Global Langlands for Function Fields - Sophie Morel - Spring 2014
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1 Lecture - 02/05/2014

Fix a finite field $\mathbb{F}_q$, prime number $\ell \nmid q$, let $X/\mathbb{F}_q$ be a smooth projective geometrically connected curve. Let $F = \mathbb{F}_q(X)$ be the function field of the curve (e.g. $F = \mathbb{F}_q(T)$ for $X = \mathbb{P}^1$). Let $|X|$ be the closed points of X (the places of F), let $\mathbb{A} = \prod'_{v \in |X|} F_v$ which contains $\mathbb{O} = \prod_{v \in |X|} \mathcal{O}_v$. Take $G/\mathbb{F}_q$ a split connected reductive group.

Automorphic forms: Levels are given by finite closed subschemes $N \subseteq X$, set $\mathcal{O}_N = \mathbb{F}_q[N]$ and $K_N = \ker(G(\mathbb{O}) \to G(\mathcal{O}_N))$. Take $Z = Z(G)$, fix a lattice $\Xi \subseteq Z(F)\backslash Z(\mathbb{A})$. Then $G(F)\backslash G(\mathbb{A})/K_N\Xi$ has finite volume (using Haar measure giving $G(\mathbb{O})$ volume 1). Consider the space of functions

$$C_c^\infty(G(F)\backslash G(\mathbb{A})/K_N\Xi, \overline{\mathbb{Q}}_\ell)$$

as our "automorphic forms". A function f in this space is called *cuspidal* if, for all parabolic $P \subseteq G$, $N_P = R_u(P)$, then

$$\int_{N_P(F)\backslash N_P(\mathbb{A})} f(nx)dn = 0$$

for all $x \in G(\mathbb{A})$. Let

$$C^{cusp} = C^{cusp}(G(F)\backslash G(\mathbb{A})/K_N\Xi, \overline{\mathbb{Q}}_\ell) \subseteq C_c^\infty(G(F)\backslash G(\mathbb{A})/K_N\Xi, \overline{\mathbb{Q}}_\ell)$$

be the space of such cusp forms; this is a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space.

Hecke algebra: Take $\mathcal{H}_N = C_c^\infty(K_N\backslash G(\mathbb{A})/K_N, \overline{\mathbb{Q}}_\ell)$ with convolution product; this acts on $C_c^\infty(G(F)\backslash G(\mathbb{A})/K_N\Xi, \overline{\mathbb{Q}}_\ell)$ by right convolution, preserving C^{cusp}.

Dual group of G: Fix $T \subseteq G$ a split maximal torus. The group of characters $X^* = X^*(T)$ has a subset of roots and the set of cocharacters $X_*(T)$ has a subset of coroots $\varphi^\vee$. The tuple $(X^*, \varphi, X_*, \varphi^\vee)$ is the *root data* of G, and completely determines the reductive group. The dual group $\widehat{G}$ is defined as the spit reductive group with root data $(X_*, \varphi^\vee, X^*, \varphi)$. Examples: if $G = \mathrm{GL}_n$ then $\widehat{G} = \mathrm{GL}_n$; if $G = \mathrm{SO}_{2n}$ then $\widehat{G} = \mathrm{SO}_{2n}$, if $G = Sp_{2n}$ then $\widehat{G} = \mathrm{SO}_{2n+1}$ and vice versa, if $G = GSp_{2n}$ then $\widehat{G} = GSpin_{2n+1}$, if $G = \mathrm{SL}_n$ then $\widehat{G} = \mathrm{PGL}_n$...

Definition: A *Langlands parameter* for G is a continuous group homomorphism $\sigma : \mathrm{Gal}(\overline{F}/F) \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$ such that:

- σ is defined over a finite extension of $\mathbb{Q}_\ell$ (automatic for GL_n; maybe for others?)
- σ is semisimple (i.e. the Zariski closure of the image of σ is a reductive subgroup of $\widehat{G}$).
- σ is almost everywhere unramified.

We say two parameters are equivalent, $\sigma \sim \sigma'$, if they're conjugate under $\widehat{G}(\overline{\mathbb{Q}}_\ell)$. (E.g. a Langlands parameter for GL_n is an n-dimensional ℓ-adic Galois representation of $\mathrm{Gal}(\overline{F}/F)$).

Main theorem (V. Lafforgue): There exists a canonical $\mathcal{H}_N$-equivariant decomposition $C^{cusp} = \bigoplus_{[\sigma], \sigma \text{ unramified}} H_\sigma$ compatible with the Satake isomorphism at places of $X \setminus N$.

Remark: For GL_n this is known (Drinfeld for $n = 2$, L. Lafforgue for $n \geq 3$).

Very rough sketch of proof: Suppose that the theorem is true. Then for every finite set I we get a functor $\mathbf{Rep}(\widehat{G}^I) \to \mathbf{Rep}(\mathcal{H}_N \times \Gamma_F^I)$ (where Γ_F is the absolute Galois group), given by sending W to $\bigoplus_\sigma H_\sigma \otimes W_{\sigma^I}$ (where W has Γ_F^I acting through σ^I). We have:

- $H_{\emptyset,1} = C^{cusp}$.
- For all $\xi : I \to H$, we have a Γ_F^J-equivariant isomorphism (functorial in W) $X_\xi : H_{I,W} \cong H_{J,W^\xi}$ where W^ξ is W with $\widehat{G}^J$ acting via $\xi^* : \widehat{G}^J \to \widehat{G}^I$.

How do we get back to Langlands parameters from this formalism? Fix I finite, $W \in \mathbf{Rep}(\widehat{G}^I)$, $(\gamma_i) \in \Gamma_F^I$. Fix $\widehat{G}$-equivariant maps $x : 1 \to W^{diag}$ and $\xi : W^{diag} \to 1$ (so $x \in W^{\widehat{G}-diag}$ and $\xi \in (W^*)^{\widehat{G}-diag}$). Then get that

$$C^{cusp} = H_{\emptyset,1} \cong H_{\{0\},1} \to H_{\{0\},W^{diag}} \cong H_{I,W} \to H_{I,W} \cong H_{\{0\},W^{diag}} \to H_{\{0\},1} \cong C^{cusp}$$

where the arrows are x, (γ_i), and ξ, respectively. This whole big thing gives an endomorphism $S_{I,W,x,\xi,(\gamma_i)}$ of $\mathrm{End}(C^{cusp})$. On the factor H_σ of C^{cusp}, trace through and find it's multiplication by the scalar $\langle \xi, (\sigma(\gamma_i)) \cdot \gamma \rangle$.

Lafforgue's crucial observations:
(1) As we vary $I, W, x, \xi, (\gamma_i)$, these scalars totally determine σ, so we get back $C^{cusp} = H_\sigma$ by simultaneously diagonalizing the $S_{I,W,x,\xi,(\gamma_i)}$.
(2) We only need to have the functors $W \mapsto H_{I,W}$ plus some basic properties to make this work. (We'll get those functors by using the cohomology of moduli stacks of shtukas).

Explanation of (1): First note that, as W, x, ξ vary the functions $(g_i) \mapsto \langle \xi, (g_i)x \rangle$ are exactly the functions in $\mathcal{O}(\widehat{G} \backslash\backslash \widehat{G}^i // \widehat{G})$ ("coarse quotient" - take ring of functions, take invariants).

If W, x, ξ correspond to f then $S_{I,W,x,\xi,(\gamma_i)}$ depends on W, x, ξ only through f; write it as $S_{I,f,(\gamma_i)}$. A simultaneous eigenvalue of these $S_{I,f,(\gamma_i)}$ gives a morphism of algebras

$$\mathcal{O}(\widehat{G} \backslash\backslash \widehat{G}^I // \widehat{G}) \to \mathrm{Cont}(\Gamma_F^I, \overline{\mathbb{Q}}_\ell).$$

Write $I = \{0, 1, \ldots, n\}$. Note that we have maps $\widehat{G}^n // \widehat{G} \to \widehat{G} \backslash\backslash \widehat{G}^I // \widehat{G}$ by $(g_1, \ldots, g_n) \mapsto (1, g_1, \ldots, g_n)$.

Proposition (Lafforgue, based on results Richardson). Let Γ be a profinite group, H a split connected reductive group, $E/\mathbb{Q}_\ell$ a finite extension. Suppose give, for all $n > 0$, we're given algebra maps $\Xi_n : \mathcal{O}(H^n//H) \to Cont(\Gamma^n, E)$ such that

1. (Ξ_n) is functorial for maps $\xi : \{1, \ldots, n\} \to \{1, \ldots, m\}$.
2. For all n, for all $f \in \mathcal{O}(H^n//H)$, if $\widehat{f} \in \mathcal{O}(H^{n+1}/H)$ is given by $\widehat{f}(g_1, \ldots, g_{n+1}) = f(g_1, \ldots, g_{n-1}, g_n g_{n+1})$ then $\Xi_{n+1}(\widehat{f})(\gamma_1, \ldots, \gamma_{n+1}) = \Xi_n(f)(\gamma_1, \ldots, \gamma_{n-1}, \gamma_n\gamma_{n+1})$.

Then, there exists E'/E finite and $\sigma : \Gamma_F \to H(E)$, a semisimple continuous group homomorphism, such that $\Xi_n(F)(\gamma_1, \ldots, \gamma_n) = f(\sigma(\gamma_1), \ldots, \sigma(\gamma_n))$ for all f. Moreover, σ is unique modulo $H(\mathbb{Q}_\ell)$-conjugacy.

Case $H = \mathrm{GL}_r$: let χ_{st} be the character of the standard representation; then $\tau = \Xi_1(\chi_{st}) : \Gamma \to E$ determines all of the $\Xi_n(F)$. Moreover, there's a condition on τ given by $\bigwedge^{r+1} St = 0$ (?).

2 Lecture - 02/07/2014

Part I. The geometric Satake equivalence.

Introduction: Classical Satake. Let k be a finite field, $K = k((t))$, $\mathcal{O} = k[[t]]$. Fix a split connected reductive group G/k. The unramified Hecke algebra is

$$\mathcal{H}_G = C_c^\infty(G(\mathcal{O})\backslash G(K)/G(\mathcal{O}), \overline{\mathbb{Q}}_\ell)$$

for some ℓ not equal to the characteristic of k. We give this a convolution product (via Haar measure on $G(K)$ with the volume of $G(\mathcal{O})$ equal to 1).

Example: $G = T \cong \mathbb{G}_m^n$ a torus. The dual group $\widehat{T}$ is the torus with cocharacters $X_*(\widehat{T}) = X^*(T)$ (so we think of $\widehat{T}$ as being "$X^*(T) \otimes_{\mathbb{Z}} \mathbb{G}_m$"). Now, we have

$$X^*(\widehat{T}) = X_*(T) \cong T(K)/T(\mathcal{O}) = T\mathcal{O})\backslash T(K)/T(\mathcal{O})$$

where the isomorphism is given by $\mu \mapsto \mu(t)$. This gives an isomorphism of $\overline{\mathbb{Q}}_\ell$-algebras

$$\mathcal{H}_T \cong \overline{\mathbb{Q}}_\ell[X^*(\widehat{T})] = \overline{\mathbb{Q}}_\ell[\widehat{T}]$$

(where the first thing is a group algebra, isomorphic to $K_0(\mathbf{Rep}_{\widehat{T}}) \otimes \overline{\mathbb{Q}}_\ell$, and the second thing is the rational functions on $\widehat{T}$. Here, $\mathbf{Rep}_{\widehat{T}}$ is the category of algebraic representations, and K_0 is the Grothendieck group, which has multiplication given by tensor product).

General case: Fix Borel subgroup $B \subseteq G$, split maximal torus $T \subseteq B$. Set $|ph = \varphi(T, G)$ and $\varphi^+ = \varphi(T, B)$. Define

$$X_*(T)^+ = X^*(\widehat{T})^+ = \{\mu \in X_*(T) : \forall \alpha \in \varphi^+, \langle \alpha, \mu \rangle > 0\}.$$

Cartan decomposition: $G(k) = \coprod_{\mu(t) \in X_*(T)^+} G(\mathcal{O})\mu(t)G(\mathcal{O})$. Then, if we set c_μ to be the indicator function of $G(\mathcal{O})\mu(t)G(\mathcal{O})$, the collection of these as μ runs over $X_*(T)^+$ is a basis of $\mathcal{H}_G$ as a $\overline{\mathbb{Q}}_\ell$-vector space.

Satake transform: Set $N = R_u(B)$, give Haar measure dn on $N(K)$ such that the volume of $N(\mathcal{O})$ is 1. Let $\delta : B(K) \to \mathbb{R}^+$ be the modular function (so $\delta(g) = |\alpha\rho(g)|$, $\alpha\rho = \sum_{\alpha \in \varphi^+} \alpha$, seen as a character of B via $B \twoheadrightarrow B/N \cong T$).

Definition: For all $f \in \mathcal{H}_G$, define $Sf : T(K) \to \overline{\mathbb{Q}}_\ell$ by

$$Sf(g) = \delta(g)^{1/2} \int_{N(K)} f(gn)dn = \delta(g)^{-1/2} \int_{N(K)} f(ng)dn.$$

Then $Sf \in \mathcal{H}_T$.

Theorem: Let $W = W(T, G)$. Then S induces an isomorphism of algebras

$$\mathcal{H}_G \cong \mathcal{H}_T^W \cong \overline{\mathbb{Q}}_\ell[X^*(\mathbb{T})]^W \cong \overline{\mathbb{Q}}_\ell[\mathbb{T}/W] \cong K_0(\mathbf{Rep}_{\widehat{G}}) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_\ell$$

(where the last isomorphism takes a class $[V]$ in K_0 to the function $g \mapsto \mathrm{tr}(g, V)$).

Corollary: Characters $\mathcal{H}_G \to \overline{\mathbb{Q}}_\ell$ are the same as $\overline{\mathbb{Q}}_\ell$-points of $\widehat{T}/W$, which are the same as semisimple conjugacy classes in $\widehat{G}(\overline{\mathbb{Q}}_\ell)$.

Aside: Now, recall last time that $C^{cusp}(G) \cong \bigoplus_\sigma H_\sigma$ by a $\mathcal{H}_N$-equivariant map; the character by which $\mathcal{H}_{G(k)}$ acts on H_σ corresponds to $\sigma(\mathrm{Frob}_x)$. (???)

Example: $G = \mathrm{GL}_n$, usual B and T... Then $\mathcal{H}_G \cong \overline{\mathbb{Q}}_\ell[T_1^{\pm 1}, \ldots, T_n^{\pm n}]$.

Idea of proof of Theorem:

(1) Show that S is a morphism of algebras ("easy" calculation using the Iwasawa decomposition).

(2) Show $S(\mathcal{H}_G) \subseteq \mathcal{H}_T^W$ (also easy).

