$\max_{i,j} |a_{ii} - a_{jj}| \leq \max_{i,j} |\lambda_i - \lambda_j|.$$

In other words, the *spread* of the diagonal part of $A$ is not greater than that of $A$ itself see also Exercise 259).

**Hint**: All $a_{ii}$ and $\lambda_i$ belong to the numerical range, but some of them are vertices.

420. (Lewis' TheoremLewis.)

Let $N$ be a norm over $\mathbf{M}_n(\mathbb{C})$. Recall that the dual norm is defined as

$$N^*(M) = \sup_{N(T) \leq 1} |\operatorname{Tr}(MT)|.$$

(a) Verify that $N(M)N^*(M^{-1}) \geq n$ for all $M \in \mathbf{GL}_n(\mathbb{C})$.

(b) Prove that $M \mapsto |\det M|$ reaches a maximum over the unit ball of $N$, say at a matrix $P$. Show that $N(P) = 1$ and $\det P \neq 0$.

(c) Prove that for every $T \in \mathbf{M}_n(\mathbb{C})$, we have

$$|\det(I_n + P^{-1}T)| \leq N(P+T)^n \leq (1+N(T))^n.$$

(d) Derive the inequality $N^*(P^{-1}) \leq n$.

(e) Deduce Lewis' TheoremLewis: There exists $P \in \mathbf{GL}_n(\mathbb{C})$ such that

$$N(P) = 1 \qquad \text{and} \quad N^*(P^{-1}) = n.$$

421. Recall (see Exercise 165) that a non-negative $n \times n$ matrix $A$ is *primitive* if it is irreducible and $\rho(A) > 0$ is the only eigenvalue of maximal modulus. Equivalently, $A^m$ is positive for some $m \geq 1$. We prove here Wielandt'sWielandt Theorem: $A^p$ is positive for $p = n^2 - 2n + 2 = (n-1)^2 + 1$, and this $p$ is sharp in the sense that there is an example for which $A^{p-1}$ is *only* non-negative.

(a) If $A^m$ has a positive column, verify that the same column of $A^{m+1}$ is positive.

(b) If $M \geq 0$ is irreducible and $m_{11} > 0$, prove that the first row and column of $M^q$ are positive for every $q \geq n-1$. **Hint**: it is enough to find positive products $m_{i\alpha}m_{\alpha\beta}\cdots m_{\gamma\tau}m_{\tau 1}$ and $m_{1\alpha}m_{\alpha\beta}\cdots m_{\gamma\tau}m_{\tau i}$ of length $n-1$, for every $i$.

(c) Suppose that for every $i = 1, \dots, n$, there exists an exponent $\ell = \ell_i \in \{1, \dots, n-1\}$ such that the matrix $A^\ell$ has a positive diagonal entry $a_{ii}^{(\ell)}$. Deduce from above that $A^p > 0$.

(d) There remains the case where one of the diagonal entries of all of $A, A^2, \dots, A^{n-1}$ vanishes. Say the upper-left entry : $a_{11} = \cdots = a_{11}^{(n-1)} = 0$. Prove that $a_{11}^{(n)} > 0$. **Hint**: Cayley–HamiltonCayleyHamilton. Deduce that, after a possible reordering, one has $a_{12}a_{23}\cdots a_{n1} > 0$, that is

$$A \geq \epsilon \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & & \ddots & 1 \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix}, \qquad \epsilon > 0.$$

(e) Then, if for some $q \geq 1$, $A^q$ has a positive column, prove that $A^{q+1}, \dots, A^{q+n-1}$ also have a positive column, a different one at each step, and therefore $A^{q+n-1}$ is positive.

(f) If $\operatorname{Tr} A^\lambda > 0$ for some $\lambda \in \{1, \dots, n-2\}$, conclude.

(g) There remains the case where $\operatorname{Tr} A = \operatorname{Tr} A^2 = \cdots = \operatorname{Tr} A^{n-2} = 0$. Show that $A$ satisfies an equation

$$A^n = \frac{\operatorname{Tr} A^{n-1}}{n-1} A + \frac{\operatorname{Tr} A^n}{n} I_n.$$

**Hint**: Apply Newton'sNewton relations.

Deduce $a, b > 0$ from the fact that $A$ is primitive.

(h) Let $c = \min(a,b)$. Show that $A^p \geq c^{n-2}(A^2 + \cdots A^n)$, whence $A^p \geq d(I_n + A + \cdots + A^{n-1}$ for some $d > 0$. Deduce $A^p > 0_n$.

422. We prove sharpness of Wielandt's Theorem. Define

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & & \ddots & 1 \\ 1 & 1 & 0 & \dots & 0 \end{pmatrix}.$$

Show that $A^n = A + I_n$. Deduce that $A^{p-1} \leq g(A + \cdots + A^{n-1})$ for some $g > 0$. Conclude that the upper-left entry of $A^{p-1}$ is zero.

423. Given $S \in \mathbf{SPD}_n$ and $x \in \mathbb{R}^n$, show that $\widehat{S + xx^T} \geq \hat{S}$, where $\hat{S} = (\det S)S^{-1}$ denotes the matrix of cofactors. **Hint**: Apply ShermanSherman–MorrisonMorrision Formula.

Deduce that the map $S \mapsto \hat{S}$ is monotonous over $\mathbf{SPD}_n$. If $S \leq T$ and the rank of $T - S$ is $\geq 2$, show that actually $\hat{S} < \hat{T}$.

Of course, these results have a counterpart in the realm of positive Hermitian matrices.

424. Let $B, C \in \mathbf{SPD}_n$ be given. Let us define a function

$$\begin{array}{rcl} \mathbf{SPD}_n & \xrightarrow{\phi} & (0, +\infty) \\ A & \mapsto & \det(A + B + C) + \det C - \det(A + C) - \det(B + C). \end{array}$$

(a) Show that the differential of $\phi$ is

$$\mathrm{d}_A\phi = \widehat{A + B + C} - \widehat{A + C}.$$

(b) Prove that $\phi$ is monotonous. **Hint**: Use Exercise 423.

(c) Deduce the inequality

$$\det(A + B + C) + \det C \geq \det(A + C) + \det(B + C).$$

425. Let $F_0 = 0$, $F_1 = F_2 = 1$, etc ... be the FibonacciFibonacci sequence. Prove the identity

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

Deduce $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$.

426. (From Z. BradyBrady.) In $\mathbf{Sym}_3(\mathbb{R})$, define the set

$$C = \{S \mid \mathrm{Tr}(SO) \leq 3, \forall O \in \mathbf{O}_3\}.$$

Show that $C$ is a convex subset, with $-C = C$ (symmetric subset), such that

$$(S \in C) \Longrightarrow (|\det S| \leq 1).$$

**Hint**: Use Schur'sSchur triangularization.

427. Let $K$ be a field with characteristic 0 and $V$ be a finite-dimensional vector space over $K$.

(a) Let $\{e^1, \dots, e^n\}$ be a basis of $V$. For $k \in \mathbb{Z}$, define

$$v_k = e^1 + ke^2 + \cdots + k^{n-1}e^n.$$

If $W$ is a proper subspace, show that $W$ contains at most $n-1$ vectors of the form $v_k$.

(b) Deduce that $V$ cannot be the finite union of proper subspaces.

428. Let $x_1, \dots, x_d \in \mathbb{C}^n$ be vectors. If $m \in \mathbb{N}$, let us form the Hermitian matrix

$$H = ((mx_j^* x_i I_n - x_i x_j^*))_{1 \le i,j \le n}.$$

Denote also $q(\xi) := \xi^* H \xi$ the form associated with $H$.

(a) If $\xi = (\xi_1, \dots, \xi_d)$ is given blockwise, with $\xi_j \in \mathbb{C}^n$, develop $q(\xi)$ and show that it equals $m \operatorname{Tr} FF^* - |\operatorname{Tr} F|^2$ for some matrix $F(\xi, x)$.

(b) If $m \ge \min(n, d)$, deduce that $H$ is positive semi-definite. **Hint**: Recall (see Eexercise 49) the inequality

$$|\mathrm{Tr} M|^2 \le \mathrm{rk} M \cdot \mathrm{Tr} MM^*.$$

429. A matrix $A \in \mathbf{M}_n(\mathbb{R})$ acts upon the space $\mathbb{R}^2$ by $X \mapsto AX$. Let us identify $\mathbb{R}^2 \sim \mathbb{C}$ : if $X = \begin{pmatrix} x \\ y \end{pmatrix}$, then $X \sim z = x + iy$. We therefore may write $Az$ instead of $AX$.

(a) Show that there exist uniquely defined complex numbers $a_\pm$ such that

$$Az = a_+ z + a_- \bar{z} \qquad \text{for all } z \in \mathbb{C}.$$

Write the entries of $A$ in terms of the real and imaginary parts of $a_+$ and $a_-$.

(b) Prove the formulæ

$$\det A = |a_+|^2 - |a_-|^2, \qquad \|A\|_F^2 = 2|a_+|^2 + 2|a_-|^2.$$

(c) Compute the singular values of $A$ in terms of $a_\pm$. Deduce $\|A\|_2 = |a_+| + |a_-|$. If $A$ is non-singular, compute $\|A^{-1}\|_2$.

(d) At which condition does $A$ have two real distinct eigenvalues ?

430. Recall that $\mathbb{F}_q$ denotes the finite field with $q$ elements ($q$ is a power of a prime number). We consider matrices chosen randomly in $\mathbf{M}_n(\mathbb{F}_q)$, when the entries are independent and uniformly distributed over $\mathbb{F}_q$.

(a) Show that $\mathbf{GL}_n(\mathbb{F}_q)$ is of order

$$(q^n - 1)(q^n - q) \cdots (q^n - q^{n-1}).$$

(b) Show that the cardinal of the class

$$D_a := \{M \in \mathbf{M}_n(\mathbb{F}_q) \mid \det M = a\}$$

does not depend on $a \neq 0$. Deduce the value of this cardinal.

(c) Show that the probability $p_n(q)$ that $\det M = 1$ for $M$ chosen randomly in $\mathbf{M}_n(\mathbb{F}_q)$ is stricly less than $\frac{1}{q}$. What is the probability that $\det M = 0$?

(d) Show that $\lim_{n\to+\infty} p_n(q) =: p(q)$ exists and is non-zero.

**Comment**: $p(q)$ is the probability that a *large* random matrix with entries in $\mathbb{F}_q$ have determinant 1 (or any other number $a \neq 0$). This probability can be expressed in terms of Dedekind'sDedekind *eta* function $\eta$.

431. If $m \in \mathbb{N}^*$ and $\lambda \in k$, we denote $J_m(\lambda)$ the Jordan block of size $m$ with eigenvalue $\lambda$.

(a) If $\lambda \neq 0$, prove that the minimal polynomial of $J_m(\lambda)^2$ is $(X - \lambda^2)^m$. Deduce that $J_m(\lambda)^2$ is similar to $J_m(\lambda^2)$.

(b) On the contrary, verify that $J_m(0)^2$ is permutation-similar to

$$J_{\left[\frac{m+1}{2}\right]}(0) \oplus J_{\left[\frac{m}{2}\right]}(0).$$

432. Let $k$ be a field of characteristic 0 (that is $\mathbb{Q} \subset k$). If $n \geq 1$, we denote $\phi_n$ the $n$th cyclotomic polynomial:

$$X^n - 1 = \prod_{d|n} \phi_d(X).$$

Remark that if $n \neq m$, then $\phi_n \wedge \phi_m = 1$, because the roots of $X^\ell - 1$ are simple and $\phi_n\phi_m$ divides $X^\ell - 1$, where $\ell = \mathrm{lcm}(m, n)$. With $\sigma \in \mathfrak{S}_n$, we associate the permutation matrix $P_\sigma$ as usual.

(a) If $\sigma$ and $\rho$ are conjugated in $\mathfrak{S}_n$, prove that $P_\sigma$ and $P_\rho$ are similar.

(b) Let $c$ be an $n$-cycle.

i. Show that the similarity invariants of $P_c$ are $1, \dots, 1, X^n - 1$. **Hint**: there is a basis in which $P_\sigma$ becomes a companion matrix.

ii. Let $d_1, \dots, d_t = n$ be the divisors of $n$. Show that $XI_n - P_c$ is equivalent, in $\mathbf{M}_n(k[X])$, to $\mathrm{diag}(1, \dots, 1, \phi_{d_1}, \dots, \phi_{d_t})$.

(c) Let $\sigma \in \mathfrak{S}_n$ be given, and $n_1 \leq \cdots \leq n_r$ be the cardinals of the $\sigma$-orbits. If $d \geq 1$, we denote $m(d)$ the number of lengths $n_j$ that are multiple of $d$. Prove that the similarity invariants of $P_\sigma$ are $p_n, \dots, p_1$ defined by

$$p_s = \prod_{d\,;\,m(d)\geq s} \phi_d.$$

(d) Show that the map $(n_1, \dots, n_r) \mapsto (p_n, \dots, p_1)$ which, to a partition of $n$, associates the similarity invariants of $P_\sigma$, is injective. **Hint**: The list $(p_n, \dots, p_1)$ determines the list of $\phi_d$'s with multiplicities. If $d$ is maximal for division, then it is one of the $n_j$. Argue by induction over $r$.

(e) Deduce Brauer'sBrauer Theorem: if $P_\sigma$ and $P_\rho$ are similar, then $\rho$ and $\sigma$ are conjugated. **Hint**: It amounts to proving that the $\sigma$-orbits and the $\rho$-orbits have the same cardinals.

433. Let $A \in \mathbf{M}_{n\times m}(\mathbb{R})$ be a matrix with non-negative entries: $a_{ij} \geq 0$ for every $i, j$. The *positive rank* of $A$ is the minimal number $\ell$ such that $A$ can be written as the sum of $\ell$ rank-one matrices with non-negative entries:

$$A = \sum_{\alpha=1}^{\ell} x^\alpha (y^\alpha)^T, \qquad x_i^\alpha \geq 0,\, y_j^\alpha \geq 0, \quad \forall \alpha, i, j.$$

(a) Verify that $\mathrm{rk}(A) \leq \ell \leq \min(n, m)$.

(b) For

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

verify that $\mathrm{rk}(A) < \ell$.

434. We consider $n \times n$ matrices $A$ and view their entries $a_{ij}$ as indeterminates. Thus $\det A$ is a homogeneous polynomial of degree $n$. Let us define the CayleyCayley operator

$$\Omega = \det \begin{pmatrix} \frac{\partial}{\partial a_{11}} & \cdots & \frac{\partial}{\partial a_{1n}} \\ \vdots & & \vdots \\ \frac{\partial}{\partial a_{n1}} & \cdots & \frac{\partial}{\partial a_{nn}} \end{pmatrix},$$

which is a differential operator.