(3) Show $S : \mathcal{H}_G \to \mathcal{H}_T^W$ is an isomorphism of vector spaces. Had a basis basis c_μ defined earlier of $\mathcal{H}_G$ for $\mu \in X_*(T)^+$. Define a basis $\{d_\mu\}$ of $\mathcal{H}_T^W$ by setting, $W(\mu) = \{w \in W : w\mu = \mu\}$ and

$$d_\mu = \frac{1}{|W(\mu)|} \sum_{w \in W} w\mu \in \overline{\mathbb{Q}}_\ell[X^*(\widehat{T})]^W.$$

Define the "matrix" $(\alpha_{\lambda\mu})$ by

$$Sc_\lambda = \sum_{\mu \in X^*(\widehat{T})^+} a_{\lambda\mu} d_\mu.$$

Order $X * (\widehat{T})$ by $\lambda \le \mu$ iff $\mu - \lambda$ is a positive sum of stuff in $(\varphi^\vee)^+$. Calculate

$$a_{\lambda\mu} = Sc_\lambda(\mu(t)) = \delta(\mu(t))^{-1/2} \cdot \mathrm{vol}(G(\mathcal{O})\lambda(t)G(\mathcal{O}) \cap N(K)\mu(t)G(\mathcal{O})).$$

Lemma (F. Herzig): This volume is zero unless $\mu \le \lambda$, and is $|k|^{\langle \rho, \lambda \rangle}$ if $\mu = \lambda$. So $(\alpha_{\lambda,\mu})$ is "lower-triangular" and has nonzero diagonal entries so is invertible. Thus $\mathbb{H}_G \cong K_0(\mathbf{Rep}_{\widehat{G}}) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_\ell$.

Question: Can we upgrade this to an equivalence of (Tannakian) categories? Something isomorphic to $\mathbf{rep}_{\widehat{G}}$? Sheaf-to-function dictionary. If $X/\mathbb{F}_q$ is a scheme of finite type, K a constructible ℓ-adic complex on X, this passes to $t_K : X(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell$ given by $x \mapsto \mathrm{tr}(\mathrm{Frob}_x)$. The set of all $T_{K/X \otimes \mathbb{F}_{q^n}}$'s determines the class of K in K_0 of the ℓ-adic complexes.

Suggests: On LHS, use ℓ-adic complexes on some scheme-line object X/k such that $X(k) = G(\mathcal{O})\backslash G(K)/G(\mathcal{O})$. Issues: this quotient is $X_*(T)^+$ which is discrete for any reasonable geometric structure. Instead, use ℓ-adic sheaves on $G(K)/G(\mathcal{O})$ (affine Grassmannians) and take $G(\mathcal{O})$-equivariant sheaves. But $|G(K)/G(\mathcal{O})|$ is infinite so isn't $X(k)$ for any scheme of finite type; thus our affine Grassmannian will be an ind-scheme.

The RHS is $\mathbf{Rep}_{\widehat{G}}$, a semisimple abelian category. The LHS thus cannot be D_c^b (ℓ-adic complexes) or Sh_c (constructible sheaves). We'll have to use perverse sheaves. Bonus: k can be any field (We'll take $k = \overline{k}$ to simplify).

The affine Grassmannian. Schemes as functors of points. Fix a commutative ring R (later we'll take $R = k$), and $\mathbf{Aff}_R$ to be the category of affine schemes over $\mathrm{Spec}\, R$. The Zariski (étale, fppf, fpqc) topology on $\mathbf{Aff}_R$ is defined by taking as covering families as the families $\{f_i : S_i \to S\}_{i \in I}$ such that

- All of the f_i's are (open immersions, étale, flat of finite presentation, flat).
- There exists $J \subseteq I$ finite such that $S = \bigcup_{j \in J} f_j[S_j]$.

Then, take $PSh(\mathbf{Aff}_R)$ to be all functors $\mathbf{Aff}_R^{op} \to \mathbf{set}$, and $Sh(\mathbf{Aff}_R)$ the fpqc sheaves of sets on $\mathbf{Aff}_R$. There's an exact sheafification functor $PSh(\mathbf{Aff}_R) \to Sh(\mathbf{Aff}_R)$ which we'll denote $\mathcal{F} \mapsto \mathcal{F}^{sh}$.

Define an *R-space* as an fpqc sheaf on $\mathbf{Aff}_R$. Example: if X/R is a scheme, then $\underline{X} : S \mapsto \mathrm{Hom}_R(S, X)$ is an R-space. The map $X \mapsto \underline{X}$ is functorial and induces a fully faithful functor $\mathbf{Sch}_R \to Sh(\mathbf{Aff}_R)$ (by Yoneda + a bit more).

Faithfully flat descent (Grothendieck): Let $\{f_i : S_i \to S\}_{i \in I}$ be a family of morphisms of R-schemes. A *descent datum* for quasicoherent sheaves WRT this family consists of quasicoherent sheaves $\mathcal{F}_i$ on S_i for all i and isomorphisms $\varphi_{ij} : \pi_i^*\mathcal{F}_i \cong \pi_j^*\mathcal{F}_j$ as schemes on $S_i \times_S S_j$. These must satisfy the usual cocycle condition: for every i, j, k the following diagram commutes (where $\pi_i : S_i \times_S S_j \times_S S_k \to S_i$ and $\pi_{ij} : S_i \times_S S_j \times_S S_k \to S_i \times_S S_j$ are the obvious things):

$$\begin{tikzcd}
\pi_i^*\mathcal{F}_i \arrow[rr, "\pi_{ik}^*\varphi_{ik}"] \arrow[dr, "\pi_i^*\varphi_{ij}"'] & & \pi_k^*\mathcal{F}_k \\
& \pi_j^*\mathcal{F}_j. \arrow[ur, "\pi_{jk}^*\varphi_{jk}"'] &
\end{tikzcd}$$

A morphism $\psi : (\mathcal{F}_i, \varphi_{ij}) \to (\mathcal{F}'_i, \varphi'_{ij})$ of descent data is a family of morphisms $\psi_i : \mathcal{F}_i \to \mathcal{F}'_i$ such that

$$\begin{tikzcd}
\pi_i^*\mathcal{F}_i \arrow[r, "\varphi_{ij}"] \arrow[d, "\pi_i^*\varphi_i"] & \pi_j^*\mathcal{F}_j \arrow[d, "\pi_j^*\varphi_j"] \\
\pi_i^*\mathcal{F}'_i \arrow[r, "\varphi'_{ij}"] & \pi_j^*\mathcal{F}'_j
\end{tikzcd}$$

Then get a category of descent data for our family $\{f_i : S_i \to S\}$. If $\mathcal{F}$ is a quasicoherent sheaf on S, then we get a descent datum consisting of the $f_i^*\mathcal{F}$ and the canonical map. This assignment is functorial.

Theorem (Grothendieck): Assume that $\{f_i : S_i \to S\}$ is a fpqc cover. Then the functor from quasicoherent sheaves to descent data above is an equivalence of categories. (I.e. a descent datum "glues" together to a quasicoherent sheaf). (Later we'll see this means the functor $S \mapsto \mathbf{QCoh}(S)$ is a fpqc stack).

In particular, affine schemes over S are given as relative Spec's of quasicoherent $\mathcal{O}_S$-algebras. So you can descend affine schemes.

3 Lecture - 02/14/2014

Last time: Defined R-spaces as the category $Sh(\mathbf{Aff}_R)$ for the fpqc topology. Also had $PSh(\mathbf{Aff}_R)$ with sheafification functor (that's exact) to $Sh(\mathbf{Aff}_R)$. Yoneda: Have fully faithful embedding $\mathbf{Sch}_R$ into R-spaces by mapping X to the sheaf $\operatorname{Spec} A \mapsto X(A)$.

How about projective and inductive limits? In $Psch(\mathbf{Aff}_R)$ these are calculated term-by-term, i.e.

$$(\varinjlim \mathcal{P}_i)(\operatorname{Spec} A) = \varinjlim \mathcal{P}_i(\operatorname{Spec} A).$$

What about in $Sh(\mathbf{Aff}_R)$? Projective limits and directed (= filtered) inductive limits are calculated term by terms. General inductive limits: calculate limit in $PSh(\mathbf{Aff}_R)$ and sheafify. (Example of where we need to do this: quotients).

Digression: How do you define "geometric" properties of R-spaces and morphisms of them? A few ways: (1) Schematic morphisms. If $f : X \to Y$ is a morphism of R-spaces. Say f is *schematic* (or representable, but some people define that using algebraic spaces) if, for all morphisms $S \to Y$ with S a R-scheme, $S \times_Y X$ is also a scheme. Then, if $f : X \to Y$ is schematic, say it satisfies a property (P) of morphisms of schemes (that's stable under base change and fpqc local on the target), we say f has property (P) if for every $S \to Y$ with S a scheme, the base change $S \times_X Y \to S$ has property (P). Examples of these: closed/open/locally closed immersions, quasicompact, universally closed, affine, proper, quasi-affine, (locally) quasi-finite / finite type / finite presentation (fibers of dimension d), flat, smooth, unramified, étale, ...

(2) Properties defined directly on functors of points. Most important are formally smooth/unramified/étale: Let $f : X \to Y$ be a map of R-spaces. Then f is formally smooth/unramified/étale if for every surjective R-algebra $B \to A$ with nilpotent kernel, and for every commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} B \arrow[r] \arrow[ur, dotted, "g"] & Y \end{tikzcd}$$

there exists (at least one / at most one / exactly one) g that fits in the diagram.

(3) Some properties are originally defined for schemes but can be checked on functors of points. Definition: An R-space X is *locally of finite presentation (type)* if for every directed inductive system (A_i) of R-algebras of finite presentation/type, if $A = \varinjlim A_i$ then the map $\varinjlim X(\operatorname{Spec} A_i) \to X(\operatorname{Spec} A)$ is an isomorphism. A map $f : X \to Y$ is locally of finite presentation/type if for every $\operatorname{Spec} A$ in $\mathbf{Aff}_R$ and for every point $y : S = \operatorname{Spec} A$ (i.e. $y \in Y(S)$) then the A-space $f^{-1}[y]$ given by

$$\operatorname{Spec} B \mapsto \{x \in X(\operatorname{Spec} B) : f(x) = y \text{ on } \mathbf{SpecB}\}.$$

Theorem: This is compatible with the usual definitions for schemes (and compatible with the previous definition for schematic morphisms, but more general since works for non-schematic ones!!!).

Other examples: quasi-compact, separated, proper, smooth/unramified/étale, open immersions (= étale + universally injective), ...

Ind-schemes: A (strict $\mathbb{N}$-)*ind-scheme* is an R-space X that can be written as

$$X = \varinjlim_{n \in \mathbb{N}} X_n$$

with the X_n's R-schemes and the transition maps $X_n \to X_{n+1}$ closed immersions. Will just say "ind-scheme" to mean this in our context (in general could define ind-schemes over other index sets, and drop the "strict" = closed immersion). If X is an ind-scheme, an equality (isomorphism?) $X = \varinjlim X_n$ with the X_n's schemes is called an *ind-presentation.*

Examples: $\coprod_{n \in \mathbb{N}} \mathbb{P}^1$ (is this the same as the disjoint union of schemes???), $\varinjlim \mathbb{A}^n$ (this is definitely not a scheme!) By following stuff: this is ind-affine and ind-finite type.

Definition: We say that an ind-scheme X is of ind-finite type (ind-affine, ind-proper, ind-projective, ...) if for every ind-presentation $X = \varinjlim X_n$, the schemes X_n are of finite type (affine, proper, projective)... We say X is reduced (integral) if there exists an ind-presentation with X_n reduced (integral) - not necessarily true for all!

Lemma: Let X be an ind-scheme, $Y \to X$ a schematic closed immersion. Then:
(i) Y is an ind-scheme.
(ii) If Y is a quasicompact scheme and $X = \lim_n X_n$ is an ind-presentation, then $Y \to X$ factors through some X_n.
(iii) If $X = \lim X_n = \lim X'_n$ are two ind-presentations with X_n, X'_n quasicompact, then for all n, there exists m such that $X_n \to X$ factors through $X'_m \to X$.

Examples: Define:

- $\mathbb{G}_a[[t]]$ as the R-space $\operatorname{Spec} A \mapsto A[[t]]$.
- $\mathbb{G}_a((t))$ as the R-space $\operatorname{Spec} A \mapsto A((t)) = A[[t]](1/t)$.
- $\mathbb{G}_a[t]/(t^n)$ as the R-space $\operatorname{Spec} A \mapsto A[t]/(t^n)$.

Then $\mathbb{G}_a[t]/(t^n)$ is an affine group that's a scheme of finite type (isomorphic to $\mathbb{A}^n_R$). Also, $\mathbb{G}_a[[t]] = \varprojlim \mathbb{G}_a[t]/(t^n)$ is an affine group scheme (isomorphic to $\prod_{\mathbb{N}} \mathbb{A}^1_R \cong \operatorname{Spec} A[x_n : n \in \mathbb{N}]$). Finally, $\mathbb{G}_a((t))$ is a group ind-scheme but not a scheme: for all n, let $t^{-n}\mathbb{G}_a[[t]]$ be $\mathbb{G}_a[[t]]$ and let $t^{-n}\mathbb{G}_a[[t]] \hookrightarrow t^{-(n+1)}\mathbb{G}_a[[t]]$ be the embedding corresponding to multiplication by t on $\mathbb{G}_a[[t]]$. Then $\mathbb{G}_a((t)) = \varinjlim t^{-n}\mathbb{G}_a[[t]]$. This is not of ind-finite type. But, what we can do is take a quotient of R-spaces (i.e. fpqc quotient, quotient as sheaves so computed as quotient as presheaves then sheafifying)

$$Gr_{\mathbb{G}_a} = \mathbb{G}_a((t))/\mathbb{G}_a[[t]] = \varinjlim t^{-n}\mathbb{G}_a[[t]]/\mathbb{G}_a[[t]] \cong \varinjlim \mathbb{A}^n$$

which is a group ind-scheme of ind-finite type that's ind-affine.

Remark: If R is Noetherian, then ind-finite type implies locally of finite presentation.

Loop and arc spaces, the affine Grassmannian. Let X be an R-scheme. Its *arc space* is $X[[t]]$ given by $\operatorname{Spec} A \mapsto X(A[[t]])$, its *loop space* $X((T)$ is $\operatorname{Spec} A \mapsto X(A((t)))$, and its space of *$n$-th order jets* is $X[t]/(t^{n+1})$ taking $\operatorname{Spec} A \mapsto X(A[t]/(t^{n+1}))$.

Why the names? The infinitesimal (pointed) disc over R is $D_R = \operatorname{Spec} R[[t]]$ (and $D_R^\circ = \operatorname{Spec} R((t))$). Then $X[[t]](\operatorname{Spec} A) = \operatorname{Hom}(D_A, X_A)$ and $X((t))(\operatorname{Spec} A) = \operatorname{Hom}(D_A^\circ, X_A)$.

Facts: The functors $X \mapsto$ arc/loop/jet space commute with projective limits. Also, $X[[t]] = \varprojlim X[t]/(t^n)$. If $X \to Y$ is an étale morphism of schemes, then the following squares are Cartesian:

$$\begin{tikzcd} X[[t]] \arrow[r] \arrow[d] & Y[[t]] \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

$$\begin{tikzcd} X[t]/(t^n) \arrow[r] \arrow[d] & Y[t]/(t^n) \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

(this follows from infinitesimal lifting property of étale maps). So $X[[t]] \to Y[[t]]$ and $X[t]/(t^n) \to Y[t]/(t^n)$ are schematic and étale, and they are open immersions if the original map $X \to Y$ is.

Proposition (1) $X[t]/(t^n)$ and $X[[t]]$ are schemes, affine if X is affine, $X[t]/(t^n)$ is of finite type if X is. The maps $X[[t]] \to X[t]/(t^{n+1}) \to X[t]/(t^n)$ are affine.
(2) If X is affine of finite presentation (over R) then $X((t))$ is an ind-scheme. Moreover the map $X[[t]] \to X((t))$ is a schematic closed immersion.

Remarks: (1) $X((t))$ is almost never ind-finite type. (2) It's not an ind-scheme in general if X is not affine (even if $X = \mathbb{P}^1$ for instance). Maybe even not true if X is affine but not finite type/presentation?