When $s \in N$, the CapelliCapelli identity says that

$$\Omega[(\det A)^s] = s(s+1)\cdots(s+n-1)(\det A)^{s-1}.$$

Prove the cases $n \leq 2$ (and, why not ?, $n = 3$).

**Nota**: The polynomial $b(s) = s(s+1)\cdots(s+n-1)$ is called the BernsteinBernstein–SatoSato polynomial of the determinant.

435. Let $A, B \in \mathbf{M}_n(k)$ be such that $\sigma(A) \cap \sigma(B) = \emptyset$. Let $C \in \mathbf{M}_n(k)$ be commuting with both $A + B$ and $AB$.

Show that $AC - CA$ belongs to the kernel of $X \mapsto AX - XB$. Deduce that $C$ commutes with both $A$ and $B$ (Embry'sCapelli Theorem). **Hint**: see Exercise 167.

436. Let $q_1$ and $q_2$ be two non-degenerate quadratic forms over $\mathbb{R}^n$. Prove that the orthogonal groups $\mathbf{O}(q_1)$ and $\mathbf{O}(q_2)$ are isomorphic if and only if $q_1$ and $q_2$ are isomorphic, that is, if there exist $u \in \mathbf{GL}(\mathbb{R}^n)$ such that $q_2 = q_1 \circ u$.

437. We recall the numerical radius of a complex $n \times n$ matrix :

$$r(A) = \sup\{|x^* Ax| \mid \|x\|_2 = 1\}.$$

   (a) If $U \in \mathbf{U}_n$ is unitary, verify that $r(U^{-1}) = r(U) = 1$.

   (b) Let $T$ be triangular, with unitary diagonal : $|t_{jj}| = 1$ for every $j \le n$. Show that $r(T) > 1$ unless $T$ is diagonal. **Hint**: Start with the $2 \times 2$ case.

   (c) Let $A \in \mathbf{M}_n(\mathbb{C})$ be such that $r(A) \le 1$ and $r(A^{-1}) \le 1$.

      i. Prove that the spectrum of $A$ is contained in the unit circle.

      ii. Show the converse property : $A$ is unitary. **Hint**: Start with the triangular case.

438. This is an other proof of the fact that if $A, B$ and $AB$ are normal, then $BA$ is normal too (Exercise 281). We use the Schur–Frobenius norm over $\mathbf{M}_n(\mathbb{C})$,

$$\|M\| = \sqrt{\operatorname{Tr} M^* M} = \left(\sum_{i,j} |m_{ij}|^2\right)^{1/2}.$$

We denote $\lambda_1(M), \dots, \lambda_n(M)$ the eigenvalues of $M$ (the order is not important).

   (a) Show that $\|AB\| = \|BA\|$.

   (b) Deduce that

$$\sum_i |\lambda_i(BA)|^2 = \|BA\|^2.$$

   (c) Conclude.

439. Let $A \in \mathbf{M}_n(\mathbb{C})$ be given. We suppose that an eigenvalue $\lambda$ of (algebraic) multiplicity $\ell$ belongs to the boundary of the numerical range $\mathcal{H}(A)$.

   (a) In the case where $A$ is triangular with $a_{11} = \cdots = a_{\ell\ell} = \lambda$, show that actually $A$ is block diagonal, $A = \operatorname{diag}(\lambda I_\ell, A')$.

   (b) Deduce that in the general case, $\lambda$ is semi-simple ($\dim \ker(A - \lambda I_n) = \ell$), and the orthogonal of $\ker(A - \lambda I_n)$ is stable under $A$. In other words, $\lambda$ is a *normal* eigenvalue, in the sense of normal matrices.

440. (From L. LessardLessard.) Let $A \in \mathbf{M}_n(k)$ be given, and $X^n - a_1 X^{n-1} + \cdots + (-1)^n a_n$ be its characteristic polynomial. Denote adj$A$ the adjugate matrix (the transpose of the matrix of cofactors).

Prove

$$\operatorname{adj} A = a_{n-1} I_n - a_{n-2} A \cdots + (-1)^{n-1} A^{n-1}.$$

**Hint**: When $A$ is non-singular, Cayley–HamiltonCayleyHamilton. Then it is nothing but a polynomial identity in $n^2$ indeterminates.

441. If $X \in \mathbf{M}_n(k)$, we denote $L_X \in \mathcal{L}(\mathbf{Sym}_n(k))$ the linear operator

$$S \mapsto X^T M X.$$

(a) If $\lambda, \mu$ are eigenvalues of $X$, show that $\lambda\mu$ is an eigenvalue of $L_X$.

(b) If $L_X$ is diagonalisable with eigenvalues $\mu_1, \dots, \mu_n$, and if the products $\mu_i \mu_j$ are pairwise distinct for $1 \le i \le j \le n$, prove that the characteristic polynomial $\Pi_X(t)$ equals

$$\prod_{1 \le i \le j \le n} (t - \mu_i \mu_j). \tag{38}$$

(c) Remarking that the expression in (38) is a polynomial in the entries of $X$, with integer coefficients, deduce that the formula holds true for every field $k$ and every $X \in \mathbf{M}_n(k)$.

442. Let $A, B \in \mathbf{U}_n$ be unitary matrices. If $Y := (AB^*)^{1/2} \in \mathbf{U}_n$ is a solution of $Y^2 = AB^*$ (a *square root*), show that $X := A^* Y$ is a solution of the quadratic equation

$$XAXB = I_n.$$

443. This exercise shows that $O(n^3)$ operations suffice to compute the characteristic polynomial of a real or complex square matrix.

(a) Let $R$ be a commutative ring and $M \in \mathbf{M}_n(R)$ be an upper Hessenberg matrix. Denote $a_1, \dots, a_{n-1}$ the sub-diagonal entries, $x_1, \dots, x_n$ those of the last column and $M_1, \dots, M_n = M$ the principal square submatrices ; $M_k$ is obtained by keeping only the $k$th first rows and columns of $M$. Prove the formula

$$\det M = x_n \det M_{n-1} - a_{n-1} x_{n-1} \det M_{n-2} + a_{n-1} a_{n-2} x_{n-2} \det M_{n-3} - \cdots .$$

(b) If $A \in \mathbf{M}_n(k)$, with $A_1, \dots, A_n = A$ its principal submatrices, and if one knows the characteristic polynomials of $A_1, \dots, A_{n-1}$, show that $P_A$ can be deduced in $n^2 + O(n)$ elementary operations in $k$.

(c) Deduce that the calculation of $P_A(X)$ can be done in $\frac{1}{2} n^3 + O(n^2)$ elementary operations.

(d) For a general real or complex $n \times n$ matrix $B$, show that that there is a calculation of $P_B$ in $O(n^3)$ operations.

444. We say that $A, B \in \mathbf{M}_n(k)$ are *orthogonally similar* if there exists an $O \in \mathbf{O}_n(k)$ such that $AO = OB$.

(a) Suppose that $A$ and $B$ are orthogonally similar, say $O^T A O = B$. Verify that for every word $w(X, Y)$ in two letters, one has $O^T w(A, A^T) O = w(B, B^T)$.

(b) Deduce the necessary condition for $A$ and $B$ to be orthogonally similar:

$$\text{(39)} \qquad \operatorname{Tr} w(A, A^T) = \operatorname{Tr} w(B, B^T), \qquad \text{for all word } w(X, Y).$$

(c) If $n = 2$, show that (39) is equivalent to

$$\operatorname{Tr} A^2 = \operatorname{Tr} B^2, \qquad \operatorname{Tr}(AA^T)^k = \operatorname{Tr}(BB^T)^k, \quad \operatorname{Tr}((AA^T)^k A) = \operatorname{Tr}((BB^T)^k B), \qquad \forall\, k \geq 0.$$

**Hint**: Use Cayley–Hamilton.

(d) Still, when $n = 2$, deduce that for every $A \in \mathbf{M}_2(k)$, the pair $(A, B := A^T)$ satifies (39).

(e) Yet, prove that if $k = \mathbb{F}_p$ ($p$ an odd prime), and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $(d-a)^2 + (b+c)^2$ is not a square, then $A$ and $A^T$ are not orthogonaly similar.

(f) On the contrary, show that if $k = \mathbb{R}$ and $n = 2$, then $A$ and $A^T$ are always orthogonally similar.

445. We endow $\mathbf{M}_n(\mathbb{R})$ with the standard scalar product $\langle M, N \rangle = \operatorname{Tr}(M^T N)$.

(a) Let $E$ be the vector space spanned by the bistochastic matrices. Show that $E^\perp$ consists of matrices for which every sum

$$\sum_{i=1}^{n} a_{i\sigma(i)}, \qquad \forall \sigma \in \mathfrak{S}_n$$

vanishes.

(b) Verify: The entries $a_{1j}$ ($2 \leq j \leq n$) and $a_{ii}$ ($1 \leq i \leq n-1$) form a coordinate system in $E^\perp$. Actually, the other entries are determined through

$$a_{ij} = a_{ii} + a_{1j} - a_{1i}, \qquad \text{if } i \leq n-1$$

and

$$a_{nj} = a_{1j} - a_{1n} - \sum_{k=1}^{n-1} a_{kk}.$$

(c) Let $K$ be set of matrices $M \in \mathbf{M}_n(K)$, entrywise non-negative, whose diagonals sum to one:

$$\sum_{i=1}^{n} a_{i\sigma(i)} = 1, \qquad \forall \sigma \in \mathfrak{S}_n.$$

Show that $K$ is convex and compact. Prove that its extremal points are the matrices $\mathbf{1} \otimes \vec{e}^j$ and $\vec{e}^i \otimes \mathbf{1}$.

446. We begin by completing one entry of a positive definite symmetric matrix, and then consider a more general completing problem.

(a) Let $A \in \mathbf{Sym}_n(\mathbb{R})$ be given with $a_{1n} = 0$. We denote $S(x)$ the symmetric matrix obtained from $A$ by replacing $a_{1n}$ by $x \in \mathbb{R}$. We ask whether it is possible to find $x$ such that $S(x)$ be positive definite.

i. A necessary condition is that the sub-matrices

$$A\begin{pmatrix} 1 & \dots & n-1 \\ 1 & \dots & n-1 \end{pmatrix} \qquad \text{and} \qquad A\begin{pmatrix} 2 & \dots & n \\ 2 & \dots & n \end{pmatrix}$$

are positive definite.

ii. Express $\det S(x)$ as a polynomial in $x$.

iii. We suppose that the necessary condition above is fulfilled. Show that $S(x)$ can be made positive definite if, and only if

$$(\det A) \det A\begin{pmatrix} 2 & \dots & n-1 \\ 2 & \dots & n-1 \end{pmatrix} + \left[\det A\begin{pmatrix} 1 & \dots & n-1 \\ 2 & \dots & n \end{pmatrix}\right]^2 > 0.$$

iv. Conclude, with the help of the Desnanot–JacobiDesnanotJacobi formula (Exercise 24).

(b) Let $T$ be an $n \times n$ tri-diagonal symmetric matrix. Assume that

$$a_{ii}a_{i+1,i+1} - a_{i,i+1}^2 > 0, \qquad i = 1, \dots, n-1.$$

Prove that the remaining diagonals can be completed in such a way that the new matrix is positive definite.

447. (G. StrangStrang; thanks to B. SévennecSévennec.) This is a sequel of the previous exercise. Let $T \in \mathbf{SPD}_n$ be tridiagonal.

(a) Prove that there exists a unique $S \in \mathbf{SPD}_n$ maximizing the determinant among those matrices whose three main diagonals coincide with those of $T$ :

$$(|i-j| \leq 1) \Longrightarrow (s_{ij} = t_{ij}).$$

(b) Show that the inverse $T' := S^{-1}$ is tridiagonal.

**Hint**: the differential of the map $S \mapsto \det S$ is the cofactor matrix $\hat{S}$.

(c) If $T$ is diagonal, show that $T'$ is diagonal too.

**Hint**: Hadamard'sHadamard inequality.

(d) Likewise, if $t_{i,i+1} = 0$ for some $i < n$, prove that $t'_{i,i+1} = 0$.

448. We know that the function $x \mapsto x^2$ is not operator monotone over $(0, +\infty)$: there exist Hermitian matrices $0_n \leq A \leq B$ such that $B^2 - A^2$ is not positive semidefinite. This does not tell what happens for projections. Therefore, let $P$ and $Q$ be orthogonal projections over $\mathbb{C}^n$:

$$P = P^*, \quad Q = Q^*, \quad P^2 = P, \quad Q^2 = Q.$$

Let us point out that $P$ and $Q$ are positive, and therefore $P + Q \geq P \geq 0_n$. We look for a necessary and sufficient condition in order that $(P + Q)^2 \geq P^2$. So suppose that this inequality holds true.

(a) Show that

$$(Px = 0,\, Qy = 0) \Longrightarrow (\Re\langle Py, Qx\rangle = 0).$$

(b) Deduce that $(I_n - Q)PQ(I_n - P) = 0_n$.

(c) Show that $X = PQ$ satisfies $X^3 - 2X^2 + X = 0_n$, that is $(X - I_n)^2 X = 0$.

(d) Prove that the eigenvalue 1 of $X$ is semi-simple. **Hint**: $\|X\|_2 \leq 1$. Deduce $X^2 = X$.

(e) Deduce that $R(X) \perp \ker(X)$ and therefore $X$ is Hermitian. Conclude that $[P, Q] = 0_n$.

(f) Conversely, verify that $[P, Q] = 0_n$ is sufficient in order that $(P + Q)^2 \geq P^2$.

(g) What does the property $[P, Q] = 0_n$ mean geometrically, for orthogonal projections ?

449. (After Ilya BogdanovBogdanov.) Let $p$ be a prime number and $n \geq 2$ be an integer. If $M \in \mathbf{M}_n(\mathbb{F}_p)$, we denote $\mathcal{A}(M)$ the algebra spanned by $M$.

(a) Show that $\dim \mathcal{A}(M) \leq n$. If $M \in \mathbf{GL}_n(\mathbb{F}_p)$, deduce that the order of $M$ is at most $p^n - 1$.