Proof: (1) Did example of $X = \mathbb{A}^1$ above; idea now is to reduce proof to that example. Case where X affine: $X = \operatorname{Spec} R[t_i]/(f_j)$ for some set of variables $(t_i)_{i\in I}$ and relations $(f_j)^{j\in J}$. Then $X = \operatorname{Spec} R \otimes_{\mathbb{A}^J} \mathbb{A}^I$; since forming arc spaces commutes with projective limits, get

$$X[[t]] = \operatorname{Spec} R \times_{\mathbb{A}^1[[t]]^J} \mathbb{A}^1[[t]]^I.$$

So $X[[t]]$ is an affine scheme (same for $X[t]/(t^n)$). If X is finite type then we can take I finite so this gives $X[t]/(t^n)$ of finite type. General case: Choose affine cover $X = \coprod V_i$. Then the $V_i[[t]]$ are an affine cover of $X[[t]]$, and same for $X[t]/(t^n)$.

(2) Again, done the $X = \mathbb{A}^1$ case. Generalize to the X affine of finite presentation again; get $X = \operatorname{Spec} R \otimes_{\mathbb{A}^J} \mathbb{A}^I$ for I, J finite. But finite projective limits commute with direct inductive limits so $X((t))$ is an ind-scheme.

Remark: If X is is smooth over R, then $X[t]/(t^2)$ is the tangent bundle TX of X. Also, $X[t]/(t^{n+1}) \to X[t]/(t^n)$ is a torsor under $TX \times_x X[t]/(t^n)$.

Example we care most about: $R = k$ a field, G is linear algebraic group over k. Then $G[t]/(t^n)$ is a linear algebraic group that projects to G; the kernel of this projection is a unipotent group (because they're successive extensions of $\operatorname{Lie}(G)$). The arc space $G[[t]]$ is a pro-algebraic group, and the kernel of $G[[t]] \to G$ is pro-unipotent (think of $G[[t]]$ as "$G(\mathcal{O})$"). Then $G((t))$ is a group ind-scheme that's ind-affine with $G[[t]]$ a closed subgroup (think of as "$G(K)$").

Definition: The affine Grassmannian of G is $Gr_G = G((t))/G[[t]]$ (think of as "$G(K)/G(\mathcal{O})$").

4 Lecture - 02/26/2014

First, some comments from last time: disjoint unions are schemes and are the same as schemes and R-schemes. Last time tried to show that $\mathbb{A}^\infty((t))$ was ind-affine, but actually the argument wouldn't work and it's not $\mathbb{N}$-ind-affine.

The affine Grassmannian of GL_n. Let k be a field, $\mathbf{Aff}_k$ the affine schemes over k with the fpqc topology. Recall a k-space is a fpqc sheaf on $\mathbf{Aff}_k$. The affine Grassmannian for GL_n, Gr_{GL_n}, is the fpqc quotient $\mathrm{GL}_n((t))/\mathrm{GL}_n[[t]]$ of group-ind-schemes in the category of k-spaces. In other words, it's the fpqc sheafification of the presheaf

$$\operatorname{Spec} A \mapsto \frac{\mathrm{GL}_n(A((t)))}{\mathrm{GL}_n(A[[t]])}.$$

Goal: Show this is an ind-projective ind-scheme (i.e. an inductive limit over $\mathbb{N}$ of projective schemes, with the transition maps immersions).

Definition: Let X be a scheme. A vector bundle on X is a locally free $\mathcal{O}_X$-module of finite rank (i.e. a finite locally free $\mathcal{O}_X$-module). Remark: Usually we should be careful about what topology we mean for "locally" but here it's okay because finite locally rank is the same as flat of finite presentation and as projective of finite presentation, so we get the same for the Zariski and fpqc topologies (and the fppf and étale topologies too). Let $\mathbf{Vect}(X)$ be the category of vector bundles on X. It's an exact category (i.e. can talk about exact functors) but not an abelian category.

Lattices: If A is a commutative ring, a *lattice* in $A((t))^n$ is a sub-$A[[t]]$-module $\mathcal{L}$ such that there exists N with $t^N A[[t]]^n \subseteq \mathcal{L} \subseteq t^{-N} A[[t]]^n$, and $t^{-N}A[[t]]^n/\mathcal{L}$ is a locally free A-module. Define $Latt_n$ to be the functor taking $\operatorname{Spec} A$ to the set of lattices in $A((t))^n$; this is actually a k-space (exercise). If we fix N in the definition, we have subfunctor $Latt_n^N$; note

$$Latt_n = \varinjlim Latt_n^N.$$

Fix N, set $V_N = t^{-N}k[[t]]^n/t^N k[[t]]^n$, a k-vector space of dimension $2Nn$ together with a nilpotent endomorphism t. Remember that the Grassmannian $\mathrm{Gr}(r, V_N)$ is the k-space taking $\operatorname{Spec} A$ to the set of quotients of $V_N \otimes_k A$ that are finite locally free of rank r (following Grothendieck, use quotients rather than subspaces). Grothendieck: $\mathrm{Gr}(r, V_N)$ is representable by a projective scheme over k.

Fact:

$$Latt_n^N \hookrightarrow \coprod_{0 \le r \le 2N} \mathrm{Gr}(r, V_N)$$

by $\mathcal{L} \mapsto t^{-N}k[[t]]^n/\mathcal{L}$, and claim this is a representable closed immersion. A corollary of this will be that $Latt_n^N$ is a projective scheme and that the transition maps $Latt_n^N \to Latt_n^{N+1}$ are closed immersions, so $Latt_n$ is an ind-projective ind-scheme.

Proof of Fact: Fix r and try to describe the subfunctor $Latt_n^N \cap Gr(r, V_N)$. This is the subfunctor of M such that $\ker V_N \twoheadrightarrow M$ is stable by t. How do you prove that this is a representable closed immersion? Take a scheme $\operatorname{Spec} A$, take a map $f : \operatorname{Spec} A \to Gr(r, V)$, look at the fiber product, P and check that P is a closed subscheme of $\operatorname{Spec} A$. So suppose that f corresponds to $M \in Gr(r, V_N)(\operatorname{Spec} A)$ and let $K = \ker(V_N \otimes A \to M)$. Then the fiber product P is the A-space given by mapping $\operatorname{Spec} B$ to a point if $K \otimes B$ is stable by t, and $\emptyset$ otherwise. We may assume that A is Noetherian (all spaces are locally of finite type). Then K is of finite presentation and M is a vector bundle, so $\mathcal{H}om(K, M)$ is represented by a vector bundle $\mathcal{V}$. Then the composition $K \otimes B \to V_N \otimes B \to M \otimes B$ is a global section of the vector bundle. Since $\mathcal{V}$ is a scheme, has the zero section $\mathcal{V}_0$ as a closed subscheme, and $P = \mathcal{V}_0 \times_{\mathcal{V}} \operatorname{Spec} A$ is a closed subscheme of $\operatorname{Spec} A$.

If we just cared about GL_n we'd be basically done; show that $Latt_n$ is isomorphic to the affine Grassmannian for GL_n directly. But for other groups lattices don't work so well, so we want another perspective that generalizes. Consider the perspective of *local bundles*. Recall that the "infinitesimal disk over k" is

$D = \operatorname{Spec} k[[t]]$, which contains $D^\circ = \operatorname{Spec} k((t))$. Definition: If A is a k-algebra, an A-family of vector bundles over D (or D°) is a module over $A[[t]]$ (or $A((t))$, respectively) that is finite locally free, where the "locally" is fpqc locally on $\operatorname{Spec} A$ (i.e. there exists $A \to B$ faithfully flat such that $M \otimes_{A[[t]]} B[[t]]$ is free over $B[[t]]$, and similarly for $B((t))$).

Define categories $\mathbf{Vect}_A(D)$ and $\mathbf{Vect}_A(D^\circ)$ as the A-families of vector bundles over the appropriate space (both are exact categories, and there's a restriction from one to the other). Lemma: $\mathbf{Vect}_A(D)$ is equivalent to the category of projective systems $(M_n, \alpha_n)_{n \in \mathbb{N}}$ with M_n finite locally free over $A[t]/t^n$ and $\alpha_n M_{n+1}/t^n \cong M_n$. (Proof: Nakayama's lemma).

Then, define $Gr^{loc}_{\mathrm{GL}_n}$ as the functor taking $\operatorname{Spec} A$ to the set of pairs (M, γ) moduli equivalence, with $M \in \mathbf{Vect}_A(D)$ and γ an isomorphism $M|_{D^\circ} \cong M^\circ_A|_{D^\circ}$. (Here $M^\circ = k[[t]]^n$). Note there are no nontrivial automorphism, and remark that this is a sheaf by faithfully flat descent.

Global bundles: Fix a smooth curve X, and $x \in |X|$ with $k(x) = k$. Fix $D \cong \operatorname{Spec} \widehat{\mathcal{O}}_{X,x}$ and $X^\circ = X \setminus x$. Then have diagram

$$\begin{array}{ccc} \mathbf{Vect}(X_A) & \longrightarrow & \mathbf{Vect}_A(D) \\ \downarrow & & \\ \mathbf{Vect}(X_A^\circ) & \longrightarrow & \mathbf{Vect}_A(D^\circ). \end{array}$$

Define $Gr^{glob}_{\mathrm{GL}_n}$ by taking $\operatorname{Spec} A$ to pairs (M_X, γ) with $M_X \in \mathbf{Vect}(X_A)$ and $\gamma : M_X|_{X_A^\circ} \cong M_X^\circ|_{X_A^\circ}$ where $M_X^\circ = \mathcal{O}_X^n$. Remark: This is a sheaf, and we have an obvious restriction map $Gr^{glob}_{\mathrm{GL}_n} \to Gr^{loc}_{\mathrm{GL}_n}$

The Beauville-Laszlo Theorem: Define the category of gluing data (over a fixed A) as the category of triple $(M_{X^\circ}, M_D, \beta)$ where $M_{X^\circ} \in \mathbf{Vect}(X_A^\circ)$, $M_D \in \mathbf{Vect}_A(D)$, and β an isomorphism $M_X^\circ|_{D_A^\circ} \cong M_D|_{D_A^\circ}$. Then the theorem (due to B-L over $\mathbb{C}$ at least) is that the obvious functor from $\mathbf{Vect}(X_A)$ to this category of gluing data is an equivalence. Remark: if A is Noetherian this is faithfully flat descent (A Noetherian implies $D_A \to X_A$ is flat, but not true in general). Will omit the proof of this because, if we're careful, we can just reduce to the Noetherian case everywhere.

Corollary: $Gr^{glob}_{\mathrm{GL}_n} \cong Gr^{loc}_{\mathrm{GL}_n}$.

Vector bundles vs. Lattices: Remark: $Gr^{loc}_{\mathrm{GL}_n}(\operatorname{Spec} A)$ is the set of sub-$A[[t]]$-modules $M_D \subseteq M_D^\circ \otimes A((t))$ such that M_D is locally on A free of rank n and such that there exists N with $t^N M^\circ_{D_A^\circ} \subseteq M_D \subseteq t^{-N} M^\circ_{D_A^\circ}$. Can define $Gr^{loc,N}_{\mathrm{GL}_n}$ to be the subfunctor for fixed N. Similarly for global ones: if $j : X^\circ \hookrightarrow X$ is the inclusion, then $Gr^{loc}_{\mathrm{GL}_n}(\operatorname{Spec} A)$ is the same as the set of sub-$\mathcal{O}_X$-modules $M_X \hookrightarrow j_* j^* \mathcal{O}_X^n$ that is a vector bundle such (automatic that there exists N with $\mathcal{O}_X^n(-Nx) \subseteq M_X \subseteq \mathcal{O}_X^n(Nx)$).

We have four maps:
(1) $Gr^{loc,N} \to Latt_n^N$ given by $M_D \to M_D$.
(2) $Gr^{glob,N} \to Latt_n^N$ given by $M_X \to M_X|_D$.
(3) $Latt_n^N \to Gr^{loc,N}$ given by $\mathcal{L} \mapsto \mathcal{L}$.
(4) $Latt_n^N \to Gr^{glob,N}$ given by taking $\mathcal{L}$ to the preimage M_X of $\mathcal{L}/t^N A[[t]]^n$ by the map $M_X^\circ(-N_X) \to M_X^\circ(-Nx)/M_X^\circ(Nx) \cong t^{-N} A[[t]]^n / t^N A[[t]]^n$.

Proposition: These are all well-defined bijections.
Proof: We just need to prove they're well-defined since they're evidently mutual inverses. Actually sufficient to prove (1) and (4) are well-defined since (2) and (3) come from composing those with restriction maps from global Grassmannian to local.

(1): Let $M \in Gr^{loc}(\operatorname{Spec} A)$ with $t^N A[[t]]^n \subseteq M \subseteq t^{-N} A[[t]]^n$. We want to show $N = t^{-N} A[[t]]/M$ is a finite locally free A-module. But $t^{-2N} M/M$ is a locally free A-module, so enough to show that $N \hookrightarrow t^{-2N} M/M$ splits. Enough to show that $t^{-N} A[[t]] \hookrightarrow t^{-2N} M$ splits as a map of A-modules. But

$$t^{-N} A[[t]] \hookrightarrow t^{-2N} M \hookrightarrow t^{-3N} A[[t]]^n$$

obviously splits and restrict this.

(4): Take a lattice $\mathcal{L} \in Latt_n^N(\operatorname{Spec} A)$ and map it to $M_X \subseteq \mathcal{O}_X^n(-Nx)$; need to check this is a vector bundle. Since $Latt_n^N$ is of finite type, there exists $A' \subseteq A$ Noetherian and $\mathcal{L}' \in Latt_n^N(\operatorname{Spec} A')$ such that $\mathcal{L} = \mathcal{L}' \otimes_{A'[[t]]} \otimes A[[t]]$. Question: Is $M_X = f^* M'_X$ for $f : X_A \to X_{A'}$? Answer: Yes, use that $t^{-N} A'[[t]]^n/\mathcal{L}$ is A'-flat and exact sequence

$$0 \to M_X \to \mathcal{O}_{X_A}^n(-Nx) \to t^{-N} A[[t]]^n/\mathcal{L} \to 0.$$

So we may assume that A is Noetherian. Then M_X is coherent and we just need to show it's flat. Enough to show that for all maximal ideals $\mathfrak{m} \otimes k(\mathfrak{m})$ is flat over $\mathcal{O}_{X_{k(\mathfrak{m})}}$; using flatness of $t^{-N} A[[t]]^n/\mathcal{L}$ we may assume that A is a field. If A is a field, then M_x is flat iff it's torsion-free, bu $M_x \subseteq \mathcal{O}_{X_A}^n(-Nx)$.

Next time: $Gr_{\mathrm{GL}_n} \cong Gr_{\mathrm{GL}_n}^{loc} \cong Gr_{\mathrm{GL}_n}^{glob} \cong Latt_n$.