(b) Conversely, let $P \in \mathbb{F}_p[X]$ be a monic polynomial such that $\mathbb{F}_{p^n} \sim \mathbb{F}_p[X]/(P)$ (we know that $\mathbb{F}_{p^n}$ is an extension of degree $n$ of $\mathbb{F}_p$ ; actually, any irreducible polynomial of degree $n$ works). Let $C_P$ be its companion matrix. Show that $C_P$ has order $p^n - 1$ in $\mathbf{GL}_n(\mathbb{F}_p)$.

450. (After L. BorisovBorisov, A. FischleFischle & P. NeffNeff.) We are going to prove the following assertion:

*If $M \in \mathbf{M}_n(\mathbb{R})$ is such that $M^2$ is symmetric, then $M$ is orthogonally similar to a block-diagonal matrix whose diagonal blocks have sizes $1 \times 1$ and/or $2 \times 2$.*

(a) Show that it is enough to treat the case where $M^2 = \lambda I_n$.

This identity is supposed from now on.

(b) Suppose first that $\lambda \neq 0$. We denote $M = USV$ the singular value decomposition and $N := SQ$, where $Q = VU \in \mathbf{O}_n(\mathbb{R})$.

i. Show that $SQS = \lambda Q^T$, then $S^2QS^2 = \lambda^2 Q$. Deduce that either $s_i s_j = |\lambda|$ or $q_{ij} = 0$.

ii. Show that $S$ and $Q$ can be (up to a permutation of indices) written simultatneously in block-diagonal form, where the diagonal blocks are respectively

$$\begin{pmatrix} sI_\ell & 0 \\ 0 & tI_\ell \end{pmatrix}, \qquad \begin{pmatrix} 0_\ell & P \\ P^T & 0_\ell \end{pmatrix} \quad \text{with } P \in \mathbf{O}_\ell(\mathbb{R}), \qquad \text{if } s < t,\ st = |\lambda|,$$

or

$$\sqrt{\lambda}\, I_m, \qquad P \in \mathbf{O}_m(\mathbb{R}), \qquad \text{but only if } \lambda > 0.$$

iii. In the first case above ($s < t$ and $st = |\lambda|$), show that there exist $R \in \mathbf{O}_{2\ell}(\mathbb{R})$ such that

$$R^T \begin{pmatrix} sI_\ell & 0 \\ 0 & tI_\ell \end{pmatrix} \begin{pmatrix} 0_\ell & P \\ P^T & 0_\ell \end{pmatrix} R = \begin{pmatrix} 0_\ell & sI_\ell \\ tI_\ell & 0_\ell \end{pmatrix}$$

and conclude the case $\lambda \neq 0$.

(c) There remains the case where $M^2 = 0_n$.

i. Proceeding as before, show that

$$S = \begin{pmatrix} \Sigma & 0 \\ 0 & 0_k \end{pmatrix}, \qquad Q = \begin{pmatrix} 0_{n-k} & P \\ P' & X \end{pmatrix}, \qquad k = \dim\ker M,\ P'^T P' = I_{n-k}$$

(mind that $P, P'$ need not be square matrices).

ii. Deduce that $N$ is block-triangular,

$$N = \begin{pmatrix} 0_{n-k} & K \\ 0 & 0_k \end{pmatrix}, \qquad K \in \mathbf{M}_{n-k\times k}(\mathbb{R}).$$

iii. Finally, show that $N$ is orthogonally equivalent to a block-diagonal matrix with diagonal blocks of sizes $1\times 1$ or $2\times 2$. **Hint**: Use the singular value decomposition of $K$.

451. Prove the identity

$$\forall A, B \in \mathbf{M}_n(\mathbb{R}), \qquad \det \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = |\det(A + iB)|^2.$$

**Hint**: the matrix in the left-hand side is conjugated to $\mathrm{diag}(A + iB, A - iB)$.

452. Here is the rather simple proof of Toeplitz–HausdorffToeplitzHausdorff Theorem found by Chandler DavisDavis. Recall that it is enough to prove the theorem for $2 \times 2$ matrix. Therefore, let $A \in \mathbf{M}_2(\mathbb{C})$ be given, and denote $W(A) = \{x^*Ax\,;\, x^*x = 1\}$ its numerical range.

(a) Verify that $W(A)$ is the image of the set

$$S := \{xx^*\,;\, x^*x = 1\}$$

under a real-linear map $f_A : \mathbb{H}_2 \to \mathbb{C}$.

(b) Verify that $S$ is a sphere in $\{H \in \mathbb{H}_2\,;\, \operatorname{Tr} H = 1\}$ : it is parametrized by

$$\begin{pmatrix} a & z \\ \bar{z} & b \end{pmatrix}, \qquad \left(a - \frac{1}{2}\right)^2 + \left(b - \frac{1}{2}\right)^2 + 2|z|^2 = \frac{1}{4}\,.$$

(c) Conclude.

**Remark**: We infer that in dimension $n$, the set $S_n := \{xx^*\,;\, x \in \mathbb{C}^n \text{ and } x^*x = 1\}$ can be viewed as a rather complicated union of 2-spheres.

453. Let $A \in \mathbf{M}_n(\mathbb{R})$ be an alternate matrix. We form the $2n \times 2n$ alternate matrix

$$B = \begin{pmatrix} A & -I_n \\ I_n & A \end{pmatrix}.$$

(a) Verify that $\det B = \det(A^2 + I_n)$.

(b) Prove $\det B = (\det(I_n + iA))^2$.

(c) Show that the polynomial $A \mapsto \det(I_n + iA)$ is a polynomial with *real* coefficients. We denote this polynomial Qf.

(d) Conclude that $\operatorname{Pf}(B) = \operatorname{Qf}(A)$.

(e) Show that the degree of Qf is $n$ if $n$ is even, but $n - 1$ if $n$ is odd. Actually, Qf is an even polynomial.

454. Let $K \subset \mathbf{M}_n(\mathbb{R})$ be the convex cone formed by the matrices $S + A$ where $S$ is symmetric positive semi-definite, and $A$ is alternate.

(a) Show that with $S$ and $A$ as above, $\det(S + A) \geq \det S$. **Hint**: Start with the case where $S$ is positive definite, and use the square root.

(b) Deduce that the determinant takes only non-negative values over $K$.

(c) Show that $K$ is maximal with these properties: if $K$ is contained in a convex cone $C$, and if $\det \geq 0$ over $C$, then $C = K$.

455. (After J. BochiBochi.) Let $A \in \mathbf{M}_n(\mathbb{C})$ be given, $\rho(A)$ its spectral radius and

$$r(A) := \lim_{m \to +\infty} \|A^m\|^{1/m},$$

where $\|\cdot\|$ is some operator norm over $\mathbf{M}_n(\mathbb{C})$. Because of $\rho(B) \leq \|B\|$, we have easily $\rho(A) \leq r(A)$. We now prove the converse (Banach'sBanach Formula), using the Cayley–HamiltonCayleyHamilton's Theorem. We may assume $n \geq 2$.

(a) Show that there exists a finite constant $C_n$ such that $\|A^n\| \leq C_n \rho(A) \|A\|^{n-1}$. **Hint**: CH and, again, $\rho(B) \leq \|B\|$.

(b) Apply the previous question to $A^m$ and deduce $r(A)^n \leq \rho(A) r(A)^{n-1}$. Conclude.

456. (after HörmanderHörmander & MelinMelin.)

(a) Let $M$ be a square matrix, bloc-diagonal:

$$M = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}.$$

Show that its cofactor matrix is given by the formula

$$\hat{M} = \begin{pmatrix} (\det C)\hat{B} & 0 \\ 0 & (\det B)\hat{C} \end{pmatrix}.$$

Generalize this formula to a larger number of diagonal blocs.

(b) Let $D \in \mathbf{M}_n(k)$ be Jordan matrix: it is block-diagonal and the diagonal blocks are Jordan blocks. Suppose that 0 is an eigenvalue, of algebraic multiplicity $m$. Let also $r$ be the size of the larger Jordan bloc of $D$, associated with this eigenvalue $\lambda = 0$. Let $uv^T \in \mathbf{M}_n(k)$ be a rank-one matrix. Show that $X^{m-r}$ divides the characteristic polynomial of $D + uv^T$.

(c) More generally, Let $A \in \mathbf{M}_n(k)$ have an eigenvalue $\lambda$ of algebraic multiplicity $\mu$. Let $r$ be the multiplicity of $\lambda$ as a root of the minimal polynomial. Finally, let $uv^T \in \mathbf{M}_n(k)$ be a rank-one matrix. If $r < m$, show that $\lambda$ is still an eigenvalue of $A + uv^T$, of algebraic multiplicity $\geq m - r$.

**Remark.** One can be much more precise. For each eigenvalue, only one of the Jordan blocks (one of the largest size) disappears after a rank-one pertubation. Under a rank-$\ell$ pertubation, up to $\ell$ blocks (of the largest sizes) may disappear.

(d) Let $\mu$ be an eigenvalue of $A + uv^T$, which is not an eigenvalue of $A$. Show that $\mu$ is geometrically simple. **Hint**: Prove instead that $\mathrm{rk}(A + uv^T - \mu) \geq n - 1$.

457. (After an anonymous answer to an MO question.) Given integers $a_i \in \mathbb{Z}$, we form the matrix

$$M = \begin{pmatrix} a_1 & -1 & \cdots & \cdots & -1 \\ -1 & a_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & -1 \\ -1 & \cdots & \cdots & -1 & a_n \end{pmatrix} \in \mathbf{M}_n(\mathbb{Z}).$$

Let $d_1, \dots, d_n$ be its elementary divisors. For $\ell \leq n - 1$, prove that $d_\ell$ is the gcd of all the products of $\ell$ factors of the form $a_i + 1$.

458. This is about the spread of a matrix $A \in \mathbf{M}_n(\mathbb{C})$. Recall that it is the number

$$s(A) = \max_{i,j} |\lambda_j - \lambda_i|$$

where the $\lambda_i$'s are the eigenvalues of $A$.

(a) Verify that the quadratic form

$$q_n(x) = \sum_{i=1}^n x_i^2 - \frac{1}{n}\left(\sum_1^n x_i\right)^2 - \frac{1}{2}(x_2 - x_1)^2$$

is positive semi-definite. **Hint**: It depends only upon $x_1 + x_2$, $x_3^2 + \cdots + x_n^2$ and $x_3 + \cdots + x_n$.

(b) Deduce that if $D \in \mathbf{M}_n(\mathbb{C})$ is diagonal, then

$$s(D) \leq \left(2\|D\|_F^2 - \frac{2}{n}(\operatorname{Tr} D)^2\right)^{1/2}.$$

(c) Deduce that

$$s(A) \leq \left(2\|A\|_F^2 - \frac{2}{n}(\operatorname{Tr} A)^2\right)^{1/2}.$$

**Hint**: Use Schur's trigonalisation Theorem.

459. Let $x, y \in \mathbb{R}^{n-1}$ be given vectors. Show that $\|x\|_2 = \|y\|_2$ is a necessary and sufficient condition in order that there exist $a \in \mathbb{R}$ and $N \in \mathbf{M}_n(\mathbb{R})$ such that the $n \times n$ matrix

$$M = \begin{pmatrix} a & y^T \\ x & N \end{pmatrix},$$

is normal.

460. For a matrix $M \in \mathbf{M}_{n\times m}(\mathbb{R})$, we recall the norm

$$\|M\|_{p\to q} := \max\{\|Mx\|_q \mid \|x\|_p \leq 1\} = \max\{|y^T Mx|_q \mid \|x\|_p, \|y\|_{q'} \leq 1\}.$$

We also define its *cut-norm*

$$\|M\|_c := \max_{S,T} \left| \sum_{i\in S, j\in T} m_{ij} \right|.$$

where $S$ (resp. $T$) runs over the subsets of $[\![1, n]\!]$ (resp. $[\![1, m]\!]$).

(a) Let $M$ be such that $M\mathbf{1} = 0$ and $M^T\mathbf{1} = 0$ (the row-wise and column-wise sums all vanish). Prove that

$$\|M\|_{\infty\to 1} = 4\|M\|_c.$$

(b) More generally, let $M \in \mathbf{M}_{n\times m}(\mathbb{R})$ be given, and let $A$ be the (uniquely defined) completed matrix

$$A = \begin{pmatrix} M & X \\ Y^T & a \end{pmatrix} \in \mathbf{M}_{(n+1)\times(m+1)}(\mathbb{R})$$

such that each row/column sums up to zero. Show that

$$\|M\|_c = \frac{1}{4}\|A\|_{\infty\to 1}.$$

461. Let $S$ be a simplex in the Euclidian space $\mathbb{R}^n$, whose vertices are $v_0, \dots, v_n$. We recall that the volume of $S$ is given by the formula

$$V(S) = \frac{1}{n!} \det(v_1 - v_0, \dots, v_n - v_0).$$

(a) Verify that

$$V(S) = \frac{1}{n!} \det \begin{pmatrix} 1 & \cdots & 1 \\ v_0 & \cdots & v_n \end{pmatrix}.$$

(b) Deduce that

$$\begin{aligned} (n!\,V(S))^2 &= \det((1 + v_i \cdot v_j))_{0 \le i,j \le n} = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 + v_0 \cdot v_0 & \cdots & \\ \vdots & \vdots & 1 + v_i \cdot v_j & \vdots \\ 0 & & \cdots & 1 + v_n \cdot v_n \end{pmatrix} \\ &= \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ -1 & v_0 \cdot v_0 & \cdots & \\ \vdots & \vdots & v_i \cdot v_j & \vdots \\ -1 & & \cdots & v_n \cdot v_n \end{pmatrix}. \end{aligned}$$

(c) Show that $\det((v_i \cdot v_j))_{0 \le i,j \le n} = 0$. Deduce that

$$(n!\,V(S))^2 = \det \begin{pmatrix} 0 & 1 & \cdots & 1 \\ -1 & v_0 \cdot v_0 & \cdots & \\ \vdots & \vdots & v_i \cdot v_j & \vdots \\ -1 & & \cdots & v_n \cdot v_n \end{pmatrix}$$

as well.