5 Lecture - 02/28/2014

Last time: Showed $Gr^{loc}_{\mathrm{GL}_n}, Gr^{glob}_{\mathrm{GL}_n}, Latt_n$ were ind-projective ind-schemes that were all isomorphic to each other. What about the affine Grassmannian Gr_{GL_n}? Recall that $Gr^{loc}_{\mathrm{GL}_n}(\operatorname{Spec} A)$ is the set of pairs (M_D, γ) with $M_D \in \mathbf{Vect}_A(D)$ and $\gamma : M_D|_{D^\circ} \cong M^\circ_{D^\circ}$. (Define $M^\circ_D = A[[t]]^n$ and $M^\circ_{D^\circ} = A((t))^\circ$). On the other hand

$$\mathrm{GL}_n((t))(\operatorname{Spec} A) = \operatorname{Aut}_{\mathbf{Vect}_A(D^\circ)}(M^\circ_{D^\circ})$$

and

$$\mathrm{GL}_n[[t]](\operatorname{Spec} A) = \operatorname{Aut}_{\mathbf{Vect}_A(D)}(M^\circ_D).$$

So can define a map

$$\pi : \mathrm{GL}_n((t)) \to Gr^{loc}_{\mathrm{GL}_n}$$

by $g \mapsto (M^\circ_D, g)$, since we can interpret g as an automorphism of $M^\circ_{D^\circ}$. Further note that π is $\mathrm{GL}_n[[t]]$-equivariant for the right action on $\mathrm{GL}_n((t))$ and the trivial action on $Gr^{loc}_{\mathrm{GL}_n}$, so π passes to $\overline{\pi} : Gr_{\mathrm{GL}_n} \to Gr^{loc}_{\mathrm{GL}_n}$.

Proposition: $\overline{\pi}$ is an isomorphism.

Proof: Let $\mathcal{P}$ be the presheaf quotient $\mathrm{GL}_n((t))/\mathrm{GL}_n[[t]]$. Then π also gives $\overline{\pi}^{psh} : \mathcal{P} \to Gr^{loc}_{\mathrm{GL}_n}$. Have that for all A, $\overline{\pi}^{psh} : \mathcal{P}(\operatorname{Spec} A) \to Gr^{loc}_{\mathrm{GL}_n}(\operatorname{Spec} A)$ is injective, so $\overline{\pi}$ is injective on points (as sheafification is exact). Note that the image of $\overline{\pi}^{psh}(\operatorname{Spec} A)$ is the set of $(M_D, \gamma) \in Gr^{loc}_{\mathrm{GL}_n}$ such that M_D is trivial. But each M_D is locally trivial on A, so $\overline{\pi}$ is surjective.

Corollary: Gr_{GL_n} is an ind-projective ind-scheme.

Example: $Gr_{\mathbb{G}_m}$ is a commutative group ind-scheme. If K/k is a field extension, $Gr_{\mathbb{G}_m}(\operatorname{Spec} K)$ is the lattices in $K((t))$, so is isomorphic to $\mathbb{Z}$ (sending a lattice to the valuation of a generator). Thus $(Gr_{\mathbb{G}_m})^{red} \cong \coprod_{\mathbb{Z}} \operatorname{Spec} k$.

Fact (proof is an exercise): $Gr^\circ_{\mathbb{G}_m}$, the connected component of the identity, is the (infinite-dimensional) formal group with Lie algebra $k((t))/k[[t]]$. (This formal group is the functor sending $\operatorname{Spec} A$ to the set of sequences $(a_n)_{n\in\mathbb{Z}}$ with $a_n \in A$ nilpotent and $a_n = 0$ for $n \ll \infty$).

G-bundles. Let G be an affine group scheme over k, and X/k a scheme which for convenience we'll assume to be quasicompact. A (principal) G-bundle over X is determined by any of the following three definitions:
(1) A sheaf $\mathcal{P}$ on $(\mathbf{Sch}/X)_{fpqc}$ (or $(\mathbf{Aff}/X)_{fpqc}$) which is a torsor under G (i.e. G acts on $\mathcal{P}$ on the left, such that $G \times \mathcal{P} \cong \mathcal{P} \times \mathcal{P}$ via the map $(g, s) \mapsto (gs, s)$, and there exists an fpqc cover $Y \to X$ with $\mathcal{P}(Y) \neq \emptyset$).
(2) A scheme $\widetilde{X} \to X$ with a left action of G (in $\mathbf{Sch}_X$) such that there exists a faithfully flat map $Y \to X$ such that $Y \times_X \widetilde{X} \cong Y \times G$ in a G-equivariant way.
(3) A faithfully flat $\widetilde{X} \to X$ with a left action of G such that $G \times \widetilde{X} \cong \widetilde{X} \times_X \widetilde{X}$ via $(g, x) \mapsto (gx, x)$.

Notation: let $\mathbf{BG}(X)$ be the groupoid of G-bundles over X (objects are G-bundles and morphisms are isomorphisms of G-bundles).

Proof that the definitions are equivalent (i.e. they give equivalent categories):
(1) $\Longrightarrow$ (2): If $\mathcal{P}$ is a G-bundle as in (1), take $Y \to X$ faithfully flat such that $\mathcal{P}(Y) \neq \emptyset$. Let $s \in \mathcal{P}(Y) = \operatorname{Hom}(Y, \mathcal{P})$. Define $c : Y \times_X Y \to G$ by letting $c(y_1, y_2)$ be the unique $g \in G$ such that $s(y_2) = gs(y_1)$. Let $\widetilde{Y} = Y \times G$, let $\varphi : \widetilde{Y} \times_X Y \to Y \times_X \widetilde{Y}$ be the map $((y_1, g), y_2) \mapsto (y_1, (y_2, c(y_1, y_2)g))$. This is a descent datum for $\widetilde{Y}$ with respect to $Y \to X$. As $\widetilde{Y}/Y$ is affine, this is effective, i.e. there's $\widetilde{X} \to X$ such that $\widetilde{Y} = \widetilde{X} \times_X \widetilde{Y}$. The left G-action on $\widetilde{Y}$ (by $h \cdot (y, g) = (y, gh^{-1})$) is compatible with φ hence also descends to $\widetilde{X}$. Then this $\widetilde{X}$ satisfies the conditions of (2).

(2) $\Longrightarrow$ (3): Let $\widetilde{X} \to X$ be as in (2). We want to check that $\widetilde{X} \to X$ is faithfully flat. But there exists $Y \to X$ faithfully flat such that $Y \times_X \widetilde{X} \to Y$ is faithfully flat. So $\widetilde{X} \to X$ is faithfully flat (since that's an fpqc-local condition). Also, $G \times_X \widetilde{X} \to \widetilde{X} \times_X \widetilde{X}$ becomes an isomorphism because it's an isomorphism after a faithfully flat base change.

(3) $\implies$ (2): Take $Y = \widetilde{X}$

(2) $\implies$ (1): Take $\mathcal{P}$ defined by $Y \mapsto \mathrm{Hom}_X(Y, \widetilde{X})$.

Fiber bundles associated to a G-scheme. Definition: Let $\mathcal{P}$ or $\widetilde{X}$ be a G-bundle over X. Let Z be a scheme with a left G-action. If the quotient

$$\widetilde{X} \times^G Z = G\backslash(\widetilde{X} \times Z)$$

exists (i.e. if the fpqc quotient is representable by a scheme), we call it $Z_{\mathcal{P}}$. For example, this is okay if Z is affine.

Proposition: If G is smooth (e.g. if k has characteristic zero), then every G-bundle is locally trivial in the étale topology.

Proof: Let $\widetilde{X} \to X$ be a G-bundle; want to trivialize it over an étale cover of X. First, it becomes smooth after a faithfully flat base change (as it becomes isomorphic to $G \times_X X$, and G is smooth). So WLOG $\widetilde{X} \to X$ is smooth, and étale (Zariski) locally on X it becomes $\widetilde{X} \to \mathbb{A}^n \times X \to X$ where the first map is étale and the second is the projection. Choose $a \in \mathbb{A}^n(k)$; then $Y = X \times_{\mathbb{A}^n \times X} \widetilde{X} \to X$ is étale and trivializes $\widetilde{X} \to X$ (where the map $X \to \mathbb{A}^n \times X$ is the section for our a).

Let $\mathcal{P}$ be a GL_n-bundle over X, let E° be the standard representation of GL_n (i.e. $E^\circ = \mathbb{A}^n_k$). Then $E^\circ_{\mathcal{P}}$ is a rank-n vector bundle.

Proposition: GL_n-bundles over X correspond to rank-n vector bundles over X via $\mathcal{P} \mapsto E^\circ_{\mathcal{P}}$ (i.e. this defines an equivalence of categories).

Proof: The inverse functor sends a rank-n vector bundle E to the GL_n-bundle $\mathcal{P}$ given by $Y \mapsto Isom_Y(E^\circ_Y, E_Y)$.

Tannakian point of view: Let $\mathbf{Rep}_G$ be the category of algebraic representations of G. Let $\mathbf{Vect}(X)$ be the vector bundles on X. If $\mathcal{P}$ is a G-bundle on X, we get a functor $F_{\mathcal{P}} : \mathbf{Rep}_G \to \mathbf{Vect}(X)$ given by $V \mapsto V_{\mathcal{P}}$.

Proposition: The category of G-bundles over X is equivalent to the category of exact tensor functors $F_{\mathcal{P}} : \mathbf{Rep}_G \to \mathbf{Vect}(X)$, via the above equivalence. (Recall $\mathbf{Vect}(X)$ isn't an abelian category, but it is an exact category).

Note: The trivial G-bundle goes to the functor $V \mapsto V \otimes_k \mathcal{O}_X$.

Proof: Construct the inverse functor. Let $F : \mathbf{Rep}_G \to \mathbf{Vect}(X)$ be an exact tensor functor. If V is a locally finite representation (i.e. $V = \varinjlim_{i\in I}$), define $F(V) = \varprojlim_{i \in I} F(V_i)$. Note $F(V)$ is a flat $\mathcal{O}_X$-module (as a limit of vector bundles). Apply this to the ring of regular functions on G, $k[G]$, with the left regular action. Then $\mathcal{A} = F(k[G])$ is a commutative $\mathcal{O}_X$-algebra because F is a tensor functor. Take $\widetilde{X} = \mathrm{Spec}_X(\mathcal{A})$. Then $\widetilde{X} \to X$ is flat.

Now, if we let 1 be the trivial representation on G, have

$$0 \to 1 \to k[G] \to k[G]/1 \to 0.$$

Since F is exact get exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{A} \to F(k[G]/1) \to 0$$

with $F(k[G]/1)$ flat over $\mathcal{O}_X$. So for all $x \in X$, $k(x) \to \mathcal{A} \otimes k(x)$ is injective, and $\mathcal{A}$ is faithfully flat over $\mathcal{O}_X$. Also, the second G-action on $k[G]$ gives a G-action on $\widetilde{X}$. Next,

$$\widetilde{X} \times_X \widetilde{X} = \mathrm{Spec}_X(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}) = \mathrm{Spec}_X(F(k[G] \otimes_k k[G])).$$

But by properties of the regular representation, $k[G] \otimes_k k[G] \cong k[G] \otimes_k \underline{k[G]}$ where $\underline{k[G]}$ has $k[G]$ as the underlying vector space but the trivial action. So $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A} \cong \mathcal{A} \otimes_{\mathcal{O}_X} \underline{k[G]}$. So

$$\widetilde{X} \times_X \widetilde{X} \cong \widetilde{X} \times \mathrm{Spec}\, \underline{k[G]} \cong \widetilde{X} \times G,$$

and can find this is G-equivariant.

Now, talk about changing the group of a G-bundle. Let $\rho : G \to H$ be a morphism of groups.

Extension of structure group: Have $BG(X) \to BH(X)$ given by $\mathcal{P} \mapsto H_{\mathcal{P}}$ (with the action of G via ρ); in Tannakian point of view this is $F \mapsto F \circ \rho^*$.

Reduction of structure group: Let $\mathcal{P}_H$ be a H-bundle on X. A reduction of the structure group of $\mathcal{P}_H$ to G is a G-bundle $\mathcal{P}$ with an isomorphism $H_{\mathcal{P}} \cong \mathcal{P}_H$. Now, $\mathcal{P}_H$ corresponds to a functor $F_{\mathcal{P}_H} : \mathbf{Rep}_H \to \mathbf{Vect}(X)$, and have $\rho^* : \mathbf{Rep}_H \to \mathbf{Rep}_G$.

Exercise: If ρ is a closed immersion, there's a natural bijection between reductions of $\mathcal{P}_H$ to G (modulo isomorphism) and sections $(H/G)_{\mathcal{P}_H} \to X$.

Remark: G-bundles over X (up to isomorphism) are classified by Čech cohomology $H^1_{fpqc}(X, G)$. If $G \hookrightarrow H$ is normal, we have an exact sequence

$$H^1_{fpqc}(X, G) \to H^1_{fpqc}(X, H) \to H^1_{fpqc}(X, H/G).$$

This is actually true even if G is not normal in H. This is how you do the exercise...

Application: SL_n-bundles over X are isomorphic to rank-n vector bundles E/X together with an isomorphism $\det(E) \cong G_X$ (where $\det(E) = \bigwedge^n E = (\mathrm{GL}_n/\mathrm{SL}_n)_E$). Along the same lines, $\mathrm{O}(n)$-bundles on X are equivalent to rank-n vector bundles E together with $\sigma : E \cong E^*$ with $\sigma = \sigma^\top$.

G-bundles on the formal disk. Recall $D = \operatorname{Spec} k[[t]]$ and $D^\circ = \operatorname{Spec} k((t))$. An A-family of G-bundles on D (or D°) is an exact tensor functor $\mathbf{Rep}_G \to \mathbf{Vect}_A(D)$ (or $\mathbf{Vect}_A(D^\circ)$). These give categories $BG_A(D)$ and $BG_A(D^\circ)$. Remark: $BG_A(D)$ is equivalent to compatible systems of G-bundles on $\operatorname{Spec} A[t]/t^n$.

Corollary (of BL theorem): Let X be a smooth curve, $x \in |X|$ such that $k(x) = k$, pick $\operatorname{Spec} \widehat{\mathcal{O}}_{X,x} \cong D$. Let $X^\circ = X \setminus \{x\}$. Then $BG(X_A)$ is isomorphic to triple (P_{X°, P_D, B) with $P_{X^\circ} \in BG(X^\circ_A)$, $P_D \in BG_A(D)$, and $\beta : P_{X^\circ}|_{D^\circ} \cong P_D|_{D^\circ}$.

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Remark: Formula for $(\mathrm{Gr}^{\circ}_{\mathbb{G}_m})(\operatorname{Spec} A)$ last time was wrong; should be the set of tuples (a_n) over $n \leq 0$ with $a_n \in A$ nilpotent and $a_n = 0$ fir $n \ll 0$ (at least for $\operatorname{Spec} A$ connected...)

Today: The affine Grassmannian for an arbitrary linear algebraic group G/k (recall this means smooth of finite type). Remember $\mathrm{Gr}_G = G(\!(t)\!)/G[\![t]\!]$. Want to prove that Gr_G is an ind-scheme of int-finite type, which is ind-projective iff G is reductive (i.e. that $R_u(G_{\overline{k}}) = 1$ in our context). Also, want to calculate $\pi_0(\mathrm{Gr}_G)$. (Remark: Gr_G is reduced iff $\operatorname{Hom}(G, \mathbb{G}_m) = 1$; lines up with us seeing that $\mathrm{Gr}_{\mathrm{GL}_n}$ is not reduced!)

Notation: Let P^0_* be the trivial G-bundle (or trivial family of G-bundles, etc.) over whatever the base $*$ is.

As in the GL_n case, we're doing to prove this by way of working with spaces $\mathrm{Gr}^{\mathrm{loc}}_G$ and $\mathrm{Gr}^{\mathrm{glob}}_G$. These are defined by

$$\mathrm{Gr}^{\mathrm{loc}}_G(\operatorname{Spec} A) = \{(P_D, \lambda) : P_D \in BG_A(D), \gamma : P_D|_{D^\circ} \cong P^0_{D^\circ}\},$$

$$\mathrm{Gr}^{\mathrm{glob}}_G(\operatorname{Spec} A) = \{(P_X, \gamma) : P_X \in BG(X_A), \gamma : P_X|_{X^\circ_A} \cong P^0_{X^\circ_A}\}.$$

where again $D^\circ = \operatorname{Spec} k(\!(t)\!) \subseteq \operatorname{Spec} k[\![t]\!] = D$ are the formal disc (with and without the origin removed) and X is a smooth curve with $x \in |X|$ such that $k(x) = k$, $\operatorname{Spec} \widehat{\mathcal{O}}_{X,x} \cong D$, and $X^\circ = X \setminus \{0\}$. As in the GL_n case, the BL theorem tells us that the restriction map $\mathrm{Gr}^{\mathrm{glob}}_G \to \mathrm{Gr}^{\mathrm{loc}}_G$ is an isomorphism.

As in the case of GL_n, we identify $G((t)) = \operatorname{Aut}(P^0_{D^\circ})$ and $G[\![t]\!] = \operatorname{Aut}(P^0_D)$. We then get a map $\pi : G(\!(t)\!) \to \mathrm{Gr}^{\mathrm{loc}}_G$ given by $\gamma \mapsto (P^0_D, \gamma)$, which passes to $\overline{\pi} : \mathrm{Gr}_G \to \mathrm{Gr}^{\mathrm{loc}}_G$.

Proposition: $\overline{\pi}$ is an isomorphism.

Proof: The proof is exactly the same as for GL_n. The nontrivial point is showing that $P_D \in BG_A(D)$ is trivial locally on A. This follows from the fact that G is of finite type (since then $\mathbf{Rep}_G$ has a $\otimes$-generator, and we can trivialize its image under $\mathbf{Rep}_G \to \mathbf{Vect}_A(D)$ and thus trivialize anything). Also need to use smoothness throughout the argument.

So we now have three isomorphic things $\mathrm{Gr}_G \cong \mathrm{Gr}^{\mathrm{loc}}_G \cong \mathrm{Gr}^{\mathrm{glob}}_G$; how do we prove that they are actually ind-schemes? Let $G_1 \to G_2$ be a map of linear algebraic groups. Then we get a map $\mathrm{Gr}_{G_1} \to \mathrm{Gr}_{G_2}$ of affine Grassmannians. Assume $G_1 \to G_2$ map is a closed immersion; then:

Proposition: In this situation, if G_2/G_1 is quasi-affine (affine) then the map $\mathrm{Gr}_{G_1} \to \mathrm{Gr}_{G_2}$ is a schematic locally closed immersion (schematic closed immersion). Thus if we know G_2 is an ind-scheme of ind-finite type, so is G_1.

Remark: In general $\mathrm{Gr}_{G_1} \to \mathrm{Gr}_{G_2}$ is strange (may not be schematic, may not be immersion...). For example, if $B \subseteq G$ is a Borel subgroup, then the induced map $\mathrm{Gr}_B(K) \to \mathrm{Gr}_G(K)$ is an isomorphism for all K/k but $\mathrm{Gr}_B \to \mathrm{Gr}_G$ is not. Note that in this case we have $G(K(\!(t)\!)) = B(K(\!(t)\!))G(K[\![]\!]t])$.

Corollary: For every G, Gr_G is an ind-scheme of ind-finite type, and moreover Gr_G is ind-proper iff it's ind-projective iff G is reductive.

Proof of Corollary: If G is reductive, pick any embedding $G \hookrightarrow \mathrm{GL}_n$; then GL_n/G is affine (proof of this that works in characteristic p in 3rd edition of GIT somewhere), so $\mathrm{Gr}_G \to \mathrm{Gr}_{\mathrm{GL}_n}$ is a schematic closed immersion. So Gr_G is an ind-projective ind-scheme. If G is not reductive then after extending k (which preserves ind-projectivity), have embedding $\mathbb{G}_a \to G$ with $G/\mathbb{G}_a$ is affine. Then $\mathrm{Gr}_{\mathbb{G}_a}$ (which is ind-affine) is closed in Gr_G, which means Gr_G cannot be ind-projective.

So it remains to show that that Gr_G is an ind-scheme of ind-finite type for a general linear algebraic group G. Pick an embedding $G \hookrightarrow \mathrm{GL}_n$. By Chevalley's theorem, there's a finite-dimensional representation V of GL_n and a line $\ell \subseteq V$ such that $G = \mathrm{Stab}_{\mathrm{GL}_n}(\ell)$. Thus G acts on ℓ by some character χ; then can embed $G \hookrightarrow \mathrm{GL}_n \times \mathbb{G}_m = G'$ by $g \mapsto (g, \chi(g)^{-1})$. Then G' acts on v (with $\mathbb{G}_m$ acting by homotheties), and for all $v \in \ell \setminus \{0\}$, $G = \mathrm{Stab}_{G'}(v)$. So $G'/G \cong G' \cdot v \subseteq V$ is quasi-affine. We know that $\mathrm{Gr}_{G'} = \mathrm{Gr}_{\mathrm{GL}_n} \times \mathrm{Gr}_{\mathbb{G}_m}$ is an ind-scheme of ind-finite type, and thus so is Gr_G (via the proposition).

Proof of Proposition: Let $\operatorname{Spec} A \in \mathbf{Aff}_k$, and fix an A-point $f : \operatorname{Spec} A \to \mathrm{Gr}_{G_2} = \mathrm{Gr}^{\mathrm{loc}}_{G_2}$, so f corresponds to a pair (P_D, γ). Let Z be the fiber product of $\mathrm{Gr}_{G_1} \to \mathrm{Gr}_{G_2}$ and f; we want to show that Z is a locally closed subscheme of $\operatorname{Spec} A$. How do you study Z? Calculate its points in any A-algebra.

If $\operatorname{Spec} B \in \mathbf{Aff}_A$, what's $Z(\operatorname{Spec} B)$? It's the set of maps $g : \operatorname{Spec} B \to \mathrm{Gr}_{G_1}$ such that we have a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r, "g"] \arrow[dr] & \mathrm{Gr}_{G_1} \arrow[r] & \mathrm{Gr}_{G_2} \\ & \operatorname{Spec} A \arrow[ur, "f"'] & \end{tikzcd}$$

Via our bundle-theoretic interpretation, this is the same as the set of pairs (Q_D, δ) with $Q_D \in BG_{1,B}(D)$, $\delta : Q_D|_{D^\circ} \cong P^0_{D^\circ}$, such that $(G_2 \times^{G_1} Q_D, G_2 \times^{G_1} \delta) \cong (P_D, \gamma)_B$. This then corresponds to reduction of the structure group of $(P_D)_B$ to G_1 that extend the obvious one on D°. This is then the same as a section of $(G_2/G_1) \times^{G_1} (P_D)_B \to \operatorname{Spec} B[\![t]\!]$ extending the obvious one over $\operatorname{Spec} B(\!(t)\!)$. (Defined $\widetilde{X} \times^G Z = G(\widetilde{X} \times Z)$ last time, if this fpqc quotient was representable). The proposition will then follow from the following lemma:

Lemma: Let Y be a quasi-affine scheme over $\operatorname{Spec} A[\![t]\!]$ for some k-algebra A. Let $s : \operatorname{Spec} A(\!(t)\!) \to Y_{A(\!(t)\!)}$ be a section of Y. Consider the A-space

$$Z_Y : \operatorname{Spec} B \mapsto \begin{cases} * & s_B \text{ extends to } \operatorname{Spec} B[\![t]\!], \\ \emptyset & \text{if not} \end{cases} .$$

Then $Z_Y \to \operatorname{Spec} A$ is a locally closed embedding, closed if Y is affine.

Proof of Lemma: If $Y \hookrightarrow Y'$ is an open embedding (so s passes to a section s' for Y'), we claim $Z_Y \to Z_{Y'}$ is a schematic open embedding. Indeed, let $f : \operatorname{Spec} \to Z_{Y'}$ correspond to $t : \operatorname{Spec} B[\![t]\!] \to Y'_{B[\![t]\!]}$. Given $h : \operatorname{Spec} C \to \operatorname{Spec} B$, we have

$$(Z_Y \times_{Z_{Y'}} \operatorname{Spec} B)(\operatorname{Spec} C)$$

is $*$ if $h^* t$ sends $\operatorname{Spec} C[\![t]\!]$ into Y and $\emptyset$ otherwise. So this fiber product is the intersection $t^{-1}[Y] \cap \operatorname{Spec} B$ in $\operatorname{Spec} B$, which is open.

So we can assume WLOG that Y is affine. Note that $Y \mapsto Z_Y$ commutes with projective limits, so we may assume $Y = \mathbb{A}^1$ (since if $Y = \operatorname{Spec} R$ is affine, $R = A[t_i : i \in I]/)(f_j : j \in J)$ so $Y = \operatorname{Spec} A \times_{\mathbb{A}^J} \mathbb{A}^I$, and fiber products and direct products are all projective limits). But in this case, $s : \operatorname{Spec} A(\!(t)\!) \to Y$ corresponds to $f = \sum_{n \gg -\infty} f_n t^n \in A(\!(t)\!)$. Thus Z_Y is the closed subscheme defined by $f_n = 0$ for $n < 0$.

So that finishes the proof of the proposition. Now onto the topological fundamental group of G and $\pi_0(\mathrm{Gr}_G)$. Assume G is connected, $k = \overline{k}$, and let $p = \operatorname{char} k$. If $f : G \to \mathbb{G}_m$ is a character, we get a map of étale fundamental groups

$$f_* : \pi_1^{\text{ét},p'}(G)(-1) \to \pi_1^{\text{ét},p'}(\mathbb{G}_m)(-1) = \widehat{Z}^{p'}$$

where the "p'" means maximal prime-to-p quotient. Define

$$\pi_1(G) = \{\alpha \in \pi_1^{\text{ét},p'}(G)(-1) : \forall f, f_*\alpha \in \mathbb{Z}\}$$

where $\mathbb{Z}$ is embedded in $\widehat{Z}^{p'}$ in the usual way. (Note that throwing away stuff at finitely many primes doesn't change this, so doing the prime-to-p quotient should be morally ok).

Then, for example, $\pi_1(\mathbb{G}_m) = \mathbb{Z}$ (matching up with the usual topological fundamental group over $\mathbb{C}$). Note: $\pi_1(G) \cong \pi_1(G/R_u(G))$, and moreover if G is reductive then $\pi_1(G)$ is (the prime-to-p quotient of?) the quotient of $X_*(T)$ by the lattice of coroots (where T is the maximal torus).

Let $F = k(\!(t)\!)$, $\mathcal{O} = k[\![t]\!]$. If $\pi : \widetilde{G} \to G$ is a finite (connected) cover with kernel A such that $p \nmid |A|$, then get

$$G(F) \to H^1(F, A) = A(-1).$$

Taking a limit, get $\varphi_G : G(F) \to \pi_1^{\text{ét},p'}(G)(-1)$. Example: If $G = \mathbb{G}_m$, then these connected covers are just $\mathbb{G}_m \to \mathbb{G}_m$ with $x \mapsto x^N$ for $p \nmid N$, and the map $\mathbb{G}_m(F) = F^\times \to A(-1) = \mathbb{Z}/N\mathbb{Z}$ takes f to $\mathrm{ord}(f) + N\mathbb{Z}$. Taking the limit we get $F^\times \to \widehat{\mathbb{Z}}^{p'}$ given by $f \mapsto \mathrm{ord}(f)$, landing in $\mathbb{Z}$. Corollary (to the example): Every φ_G actually sends $G(F)$ into $\pi_1(G)$.

Remark: If $\pi : \widetilde{G} \to G$ is as before, then $\pi : \widetilde{G}(\mathcal{O}) \to G(\mathcal{O})$ is surjective (follows from infinitesimal lifting property for π, which is étale). So φ_G gives $\overline{\varphi}_G : G(F)/G(\mathcal{O}) = \mathrm{Gr}_G(k) \to \pi_1(G)$. Note that $G(\!(t)\!)(k)$ and $\mathrm{Gr}_G(k)$ are Zariski dense in $G(\!(t)\!)$ and Gr_G, respectively. Then:

Proposition: The maps φ_G and $\overline{\varphi}_G$ are Zariski locally constant. Moreover, the induced maps $\pi_0(G(\!(t)\!)) \to \pi_1(G)$ and $\pi_0(\mathrm{Gr}_G) \to \pi_1(G)$ are bijective. (Remark: $G[\![t]\!] = \varprojlim G[t]/(t^n)$ is connected).

Proof: (i) It is enough to prove this for $\overline{\varphi}_G$. Since Gr_G is ind-finite, it's enough to show that for every $M = \operatorname{Spec} R$ that's connected affine finite-type over k and every $f : M \to G((t))$, the composition $M(k) \to G(F) \to \pi_1(G)$ is constant. Let $\pi : \widetilde{G} \to G$ be a finite cover as before, and $A = \ker \pi$. We want $M(k) \to \pi_1(G) \to A(-1)$ to be constant. Now, since M is affine, a map f to the loop group $G(\!(t)\!)$ is the same as a map $\varphi : \operatorname{Spec} R((t)) \to G$. Let $\beta \in H^1_{\text{ét}}(G, A)$ be the class of $\widetilde{G}$, so $\varphi^*\beta \in H^1_{\text{ét}}(\operatorname{Spec} R((t)), A)$. Apply:

Lemma: Let $M = \operatorname{Spec} R$ be connected affine finite type over k, let A be a finite abelian group with $p \nmid |A|$, and let

$$\alpha \in H^1_{\text{ét}}(\operatorname{Spec} R(\!(t)\!), A).$$

Then for all $x \in M(k)$, let $\alpha(x)$ be the restriction of α to the fiber of $\operatorname{Spec} R((t)) \to \operatorname{Spec} R$ over x, so $\alpha(x) \in H^1_{\text{ét}}(\operatorname{Spec} k(\!(t)\!), A) = A(-1)$. Then $\alpha(x)$ is independent of x.

Proof: We may assume M is smooth, and that $A = \mu_N$. Then α corresponds to a μ_N-torsor on $V = \operatorname{Spec} R[\![t]\!]$, so a line bundle $\mathcal{L}$ on v together with a trivialization $\mathcal{L}^{\otimes N} \cong \mathcal{O}_V$. Then $\mathcal{L}$ extends to $V' = \operatorname{Spec} R[\![t]\!]$ and $\mathcal{L}^{\otimes N} \cong t^a \mathcal{O}_{V'}$ for $a \in \mathbb{Z}$. Then $\alpha(x) = a \bmod N$ for all x.

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Erratum: The statement Gr_G reduced iff $\mathrm{Hom}(G, \mathbb{G}_m) = 1$ last time is only valid if $\mathrm{char}\, k = 0$ (or at least $\gg 0$, where this $\gg$ depends on G). This won't really matter to us since we don't care about the reducedness much anyway.

Setup: $k = \overline{k}$, G/k connected linear algebraic group, $F = k(\!(t)\!)$, $\mathcal{O} = k[\![t]\!]$. Change of notation from last time: The thing we called $\pi_1(G)$ last time will now be called $\pi_1(G)^{p'}$ (contained in $\pi_1^{\text{ét},p'}(G)(-1)$). This time, define $\pi_1(G) = \pi_1(G/R_u(G))$ in general, and if G is reductive and $T \subseteq G$ is a maximal torus define $\pi_1(G)$ as $X_*(T)$ modulo the coroot lattice. (Then the $\pi_1(G)^{p'}$ from last time is the prime-to-p part of this).