(d) Show that

$$(-1)^{n+1} 2^n (n!\,V(S))^2 = \det \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & -2v_0 \cdot v_0 & \cdots & \\ \vdots & \vdots & -2v_i \cdot v_j & \vdots \\ 1 & & \cdots & -2v_n \cdot v_n \end{pmatrix}$$

(e) Deduce the Cayley–Menger Formula:

$$V(S)^2 = \frac{(-1)^{n+1}}{2^n (n!)^2} \det \begin{pmatrix} 0 & \mathbf{1}^T \\ \mathbf{1} & E(S) \end{pmatrix}, \qquad E(S) := ((\|v_i - v_j\|^2))_{0 \le i,j \le n}.$$

(f) When $n = 2$ (case of a triangle), recover Heron's Formula for the area in terms of the lengths $p, q, r$ of the edges:

$$\frac{1}{4} \sqrt{(p+q+r)(p+q-r)(q+r-p)(r+p-q)}\,.$$

462. For $h:(a,b)\rightarrow\mathbb{R}$ a smooth function, we form the *Hankel*Hankel matrix

$$M_{n,h} := \begin{pmatrix} h & h' & \cdots & h^{(n)} \\ h' & h'' & \cdots & h^{(n+1)} \\ \vdots & & & \vdots \\ h^{(n)} & h^{(n+1)} & \cdots & h^{(2n)} \end{pmatrix},$$

which involves derivatives.

(a) If $h$ satisfies a constant-coefficient ODE of order $n$, show that $\det M_{n,h}\equiv 0$.

(b) Conversely, let us assume that $\det M_{n,h}\equiv 0$. We suppose that $n$ is minimal for this property, w.l.o.g. $\det M_{n-1,h}\neq 0$ over $(a,b)$.

i. Show the existence of smooth functions $g_0,\dots,g_{n-1}$ such that

$$h^{(j+n)} = \sum_{i=0}^{n-1} g_i h^{(j+i)}, \qquad j=0,\dots n.$$

ii. Prove

$$\sum_{i=0}^{n-1} g_i' h^{(k+i)} = 0, \qquad k=0,\dots n-1.$$

iii. Deduce that $h$ satisfies a constant-coefficient ODE of order $n$.

463. (From G. Zaimi.) Consider the $n\times n$ matrix

$$A = \begin{pmatrix} c_1 & a & \dots & a \\ b & c_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a \\ b & \dots & b & c_n \end{pmatrix}.$$

(a) Let $\mathbf{1}$ denote the $n\times n$ matrix whose entries are 1's. Compute $\det(A-b\mathbf{1})$ and $\det(A-c\mathbf{1})$.

(b) Show that $X\mapsto\det(A-X\mathbf{1})$ is affine.

(c) Deduce the value of $\det A$.

464. Prove that the function

$$\begin{array}{rcl} \mathbf{SDP}_n\times\mathbb{R}^n & \longrightarrow & \mathbb{R}^+ \\ (S,x) & \longmapsto & x^TS^{-1}x \end{array}$$

is convex.

465. We improve Proposition 6.1 by relaxing the assumption of positive definiteness.

Let $H\in\mathbf{Sym}_n(\mathbb{R})$ and $K\in\mathbf{Sym}_n^+$ be given.

(a) Verify that $\ker\sqrt{K} = \ker K$ and $R(\sqrt{K}) = R(K)$.

(b) Let $\lambda \neq 0$ be a real number. Show that $x \mapsto K^{1/2}x$ is an isomorphism from $\ker(HK - \lambda I_n)$ to $\ker(K^{1/2}HK^{1/2} - \lambda I_n)$.

(c) We now suppose that $H \in \mathbf{Sym}_n^+$ as well. Show that

$$\dim\ker(HK) = \dim\ker(K^{1/2}HK^{1/2}) = \dim\ker K + \dim(R(K) \cap \ker H).$$

(d) Deduce that if both $H$ and $K$ are positive semi-definite, then $HK$ is diagonalizable with non-negative real eigenvalues.

(e) In particular, the zero eigenvalue is semi-simple: $\ker((HK)^2) = \ker(HK)$. Find a direct proof of that point.

(f) Find a pair $H \in \mathbf{Sym}_2$ and $K \in \mathbf{Sym}_2^+$ such that $HK$ is not diagonalizable.

466. This is a complement to Proposition 6.1.

(a) Let $H, K \in \mathbf{Sym}_n^+(\mathbb{R})$ be given, prove an arithmetico-geometric inequality for the spectrum of the matrix product:

$$(\det(HK))^{\frac{1}{n}} \leq \frac{1}{n}\,\mathrm{Tr}(HK).$$

(b) Conversely, let $M \in \mathbf{M}_n(\mathbb{R})$ be such that $\det M > 0$. Suppose that

$$(\det(MK)^{\frac{1}{n}} \leq \frac{1}{n}\,\mathrm{Tr}(MK), \qquad \forall K \in \mathbf{SPD}_n.$$

Prove that $M$ is symmetric. **Hint**: use the polar factorization.

467. These are two examples taken from either gas dynamics or wave equation, where the determinant can be computed using Schur's complement formula.

(a) Let $\rho, p \in \mathbb{R}$ and $w \in \mathbb{R}^n$ be given. Show that

$$\det\begin{pmatrix} \rho & \rho w^T \\ \rho w & \rho w \otimes w + pI_n \end{pmatrix} = \rho p^n.$$

Mind that the matrix belongs to $\mathbf{M}_{n+1}(\mathbb{R})$.

(b) Let $a \in \mathbb{R}$ and $w \in \mathbb{R}^n$ be given. Show that

$$\det\begin{pmatrix} a^2 + |w|^2 & -2aw^T \\ -2aw & 2w \otimes w + (a^2 - |w|^2)I_n \end{pmatrix} = (a^2 - |w|^2)^{n+1}.$$

468. (H. R. Suleimanova.) The *Inverse Eigenvalue Problem* consists in characterizing the possible spectra ($n$-tuples of complex numbers) of non-negative matrices. We say that $V = (z_1, \dots, z_n)$ is *realizable* if there exists a matrix $M \in \mathbf{M}_n(\mathbb{R})$ with non-negative entries, whose spectrum is $V$. An obvious necessary condition is that $V$ be stable under complex conjugacy. An other one, given by the Perron–Frobenius Theorem is that the maximal modulus of $V$ belongs to $V$: $z_1$ is real and $z_1 \geq |z_j|$ for every $j$.

(a) For $k \in \mathbb{N}$, we define the Newton sums $s_k = z_1^k + \cdots + z_n^k$. If $V$ is realizable, prove that $s_k \geq 0$ for every $k \geq 1$.

*Remark.* Actually, R. Loewy & D. London, and independently C. R. Johnson, proved that $n^{k-1}s_{km} \geq s_m^k$ for $k, m \geq 1$.

In the sequel, we denote $\sigma_k$ the elementary symmetric polynomial of degree $k$.

(b) From now on, we assume that $V$ is real and only $z_1$ is positive:

$$V = (x_1, -x_2, -x_3, \dots, -x_n), \qquad \text{every } x_j \geq 0.$$

From the previous question, a necessary condition for realizability is

$$x_1 \geq \sigma_1(x_2, \dots, x_n),$$

which we assume from now on.

i. If $k \geq 1$, show that $\sigma_k(x_2, \dots, x_n) \leq x_1 \sigma_{k-1}(x_2, \dots, x_n)$. Deduce the sign of the coefficients of the polynomial

$$P(X) = \prod_{j=1}^{n} (X - z_j).$$

ii. Show that the necessary condition is also sufficient (Suleimanova): if $V$ is real and all but one element are non-positive, then $V$ is realizable if and only if $s_1 \geq 0$.

469. (After Borisov, Fischle & Neff.) Let $M \in \mathbf{M}_n(\mathbb{R})$ be such that $M^2$ is symmetric.

(a) Define $S := [M, M^T]$. Verify that $S$ is symmetric and $MS + SM = 0$.