Last time: Had $G(F) = G(\!(t)\!)(k)$ and defined a map

$$\varphi_G : G(F) \to \pi_1^{p'}(G) = \varprojlim_{\widetilde{G} \to G} \ker(\widetilde{G} \to G)(-1)$$

coming from Galois cohomology (where $\widetilde{G}$ was a connected prime-to-p cover). Know that this factors through to $\overline{\varphi}_G : \mathrm{Gr}_G(k) \to \pi_1^{p'}(G)$, and that this is Zariski locally constant. So we get maps $\pi_0(G(\!(t)\!)) \to \pi_1(G)^{p'}$ and $\pi_0(\mathrm{Gr}_G) \to \pi_1(G)^{p'}$.

Remains to be shown: These two maps are isomorphisms. In fact, we have an isomorphism $\pi_0(G(\!(t)\!)) \cong \pi_0(\mathrm{Gr}_G)$ and surjections $\pi_1(G) \to \pi_0(\mathrm{Gr}_G) \to \pi_1(G)^{p'}$.

Proof: The first isomorphism comes from $G[\![t]\!]$ being connected. So now, consider $\pi_0(G(\!(t)\!))$. If $G = \mathbb{G}_a$ then $G(\!(t)\!)$ this is connected because it's $\varinjlim_{\mathbb{N}} \mathbb{A}^1[\![t]\!]$ and each of these are connected. If G is unipotent then $G(\!(t)\!)$ is connected from the $\mathbb{G}_a$ case; so $\pi_0(G(\!(t)\!)) \cong \pi_0(G/R_v(G))(\!(t)\!)$ in general and thus we may assume WLOG that G is reductive (since this also doesn't change the π_1 of G).

Case $G = \mathrm{SL}_2$: The statement predicts $G((t))$ is connected, which is true and follows from a proof analogous to the proof that $\mathrm{SL}_2(\mathbb{R})$ is connected (every element of $G(F)$ is a product of unipotent elements so is connected to 1, and $G(F)$ is dense).

Case $G = T$ a torus: Assume WLOG that $G = \mathbb{G}_m$, and know $\pi_0(\mathbb{G}_m((t))) = \pi_0(\mathrm{Gr}_{\mathbb{G}_m}) \cong \mathbb{Z}$ given by taking the order of an element. But this is $\pi_1(\mathbb{G}_m)$.

General case (for G reductive): Choose a maximal torus $T \subseteq G$. Get diagram

$$\begin{tikzcd}
\pi_0(T(\!(t)\!)) \arrow[r] \arrow[d, "\cong"] & \pi_0(G(\!(t)\!)) \arrow[d] \\
\pi_1(T)^{p'} \arrow[r] & \pi_1(G)^{p'}.
\end{tikzcd}$$

The left vertical map is an isomorphism because $\pi_0(T((t))) = \pi_1(T) = \pi_1(T)^{p'} = X_*(T)$. The bottom horizontal map is surjective since $\pi_1(G)^{p'}$ is a quotient of $X_*(T)$. So the right vertical map is surjective. Also, note the top horizontal map is surjective because it's

$$\pi_0(T(\!(t)\!)) = \pi_0(B(\!(t)\!)) \to \pi_0(G(\!(t)\!))$$

and $G = \bigcup_{g \in G} gBg^{-1}$.

Now, let u be the map $\pi_0(T((t))) \to \pi_0(G((t)))$. Also, note that the map $\pi_0(T((t))) \to \pi_1(G)^{p'}$ factors through $\pi_1(G)$ because the bottom horizontal map does; let $v : \pi_0(T(\!(t)\!)) \to \pi_1(G)$ be this map. To finish proving the claim we need to show that $\ker(v) \subseteq \ker(u)$. But the kernel of v is just the coroot lattice in $X_*(T)$. Then let $\alpha^\vee : \mathbb{G}_m \to T$ be a coroot; it extends to $\mathrm{SL}_2 \to G$ but $\mathrm{SL}_2(\!(t)\!)$ is connected so $\alpha^\vee(\mathbb{G}_m(\!(t)\!)) \subseteq G(\!(t)\!)$ is in $G(\!(t)\!)^\circ$. This finishes the proof.

Example: Let $G = \mathrm{GL}_n$, B the usual Borel, and $T = \mathbb{G}_m^n$ the usual maximal torus. Then $\mathrm{Gr}_B \to \mathrm{Gr}_G$ is bijective on K-points for algebraically closed fields K, but $\pi_0(\mathrm{Gr}_B) \cong \mathbb{Z}^n$ while $\pi_0(\mathrm{Gr}_G) \cong \mathbb{Z}$, so these are not isomorphic!

Proposition: Let $\widetilde{G} \to G$ be a finite connected prime-to-p cover. Then $\mathrm{Gr}_{\widetilde{G}} \to \mathrm{Gr}_G$ identifies $\mathrm{Gr}_{\widetilde{G}}$ with a union of connected components of Gr_G.

Corollary: If G is reductive (and $p \gg 0$) then Gr_G is a union of connected components in $\mathrm{Gr}_{ad} \times \mathrm{Gr}_{G/G^{der}}$. (Need p to not divide the order of the kernel $G \to G_{ad} \times G/G^{der}$).

"Proof" of proposition: Let $A = \ker(\widetilde{G} \to G)$. Will only show the proposition modulo nilpotents (but could refine the argument to make it work in general). First, $\mathrm{Gr}_{\widetilde{G}} \to \mathrm{Gr}_G$ is a proper schematic map. Also, for all K/k algebraically closed, $\mathrm{img}(\mathrm{Gr}_{\widetilde{G}}(K) \to \mathrm{Gr}_G(K))$ is $\mathrm{Gr}_G(K)$ intersected with a certain union of connected components (independent of K). Moreover, $\mathrm{Gr}_{\widetilde{G}} \to \mathrm{Gr}_G$ is injective on R-points for all R. As a proper such map, it is a closed immersion.

To prove $\mathrm{Gr}_{\widetilde{G}} \to \mathrm{Gr}_G$ is injective on R-points we want to use $\mathrm{Gr}_G^{\mathrm{glob}}$. Take $X = \mathbb{P}^1_k$, $x = \{\infty\}$, $X^\circ = \mathbb{A}^1$, and consider $\mathrm{Gr}_{\widetilde{G}}^{\mathrm{glob}}(\mathrm{Spec}\, R) \to \mathrm{Gr}_G^{\mathrm{glob}}(\mathrm{Spec}\, R)$. Then for $(P_X, \gamma) \in \mathrm{Gr}_G^{\mathrm{glob}}(\mathrm{Spec}\, R)$, use SES

$$0 = H^1_{\text{ét}}(\mathbb{P}^1, A) \to H^1_{\text{ét}}(\mathbb{P}^1, \widetilde{G}) \to H^1_{\text{ét}}(\mathbb{P}^1, G)$$

(with $[P_x]$ in the last group) and map down to

$$0 = H^1_{\text{ét}}(\mathbb{A}^1, A) \to H^1_{\text{ét}}(\mathbb{A}^1, \widetilde{G}) \to H^1_{\text{ét}}(\mathbb{A}^1, G) \to H^2_{\text{ét}}(\mathbb{A}^1, A) = 0$$

and get that there exists a unique $\widetilde{P}_X$ that's a $\widetilde{G}$-bundle giving P_X and such that $\widetilde{P}_{\widetilde{X}/X^\circ}$ is trivial. Now use something like $\widetilde{G}(R[t]) \twoheadrightarrow \widetilde{G}(R]t)$ fpqc locally...

Now study $G[\![t]\!]$-orbits. Let k be any field and G a linear algebraic group. Review of orbits: Let X be a scheme of finite type over k with a left action of G. Then:

Theorem: (1): For all $x \in X(\bar{k})$, $G(\bar{k}) \cdot x \subseteq X(\bar{k})$ is open in its Zariski closure. So we may veiew it as a reduced locally closed subscheme of $X_{\bar{k}}$, call this $Orb(x)$.
(2) If $x \in X(k)$ then $Orb(x)$ is defined over k.

Now let X be an ind-scheme of ind-finite type with a left action of some pro-algebraic group H (which will be $G[\![t]\!]$ in our case). We say the action is *nice* if, for every closed subscheme Z of X, there's a closed subscheme $Z' \supseteq Z$ such that:
(1) Z' is stable under H,
(2) The action of H on Z' factors through a finite-type quotient.
In this situation we can define orbits as before.

The group $G[\![t]\!]$ (which we recall is a pro-algebraic group) acts by left translations on Gr_G. Proposition: The action is nice.

Proof: For $G = \mathrm{GL}_n$, then $\mathrm{Gr}_{\mathrm{GL}_n} = \varinjlim Latt_n^N$ and $Latt_n^N$ is stable by $\mathrm{GL}_n[\![t]\!]$, and the action factors through $\mathrm{GL}_n[t]/(t^{2N})/$ General case: Choose embedding $G \hookrightarrow G' = \mathrm{GL}_n \times \mathbb{G}_m$ with G'/G quasi-affine, so $\mathrm{Gr}_G \to \mathrm{Gr}_{G'}$ is a locally closed immersion. Write $\mathrm{Gr}_{G'} = \lim_N Z_N$ where Z_N is a closed subscheme, $G'[\![t]\!]$-stable, that factors through $G'[t]/(t^N)$. Then $\mathrm{Gr}_G = \varinjlim_N \mathrm{Gr}_G \times_{\mathrm{Gr}_{G'}} Z_N$, and this has the right properties.

So now we can talk about $G[\![t]\!]$-orbits in Gr_G. What are these orbits? Our goal is to describe these if G is connected reductive. Example: $G = T$ is a torus. Then for all $\mu \in X_*(T)$, let $t^\mu = \mu(t) \in T(k(\!(t)\!))$. Then $\mathrm{Gr}_T^{\mathrm{red}} = \coprod_\mu \mathrm{Spec}\, k$ and these $\mathrm{Spec}\, k$'s are the $T[\![t]\!]$-orbits. (??)

Now take G a general connected reductive group. Fix $T \subseteq B \subseteq G$, fix $\Phi^+ = \Phi(T, B)$ and $\partial\rho = \sum_{\alpha \in \Phi^+} \alpha$. Then set

$$X_*(T)^+ = \{\mu \in X_*(T) : \forall \alpha \in \Phi^+, \langle \alpha, \mu \rangle \geq 0\}$$

and for $\mu \in X_*(T)$ write $\mu(t) = t^\mu \in G(k(\!(t)\!))$.

Theorem: The orbits of $G[\![t]\!]$ on Gr_G are exactly the $Orb(t^\mu)$ for $\mu \in X_*(T)^+$. In particular, they are defined over k.
Proof: This is exactly the Cartan decomposition; for all extensions K/k we have

$$G(K(\!(t)\!)) = \coprod_{\mu \in X_*(T)^+} G(K[\![t]\!]) t^\mu G(K[\![t]\!]).$$

Theorem: If $\lambda \in X_*(T)_+$ then

$$\overline{Orb(t^\lambda)} = \bigcup_{\mu \leq \lambda} Orb(t^\mu),$$

where μ runs over $X_*(T)_+$ and we say $\mu \leq \lambda$ if $\lambda - \mu = \sum_{\alpha \in \Delta^+} n_\alpha \alpha$ with $n_\alpha \in \mathbb{N}$. (So $Orb(t^\mu) \subseteq \overline{Orb(t^\lambda)}$ iff μ is a weight of V_λ, the irreducible representation of $\widehat{G}$ with highest weight λ).
Proof: Assume WLOG that $k = \overline{k}$, and take $\mathcal{O} = k[\![t]\!]$. Let $\nu \in X^*(T)$ be anti-dominant, and let (ρ, W_v) be the irreducible representation of G with lowest weight v. If $\lambda \in X_*(T)_+$ and $g \in G(\mathcal{O})t^\lambda G(\mathcal{O})$ then $\rho(g) \in t^{\langle \nu, \lambda \rangle} \operatorname{End}(W_v \otimes \mathcal{O})$ but $\rho(g) \notin t^{\langle \nu, \lambda \rangle + 1} \operatorname{End}(W_v \otimes \mathcal{O})$. So if $\mu, \lambda \in X_*(T)^+$ and $Orb(t^\mu) \subseteq \overline{Orb(t^\lambda)}$, then for all $\nu \in X^*(T)$ anti-dominant taking $g \in G(\mathcal{O})t^\mu G(\mathcal{O})$ gives $\rho(g) \in t^{\langle \nu, \lambda \rangle} \operatorname{End}(W_v \otimes \mathcal{O})$ but $\rho(g) \notin t^{\langle \nu, \mu \rangle + 1} \operatorname{End}(W_v \otimes \mathcal{O})$. Thus $\langle \nu, \lambda \rangle \leq \langle \nu, \mu \rangle$ so $\langle \nu, \lambda - \mu \rangle \leq 0$. So $\mu - \lambda = \sum_{\alpha \in \Delta^+} n_\alpha \alpha$ with $n_\alpha \in \mathbb{R}_{\geq 0}$. To get $\lambda \geq \mu$ need $\lambda - \mu$ is in the coroot lattice. But this follows from the fact that t^λ and t^μ are in the same connected component.

Definition: A subset $Y \subseteq X_*(T)$ is saturated if, for every $\alpha \in \Phi$, every $\mu \in Y$, and every $0 \leq i \leq \langle \mu, \alpha \rangle$, we have $\mu - i\alpha \in Y$.

Fact: Let $\lambda \in X_*(T)_+$. Then the set of weights of V_λ is the smallest saturated subset of $X_*(T)$ containing λ.

To prove the other direction of the proposition, we need

$$Y = \{\mu \in X_*(T) : t^\mu \in \overline{Orb(t^\lambda)}\}$$

is saturated, for $\lambda \in X_*(T)_+$ fixed. Proof: Let $\alpha \in \Phi$, and let $T \subseteq L_\alpha \subseteq G$ be the associated rank-1 subgroup. Assume WLOG that $G = L_\alpha$. By a previous corollary (and replacing G by $G^{ad} \times G/G^{der}$) we may replace G by SL_2 or GL_2 and then do a direct calculation.

8 Lecture - 03/12/2014

Continuing from last time: k is a field, G/k connected reductive, T a maximal torus. Wanted to study the $G[\![t]\!]$-orbits in $\mathrm{Gr}_G = G(\!(t)\!)/G[\![t]\!]$. The orbits are exactly the sets $Orb(t^\lambda) = \mathbb{G}[\![t]\!] \cdot t^\lambda$ for $\lambda \in X_*(T)^+$ (where t^λ is just $\lambda(t)$ viewed as an element of $\mathrm{Gr}_G(k)$). Then said that

$$\overline{Orb(t^\lambda)} = \bigcup_{\mu \leq \lambda} Orb(t^\mu),$$

and reduced the proof to a GL_2 calculation that was left as an exercise.

The GL_2 calculation in a simple case: recall we have

$$\mathrm{Gr}_{\mathrm{GL}-2} = \varinjlim_N Latt_2^N$$

where $Latt_2^N$ was a space of lattices in $k(\!(t)\!)^2$ with $t^N k[\![t]\!]^2 \subseteq \mathcal{L} \subseteq t^{-N} k[\![t]\!]^2$ with some quotient condition. We saw this mapped into $\coprod \mathrm{Gr}(r, V_N)$ for $V_n = t^{-N} k[\![t]\!]^2 / t^N k[\![t]\!]^2$.