(b) Prove that there exists an orthogonal matrix $Q$ such that, if $N = Q^T M Q$ and $R = [N, N^T]$, then $R$ is block-diagonal, with diagonal blocs of the form $0_p$ (the first one), or $dI_p \ominus dI_q$ for some distinct real numbers $d > 0$.

(c) Show that $N$ is bloc-diagonal as well, the first diagonal block being normal, and the other ones of the form

$$\begin{pmatrix} 0_p & B \\ C & 0_q \end{pmatrix}.$$

(d) For such blocks, prove that $q = p$.

(e) Conclude that $M$ is orthogonally similar to a bloc-diagonal matrix whose diagonal blocks are either scalars or $2 \times 2$ matrices.

(f) Show also that these $2 \times 2$ blocks are either symmetric, or of zero trace.

470. Let $A, B \in \mathbf{M}_n(k)$ be two commuting matrices: $AB = BA$. We assume that the minimal polynomial $\pi_A$ equals its characteristic polynomial $P_A$.

(a) Prove that there exists a vector $v_1 \in k^n$ such that $(v_1, Av_1, \dots, A^{n-1}v_1)$ forms a basis of $k^n$.

(b) Show that there exists a polynomial $Q$ of degree $< n$ such that $Q(A)v_1 = Bv_1$.

(c) Verify that $BA^k v_1 = Q(A)A^k v_1$.

(d) Deduce that $B = Q(A)$, that is, $B$ is a polynomial in $A$.

471. Let $A \in \mathbf{GL}_n(\mathbb{C})$ be given. Use the $QR$ factorization to prove Hadamard's Inequality, plus the fact that the equality holds if and only if the columns of $A$ are pairwise orthogonal. **Hint**: Express the columns $A^{(j)}$ in terms of the columns $Q^{(k)}$.

472. According to Exercise 26, Chapter 4 (#23, Chap. 7 in the second edition ; the exercise does not exist in the French edition), the unit sphere $\Sigma$ of $\mathbf{M}_2(\mathbb{R})$ for the induced norm $\|\cdot\|_2$ is parametrized by $\mathbf{SO}_2 \times \mathbf{O}_2^- \times [0,1]$. Using the maps

$$\alpha \mapsto R(\alpha) := \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, \qquad \beta \mapsto S(\beta) := \begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix},$$

the sphere is actually parametrized by

$$(\alpha,\beta,t) \mapsto tR(\alpha) + (1-t)S(\beta), \qquad (\alpha,\beta,t) \in [0,2\pi) \times [0,2\pi) \times [0,1].$$

(See also exercise 153).

We equip $\mathbf{M}_2(\mathbb{R})$ with its standard Euclidian structure, that associated with the Frobenius norm $\|A\|_F = (\mathrm{Tr}(A^TA))^{1/2}$. For a piecewise smooth hypersurface, it induces an area element, with which we may define the 3-dimensional volume. Show that

$$\mathrm{vol}_3(\Sigma) = \frac{8}{3}\pi^2.$$

**Remark**: The same volume of the unit sphere of $\|\cdot\|_F$ is $2\pi^2$. But $\Sigma$ is clamped between the Frobenius spheres of radii 1 and $\sqrt{2}$.

473. Denote

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Show that the *joint spectral radius* (see Exercise 140) of $F = \{L, U\}$ equals the golden ratio $\phi = \frac{1}{2}(1+\sqrt{5})$. More generally, the joint spectral radius of $F = \{M, M^*\}$ is $\|M\|_2$.

474. (After T. Tao and others.) Let $A \in \mathbf{M}_n(\mathbb{C})$ be normal, with spectral decomposition

$$A = \sum_{k=1}^{n} \lambda_k(A) v_k v_k^*.$$

In particular, $(v_1, \dots, v_n)$ is a unitary basis. The coordinates of $v_k$ be will denoted $(v_{k,1}, \dots, v_{k,n})$ below.

(a) Write the spectral decomposition of $(\lambda I_n - A)^{-1}$, when $\lambda$ is not an eigenvalue.

(b) From the formula above, prove the formula

$$\sum_{k=1}^{n} \frac{|v_{k,j}|^2}{\lambda - \lambda_k(A)} = \frac{\det(\lambda I_{n-1} - M_j)}{\det(\lambda I_n - A)},$$

where $M_j$ is the matrix obtained from $A$ be deleting the $j$th row and column.

(c) Suppose that $\lambda_i(A)$ is a simple eigenvalue. Deduce the identity

$$|v_{i,j}|^2 \prod_{k \neq i} (\lambda_i(A) - \lambda_k(A)) = P_{M_j}(\lambda_i(A)).$$

(d) Verify that this identity is trivial when $\lambda_i(A)$ has multiplicity $\geq 2$.

475. Let $k$ be a field and $a, b, c, d \in k$ be such that $n = a^2 + b^2 + c^2 + d^2 \neq 0$.

(a) Verify that the matrix

$$R := \frac{1}{n} \begin{pmatrix} a^2 + b^2 - c^2 - d^2 & 2(-ad + bc) & 2(ac + bd) \\ 2(ad + bc) & a^2 - b^2 + c^2 - d^2 & 2(-ab + cd) \\ 2(-ac + bd) & 2(ab + cd) & a^2 - b^2 - c^2 + d^2 \end{pmatrix}$$

is orthogonal with determinant $+1$.

(b) Give an interpretation in terms of quaternions. **Hint**: think of a rotation in the subspace of pure imaginary quaternions.

(c) If $k = \mathbb{Q}$, verify that every rotation is of the form above with $a, b, c, d \in \mathbb{Z}$.

476. For $A \in \mathbf{M}_n(\mathbb{C})$, prove the identity

$$\min\{\rho(QA) \,|\, Q \in \mathbf{U}_n\} = |\det A|^{\frac{1}{n}}.$$

477. All matrices below belong to $\mathbf{SPD}_n$ ($\mathbf{HPD}_n$ might work as well).

(a) Show that the maps $S \mapsto \hat{S}$ (cofactor matrix) is invertible from $\mathbf{SPD}_n$ into itself. We denote $A \mapsto \check{A}$ the inverse map. Express $\check{A}$ in terms of $A^{-1}$ and $\det A$.

(b) Let $D$ be a diagonal matrix, with positive diagonal. Show that

$$I_n + D \leq (\det \frac{1}{2}(I_n + D)) \left( I_n + \frac{1}{\det D} D \right).$$

(c) Deduce that for two matrices $B_1, B_2 \in \mathbf{SPD}_n$, one has

$$\frac{1}{\det \frac{B_1 + B_2}{2}} (B_1 + B_2) \leq \frac{1}{\det B_1} B_1 + \frac{1}{\det B_2} B_2.$$

In other words, the map $B \mapsto \frac{1}{\det B} B$ is operator convex over $\mathbf{SPD}_n$.

(d) Given $A_1, A_2 \in \mathbf{SPD}_n$, we define a *mean-by-cofactors*:

$$m_C(A_1, A_2) = \hat{C}, \qquad C := \frac{1}{2}(\check{A}_1 + \check{A}_2).$$

Verify that $m_C(A_1, A_2) \in \mathbf{SPD}_n$. Show that

$$m_H(A_1, A_2) \leq m_C(A_1, A_2),$$

where $m_H$ denotes the harmonic mean.

# Index

authorIndex of scientists (mostly mathematicians)

# Index

stpIndex of stamps