To make things explicit, let $e_1 = (1,0)$ and $e_2 = (0,2)$ in $k(\!(t)\!)^2$. Take a basis of V_N as consisting of all $t^i e_1$ and $t^i e_2$ for $-N \leq i < N$. Let's show that

$$t^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is in the closure of the orbit

$$t^\lambda = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}.$$

We work in $Latt_2^1$. First, note

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\overline{k}) \qquad \Longrightarrow \qquad \begin{bmatrix} at & b \\ c & dt^{-1} \end{bmatrix} Orb(t^\lambda)$$

(if $ad \neq 0$). This corresponds to the subspace with basis $(be_1 t + dt^{-1} e_2, de_2)$. On the other hand, t^0 corresponds to the subspace with basis e_1, e_2. Take limits $d \to 0$ (with $b \neq 0$).

Remark: One thing we know about these strata at this point is that $Orb(t^\lambda)$ is smooth (as a finite-type quotient of $G[\![t]\!]$).

The $\mathbb{G}_m$-action. Let Y be a k-scheme. Then we have a $\mathbb{G}_m$-action on the k-space $Y(\!(t)\!)$, by

$$R^\times \times Y(R(\!(t)\!)) = \mathbb{G}_m(R) \times Y(\!(t)\!)(R) \to Y(\!(t)\!)(R) = Y(R(\!(t)\!))$$

coming from the action of $R^\times$ on $R(\!(t)\!)$, by $(a, f(t)) \mapsto f(at)$. (Note $R^\times$ acts by R-algebra automorphisms). Example: If y is quasi-affine of finite type, choose $Y \hookrightarrow \mathbb{A}^N$, then the action is $a \cdot (f_1(t), \ldots, f_N(t)) = (f_1(at), \ldots, f_N(at))$.

In particular, we get an action δ of $\mathbb{G}_m$ on $G(\!(t)\!)$. This action is by group automorphisms and it stabilizes $G[\![t]\!]$, so it gives an action δ on Gr_G. Now assume G is connected reductive (with maximal torus and Borel $T \subseteq B$). Note if $\lambda \in X_*(T)$ then $t^\lambda \in \mathrm{Gr}_G$ is fixed by $\mathbb{G}_m$. So $Orb(t^\lambda)$ is $\mathbb{G}_m$-stable.

Note Gr_G is ind-projective so the map $\mathbb{G}_m \times \{x\} \to \mathrm{Gr}_G$ extends uniquely to $\varphi : \mathbb{P}^1 \times \{x\} \to \mathrm{Gr}_G$, given by defining $\varphi(0, x) = \lim_{a \to 0} \delta(a)x$ and $\varphi(\infty, x) = \lim_{a \to \infty} \delta(a)x$.

Also, the k-algebra maps $k \hookrightarrow k[\![t]\!] \twoheadrightarrow k$ (with the latter being evaluation at 0) give $G \to G[\![t]\!] \to G$, with the latter map being $g \mapsto g(0)$.

Fact: (a) The fixed point set of $\mathbb{G}_m$ on $\mathrm{Gr}_G(\overline{k})$ is $\bigcup_{\lambda \in X_*(T)} Gt^\lambda$, and its connected components are the Gt^λ for $\lambda \in X_*(T)^+$.
(b) For all $x \in Orb(t^\lambda)$, $\lim_{a \to 0} \delta(a)x \in Gt^\lambda$. More precisely, if $x = gt^\lambda$ with $g \in G[\![t]\!]$, then this limit is $g(0)t^\lambda$.

Prove this by showing (b) and concluding (a). This in particular shows that

$$\mathrm{Gr}_G = \coprod_{\lambda \in X_*(T)^+} Orb(t^\lambda)$$

is the Bialynicki-Birula decomposition (named after one person; hereafter BB decomposition). What is this decomposition?

Theorem (BB): If X/k is a smooth proper variety with a $\mathbb{G}_m$-action, and if we let $X_1^\circ, \dots, X_r^\circ$ be the connected components of $X^{\mathbb{G}_m}$, then:
(i) Each X_i° is smooth.
(ii) We have a (unique) decomposition $X = \coprod_{i=1}^r X_i^+$ into $\mathbb{G}_m$-stable locally closed smooth subschemes with "retraction" maps $\gamma_i^+ : X_i^+ \to X_i^\circ$ satisfying:
(iii) $\gamma_i^+|_{X_i^\circ} = \mathrm{id}_{X_i^\circ}$.
(iv) $X_i^+ = \{x \in X : \lim_{a \to 0} ax \in X_i^\circ\}$.
(v) $\gamma_i^+ : X_i^+ \to X_i^\circ$ is Zariski locally on X_i° of the form $\pi_2 : V \times X_1^\circ \to X_1^\circ$, where $V \cong \mathbb{A}^n$ with the diagonal action of $\mathbb{G}_m$.
(vi) For all $x \in X_i^\circ$, $T_x(X_i^+) = (T_xX)^\circ \oplus (T_xX)^+$: this comes from T_xX having a $\mathbb{G}_m$-action so having an decomposition into eigenspaces $\bigoplus_{n\in\mathbb{Z}}(T_xX)^n$ for $\lambda \mapsto \lambda^n$, and $(T_xX)^+ = \bigoplus_{n\geq 1}(T_xX)^n$.
Moreover, if X is just smooth, we know it satisfies all of these except (ii) and (iv) in general. Also know that if $x \in X$ and $\lim ax$ exists and is in X_i° then $x \in X_i^+$. (But in our case we know that this limit exists aways so we're ok).

Back to our situation: We have retractions $Orb(t^\lambda) \to Gt^\lambda$ that satisfy (v). Fact: $Gt^\lambda = G/\operatorname{Stab}_G(t^\lambda) = G/P_\lambda$ where

$$P_\lambda = \operatorname{Stab}_G(\lambda) = \{g \in G : \lim_{t\to 0} t^{-\lambda} g t^\lambda \text{ exists}\}$$

is the standard parabolic subgroup of G corresponding to the set of simple roots α with $\langle \lambda, \alpha \rangle = 0$. In particular, Gt^λ is smooth projective rational and geometrically simply connected. (Y rational iff $k(Y)$ is purely transcendental). Consequence: $Orb(t^\lambda)$ is also rational, and $\pi_1^{geom,p'}(Orb(t^\lambda)) = 1$.

The dimension of $Orb(t^\lambda)$: For $x \in \mathrm{Gr}_G(\overline{k})$, identify $T_x \mathrm{Gr}_G$ with $\mathfrak{g}((t))/\mathfrak{g}[\![t]\!]$ in the obvious way (for $\mathfrak{g} = \mathrm{Lie}(G)_{\overline{k}}$). If $\lambda \in X_*(T)$ the action of $\mathbb{G}_m$ on $T_{t^\lambda}\mathrm{Gr}_G$ is given by

$$(a, f(t)) \mapsto t^{-\lambda} f^\lambda(at) t^\lambda = \lambda(a) f(at) \lambda(a)^{-1}$$

where $f^\lambda(t) = t^\lambda f(t) t^{-\lambda}$. Write $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$, the root space decomposition. Then if $f(t) \in \mathfrak{g}(\!(t)\!)$ is of the form

$$f(t) = \sum_{0 > i \gg -\infty} X_i t^i$$

with $X_i = H_i + \sum_\alpha X_i^\alpha$, have

$$a \cdot f(t) = \sum_{i<0} \left(H_i + \sum_\alpha a^{\langle \alpha, \lambda \rangle} X_i^\alpha \right) a^i t^i.$$

So

$$\bigoplus_{n\geq 0} (T_{t^\lambda}\mathrm{Gr}_G)^n = \bigoplus_{\substack{\alpha\in\Phi^+, i<0: \\ \langle\alpha,\lambda\rangle + i \geq 0}} \mathfrak{g}_\alpha t^i.$$

So

$$\dim Orb(t^\lambda) = \dim \bigoplus_{n\geq 0} (T_{t^\lambda}\mathrm{Gr}_G)^n = \sum_{\alpha\in\Phi^+} \langle \alpha, \lambda \rangle = \langle 2\rho, \lambda \rangle$$

where $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$.

Consequence of this formula: $Orb(t^\lambda)$ is geometrically simply connected. Proof: $\overline{Orb(t^\lambda)}$ is projective and normal (by Faltings) and rational, and thus geometrically simply-connected (see SGA1 somewhere). Then, the boundary of $\overline{Orb(t^\lambda)}$ is

$$\sum_{\mu<\lambda} Orb(t^\mu),$$

and the dimension of this thing is $\sup\{\langle\rho,\mu\rangle : \mu<\lambda\} \le \langle 2\rho,\lambda\rangle - 2$. By Grothendieck's purity theorem,

$$\pi_1^{geom}(Orb(t^\lambda)) = \pi_1^{geom}(\overline{Orb(t^\lambda)}).$$

Fact: For all λ, $Orb(t^\lambda)$ is paved by affine spaces. Proof: Remember we had evaluation-by-zero map $e_0 : G[\![t]\!] \to G$. Let I be the Iwahori subgroup $e_0^{-1}[B] \supseteq I_0 = \ker(e_0)$. The Iwasawa decomposition gives $Orb(t^\lambda) = \coprod_{w\in W} It^{w\lambda}$, and the Cartan decomposition gives $Tt^{w\lambda} = I_0t^{w\lambda}$ is a finite-type quotient of I_0 and hence on affine space as I_0 is pro-unipotent.

The Tannakian category. Start with a review of perverse sheaves. Let X/k be a scheme of finite type, and ℓ a prime not dividing char k. Let $D_c^b(X) = D_c^b(X, \overline{\mathbb{Q}}_\ell)$ be the triangulated category of bounded constructible $\overline{\mathbb{Q}}_\ell$-complexes (which can be actual complexes using the pro-étale site!) Operations on $K \in D_C^b(X)$: Can take the shift $K[n]$ with $K[n]^i = K^{n+i}$ and the Tate twist $K(n)$.

What does "bounded constructible" mean? Bounded means that sheaf cohomology H^iK is zero for $|i| \gg 0$. Constructible means that there exists a stratification $X = \bigcup X_\alpha$ such that for all α and all i, $H^iK|_{X_\alpha}$ is lisse (locally constant of finite type - here locally constant is with respect to the pro-étale topology).

Some operations on this category: Given $f : X \to Y$, have $f_*, f_! : D_C^b(X) \to D_C^b(Y)$ and $f^*, f^! \to D_C^b(Y) \to D_C^b(X)$, which form adjoint pairs (f^*, f_*) and $(f_!, f^!)$ in the usual way. (Note that we write f_*, etc. to mean the derived version, not Rf_*). Have (Poincaré-Verdier) duality functors $D : D_C^b(X)^{op} \to D_C^b(X)$ with $D\circ D \cong \mathrm{id}$. This is compatible with our other operations in that $D\circ f_* = f_!\circ D$ and likewise for others. Also, have external tensor product operation $\boxtimes$

$$D_C^b(X_1)\times D_C^b(X_2) \to D_C^b(X_1\times X_2)$$

given by $(K_1, K_2) \mapsto \pi_1^*K_1 \otimes^L \pi_2^*K_2$; this is exact.

Definition: $\mathbf{Perv}(X)$, the category of (constructible self-dual) perverse sheaves is the full subcategory of $D_C^b(X)$ with objects being complexes $K \in D_C^b(X)$ such that:
(1) There exists a stratification $X = \bigcup_\alpha X_\alpha$ such that, for every $\alpha \in A$ and every $i \in \mathbb{Z}$ we have $H^i(i_\alpha^*K) = H^iK|_{X_\alpha}$ is Lisse and is 0 if $i > -\dim X_\alpha$.
(2) There exists a stratification $X = \bigcup_\alpha X_\alpha$ such that $H^i(i_\alpha^!K)$ is Lisse and 0 for $i < -\dim X_\alpha$. (Note this is (1) for DK).

Note $D(\mathbf{Perv}(X)) = \mathbf{Perv}(X)$.

9 Lecture - 03/26/2014

Let k be a field, X/k a scheme of finite type, G/k a linear algebraic group. Had categories defined $\mathbf{Perv}(X)$ of perverse sheaves, $\mathbf{Perv}_G(X)$ of G-equivariant perverse sheaves, and a faithful forgetful functor $\mathbf{Perv}_G(X) \to \mathbf{Perv}(X)$ (fully faithful if G connected, so equivalent to a full subcategory of $\mathbf{Perv}(X)$). (Note in general: $\mathbf{Perv}_G(X)$ should be the category of perverse sheaves on the quotient $[X/G]$; this is literally true if X/G is actually a scheme rather than a stack).

What was a G-equivariant perverse sheaf? Will only talk about it in the connected case, where we said it's a full subcategory of $\mathbf{Perv}(X)$. Which one? Note we have two obvious maps $G \times X \to X$, the action map a and the projection-to-the-2nd-coordinate map p. Then $\mathbf{Perv}_G(X)$ is the full subcategory of objects K with $a^*K \cong p^*K$. It's stable by subquotients but not extensions in general.

Change of group: Assume G is connected and $H \subseteq G$ is normal and connected and acts trivially on X. Then $\mathbf{Perv}_{G/H}(X) \cong \mathbf{Perv}_G(X)$.

Homogeneous spaces. Suppose that $k = \overline{k}$ and that G acts transitively on X. Fix a point $x_0 \in X(k)$ and let $H = \mathrm{Stab}_G(x_0)$. Then $\mathbf{Perv}_G(X)$ is equivalent to the category of finite-dimensional representations of H/H°. Moreover every $K \in \mathbf{Perv}_G(X)$ is a lisse sheaf put in degree $-\dim X$. Why this is true: The stack quotient $[X/G]$ is isomorphic to $[\{x_0\}/H]$, and $\mathbf{Perv}([\{x_0\}/H])$ is isomorphic to this category. (Need to do some more to formalize this argument but that's the idea).

Now assume: $k = \overline{k}$, G is connected, X has finitely many G-orbits $X = \bigcup X_\alpha$, $j_\alpha : X_\alpha \hookrightarrow X$, $d_\alpha = \dim X_\alpha$, and for all $x \in X(k)$, $\mathrm{Stab}_G(x)$ is connected. Then:

Proposition: The simple objects of $\mathbf{Perv}_G(X)$ are the $j_{\alpha!*}(\overline{\mathbb{Q}}_{\ell,X_\alpha}[d_\alpha])$.

Proof: Obviously these are simple and mutually non-isomorphic. Then let $K \in \mathbf{Perv}(X)$ be a simple object in $\mathbf{Perv}_G(X)$. Write $K = j_{!*}(\mathcal{L}[d])$ for $j : Z \hookrightarrow X$ locally closed smooth and connected, $d = \dim Z$, $\mathcal{L}$ lisse on Z and simple. Then $\mathrm{supp}(K) = \overline{Z}$ is G-stable so $\overline{Z} = \bigcup_{\alpha \in B} X_\alpha$. Pick $\alpha \in B$ such that X_α is open in $\overline{Z}$, so X_α is dense and $K|_{X_\alpha}$ is in $\mathbf{Perv}_G(X)\alpha)$. So $K|_{X_\alpha} = \overline{\mathbb{Q}}_\ell[d_\alpha]$, but $K = j_{\alpha!*}(K|_{X_\alpha})$.

Ind-schemes: Let X be an ind-scheme of ind-finite type. Write $X = \varinjlim X_n$ with X_n of finite type. The transition maps $i_{n,m} : X_n \to X_m$ are closed immersions, so the $i_{n,m*}$ are t-exact and fully faithful. Then we can define $\varinjlim \mathbf{Perv}(X_n)$.

Now let G be a pro-algebraic group acting on X, assuming that the action is nice: $X = \varinjlim X_n$, $G = \varprojlim G_n$ with G_n linear algebraic groups such that for all n, X_n is G-stable and the action of G on X_n factors through G_n. Assume further that all of the G_n's are connected. Then, for all $m \geq n$, $\mathbf{Perv}_{G_n}(X_n) \cong \mathbf{Perv}_{G_m}(X_n)$. So, define

$$\mathbf{Perv}_G(X_n) = \varinjlim_m \mathbf{Perv}_{G_m}(X_n) = \mathbf{Perv}_{G_n}(X_n),$$

$$\mathbf{Perv}_G(X) = \varinjlim_n \mathbf{Perv}_G(X_n) = \varinjlim_n \mathbf{Perv}_{G_n}(X_n).$$

This is a full thick subcategory of $\mathbf{Perv}(X)$.

Example: G is connected reductive over k, $X = \mathrm{Gr}_G$ with $G[\![t]\!]$ acting on it, define the Satake category as $\mathbf{Sat}(G) = \mathbf{Perv}_{G[\![t]\!]}(\mathrm{Gr}_G)$. If we take $G \supseteq B \supseteq T$ as usual, recall Gr_G was the disjoint union of $\lambda \in X_*(T)_+$ of $Orb(t^\lambda)$. Let j_λ be the inclusion $Orb(t^\lambda) \hookrightarrow \mathrm{Gr}_G$; if we set $d_\lambda = \dim Orb(t^\lambda) = \langle 2\rho, \lambda \rangle$. Let

$$IC_\lambda = j_{\lambda!*}\overline{\mathbb{Q}}_{\ell,Orb(t^\lambda)}[d_\lambda].$$

Proposition: $\mathbf{Sat}(G)$ is Noetherian and Artinian, and if $k = \overline{k}$ then the simple objects are these IC_λ's.

Proof: We us the previous proposition; we just need to check that if applies. So we need to check that every $\mathrm{Stab}_{G[\![t]\!]}(x)$. Suffices to take $x = t^\lambda$ for $\lambda \in X_*(T)_+$. Then the stabilizer is

$$\{g \in G[\![t]\!] : gt^\lambda \in t^\lambda G[\![t]\!]\} = G[\![t]\!] \cap t^\lambda G[\![t]\!] t^{-\lambda}.$$

Then for all N, let K_N be the kernel of $G[\![t]\!] \to G[t]/(t^N)$; since N is connected and acts trivially on t^λ for $N \gg 0$, we just need to show that $(G[\![t]\!] \cap t^\lambda G[\![t]\!] t^{-\lambda})/K_N$ is connected, and this is the stabilizer of the action of $G[t]/(t^N)$ on t^λ.

A number of ways to do this. One way: if $\alpha \in \Phi = \Phi(T, G)$ have map $\mu_\alpha : \mathbb{G}_a \to G$ (for instance, if $\alpha > 0$ then $\mu_\alpha[\mathbb{G}_a]$ is the subgroup of $R_u(B)$ corresponding to the root space $\mathfrak{g}_\alpha \subseteq \operatorname{Lie} R_u(B)$). Choose orderings on Φ^+ and Φ^-. Then the map

$$\mu : V = \mathbb{A}^{|\Phi^+|} \times T \times \Phi^{|\Phi^-|} \to G \qquad ((x_\alpha)_{\alpha \in \Phi^+}, z, (y_\alpha)_{\alpha \in \Phi^-}) \mapsto \prod \mu_\alpha(x_\alpha) z \prod_\alpha \mu_\alpha(y_\alpha)$$

is an open embedding with dense image ("the big Bruhat cell"). We use μ on $\overline{k}[t]/(t^N)$-points. Let $a = (x_\alpha, z, y_\alpha) \in V(\overline{k}[t]/(t^N))$; then

$$t^\lambda \mu(a) t^{-\lambda} = a(t^{\langle \alpha, \lambda \rangle} x_\alpha, z, t^{\langle \alpha, \lambda \rangle} y_\alpha)$$

with the first $\langle \alpha, \lambda \rangle$'s nonnegative and the second ones nonpositive. So get

$$\mu(V[t]/t^N) \cap t^\lambda \mu(V[t]/t^N) t^{-\lambda} = \mu\Big(\{(x_\alpha, z, y_\alpha) \text{ such that } \forall \alpha \in \Phi^-, \operatorname{ord}(y_\alpha) \geq \langle \alpha, \lambda \rangle\}\Big)$$

is connected, and is open and dense in what we're looking for.

Proposition: $\mathbf{Sat}(G)$ is a semisimple abelian category. (This means that every object is a direct sum of IC_λ's).

Proof: we want to show that $\operatorname{Ext}^1_{\mathbf{Sat}(G)}(K, L) = 0$ for all $K, L \in \mathbf{sat}(G)$. We may assume $K = IC_\lambda$ and $L = IC_\mu$ (and get the general case by induction). Three cases:
(1) λ and μ are not comparable. Then Ext vanishes (even in the category of all perverse sheaves) by general properties of the $j_{!*}$'s.
(2) $\lambda = \mu$. Suppose we have an extension

$$0 \to IC_\lambda \to K \to IC_\lambda \to 0$$

in $\mathbf{Sat}(G)$. One argument we could give: since $\mathbf{Sat}(G)$ is a full subcategory of $\mathbf{Perv}(\mathrm{Gr})$, have

$$\operatorname{Ext}^1_{\mathbf{Sat}} \hookrightarrow \operatorname{Ext}^1_{\mathbf{Perv}} \cong \operatorname{Ext}^1_{Orb(t^\lambda)}(\overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}}_\ell) = H^1(Orb(t^\lambda)_{\overline{k}}, \overline{\mathbb{Q}}_\ell) = 0$$

since $Orb(t^\lambda)$ is simply connected; but simple-connectedness is hard and we didn't prove it (just stated it).

Another proof: Given our extension, K has no subobject or quotient supported in $\overline{Orb}(t^\lambda) - Orb(t^\lambda)$ so $K = j_{\lambda!*} j_\lambda^* K$. But $j_\lambda^* K$ is $G[\![t]\!]$-equivariant on $Orb(t^\lambda)$, so is constant, so $j_\lambda^* K$ splits as $\overline{\mathbb{Q}}_\ell[d_\lambda] \oplus \overline{\mathbb{Q}}_\ell[d_\lambda]$. Passing back to K via $j_{\lambda!*}$ gives what we want.

(3) $\mu \leq \lambda$ (which is equivalent to the case where $\lambda \leq \mu$ by duality). So have inclusion maps i, j of $\overline{Orb}(t^\mu), Orb(t^\lambda)$ into $\overline{Orb}(t^\lambda)$, and inclusions a, b of $Orb(t^\mu)$ and of $\overline{Orb}(t^\mu) \setminus Orb(t^\mu)$ into $\overline{Orb}(t^\mu)$. Then have $IC_\mu = i_* i^* IC_\mu$ so

$$\operatorname{Hom}(IC_\lambda, IC_\mu[1]) = \operatorname{Hom}(i^* IC_\lambda, i^* IC_\mu[1]).$$

Now, this fits into an exact sequence

$$\operatorname{Hom}(a^* i^* IC_\lambda, a^! i^* IC_\mu[1]) \to \operatorname{Hom}(i^* IC_\lambda, i^* IC_\mu[1]) \to \operatorname{Hom}(b^* i^* IC_\lambda, b^* i^* IC_\mu[1]).$$

This comes from the following: If $L \in D^b_c(\overline{Orb}(t^\mu))$ have exact triangle $a_* a^! L \to L \to b_* b^* L \to \cdots$, get exact triangle by applying the second coordinate of $RHom(-, -)$ to this, then take long exact sequence for that (?).

Now, can check that $\operatorname{Hom}(a^* i^* IC_\lambda, a^! i^* IC_\mu[1]) = 0$ because the two parts are concentrated in different degrees (≤ -1 and ≥ 0, respectively). Similarly can look at $\operatorname{Hom}(b^* i^* IC_\lambda, b^* i^* IC_\mu[1])$ and find formally that $b^* i^* IC_\lambda$ is concentrated in degrees ≤ -1 and $b^* i^* IC_\mu[1]$ in degree -1. This doesn't quite rule out that this Hom is zero; to do that we need to apply the following theorem that implies $b^* i^* IC_\lambda$ is actually concentrated in degrees ≤ -2 and finishes the proof of the theorem.

Theorem (Lusztig): For all $\lambda, \mu \in X_*(T)_+$ with $\mu \leq \lambda$, IC_λ on $Orb(t^\mu)$ is concentrated in even perverse degrees. This follows from the following lemma:

Lemma: For $x \in \overline{Orb}(t^\lambda)$, the fiber $(IC_\lambda)_x$ is concentrated in degrees $\equiv d_\lambda \pmod 2$. (This implies theorem as follows: $IC_\lambda|_{Orb(t^\mu)}$ is a $G[\![t]\!]$-equivariant complex so ${}^pH^K(IC_\lambda|_{Orb(t^\mu)})$ are $G[\![t]\!]$-equivariant perverse sheaves hence constant hence ${}^pH^k(-) = H^{k-d\mu}(-) = 0$ unless $k - d\mu \equiv d_\lambda \pmod 2$ by the lemma. But $\lambda - \mu$ is a sum of coroots hence $d_\lambda + d\mu \equiv d_\lambda - d_\mu = \langle \lambda - \mu, 2\rho \rangle = 0$).

How do you prove the lemma? $Z = \overline{Orb}(t^\lambda)$ has a resolution of singularities $\pi : \widetilde{Z} \to Z$ called the generalized Bott-Samelson resolution such that he geometric fibers of π are paved by affine spaces. By the decomposition theorem IC_λ is a direct factor of $\pi_*(\overline{\mathbb{Q}}_\ell, \widetilde{Z}[d_\lambda])$. Hence $IC_{\lambda,x}$ is a direct factor of $H^{*-d}(\pi^{-1}[x], \overline{\mathbb{Q}}_\ell)$, which is concentrated in degree $\equiv d_\lambda \pmod 2$ (by using excision exact sequence repeatedly, and that affine spaces only have things concentrated in degree 0).

10 Lecture - 03/28/2014

Continuing from last time: G is connected reductive, $k = \overline{k}$. Have action of $G[\![t]\!]$ on Gr_G, and the Satake category is the category of equivariant perverse sheaves, $\mathbf{Perv}_{G[\![t]\!]}(\mathrm{Gr}_G)$. Theorem: $\mathbf{Sat}(G) \subseteq \mathbf{Perv}(\mathrm{Gr}_G)$ is full, thick, and semisimple. The simple objects are

$$IC_\lambda = j_{\lambda!*}\mathbb{Q}_{\ell,Orb(t^\lambda)}[\langle 2\rho, \lambda\rangle].$$

The proof we gave used the generalized Bott-Samelson resolution, which we didn't talk about; but it will come up again soon and we'll discuss it then.

What if k is not algebraically closed? Then $\mathbf{Sat}(G)$ is a full and thick subcategory (only uses connectedness of G) and still Artinian and Noetherian, but is not semisimple in general. The simple objects are still the intersection complexes IC_λ, but can have nontrivial extensions; if $\lambda = 0$ is the trivial cocharacter, $t^\lambda = 1$ so $Orb(t^\lambda) = \operatorname{Spec} k$ and $IC_\lambda = \overline{\mathbb{Q}}_\ell$; then $\operatorname{Ext}^1_{\mathbf{Sat}(G)}(IC_\lambda, IC_\lambda)$ is the group of extensions of $\overline{\mathbb{Q}}_\ell$ by itself in the category of ℓ-adic representations of the absolute Galois group of k. Example: If $k = \mathbb{F}_q$ then have a nontrivial extension, namely $\overline{\mathbb{Q}}_\ell^2$ with Frob_q acting by

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Remark: If R is a k-algebra, $\mathbb{G}_m[\![t]\!](R) = R[\![t]\!]^\times$ acts by k-algebra isomorphisms on $R(\!(t)\!)$ (by $a \cdot f(t) = f(at)$), so get $a^* : \operatorname{Spec} R(\!(t)\!) \to \operatorname{Spec} R(\!(t)\!)$ for all $a \in R[\![t]\!]^\times$. If Y is a k-scheme we get an action of $\mathbb{G}_m[\![t]\!]$ on $Y(\!(t)\!)$ by $a \cdot \mu = \mu \circ a^*$ for $\mu : \operatorname{Spec} R[\![t]\!] \to Y$. If $Y = G$ is a smooth affine group scheme then $\mathbb{G}_m[\![t]\!]$ acts on $G(\!(t)\!)$ by group automorphisms and preserves $G[\![t]\!]$, so passes to an action δ of $\mathbb{G}_m[\![t]\!]$ on Gr_G extending the previous action δ of $\mathbb{G}_m$. This is the Virasoro action; combining it with the natural action of $G[\![t]\!]$ we get an action of $G[\![t]\!] \rtimes \mathbb{G}_m[\![t]\!]$ (with the action of $\mathbb{G}_m[\![t]\!]$ on $G[\![t]\!]$ what we just defined).

Corollary: If $k = \overline{k}$ and G is connected reductive then every object of $\mathbf{Sat}(G)$ is $G[\![t]\!] \rtimes \mathbb{G}_m[\![t]\!]$-equivariant. Hence $\mathbf{Sat}(G)$ is independent of the choice of t we made. (Proof: Since this group is connected we can use the simple definition of equivariant; and every object of $\mathbf{Sat}(G)$ is a sum of IC_λ's and the IC_λ's are obviously equivariant).

The convolution product. Our Satake category $\mathbf{Sat}(G)$ is supposed to be a Tannakian category but we haven't defined the tensor product! And moreover, the usual tensor product doesn't preserve perverse sheaves. So we need something else. Convolution diagram: have maps $p = \pi \times \mathrm{id} : G(\!(t)\!) \times \mathrm{Gr}_G \to \mathrm{Gr}_G \times \mathrm{Gr}_G$ and $q : G(\!(t)\!) \times \mathrm{Gr}_G \to G(\!(t)\!) \times^{G[\![t]\!]} \mathrm{Gr}_G$, both of which are $G[\![t]\!]$-torsors. Recall that $G(\!(t)\!) \times^{G[\![t]\!]} \mathrm{Gr}_G$ is the quotient $(G(\!(t)\!) \times \mathrm{Gr}_G)/G[\![t]\!]$ via the action $g \cdot (x, y) = (xg^{-1}, gy)$. This has a map $m : G(\!(t)\!) \times^{G[\![t]\!]} \mathrm{Gr}_G \to \mathrm{Gr}_G$, and the composition $m \circ q$ is just the action map $(x, y) \mapsto xy$.

Now, an incorrect explanation of what we want to do: assume that everything is a scheme of finite type. Let $K_1, K_2 \in \mathbf{Perv}(\mathrm{Gr}_G)$; then we want to define $K_1 \boxtimes K_2 \in \mathbf{Perv}(\mathrm{Gr}_G \times \mathrm{Gr}_G)$ such that $p^*(K_1 \boxtimes K_2)$ is a shifted perverse sheaf equivariant for the action $g \cdot (x, y) = (xg, y)$. If $K_2 \in \mathbf{Sat}(G)$ then $p^*(K_1 \boxtimes K_2)$ is $G[\![t]\!] \times G[\![t]\!]$-equivariant for the action $(g_1, g_2) \cdot (x, y) = (xg_1, g_2 y)$. In particular it is $G[\![t]\!]$-equivariant for the action $g \cdot (x, y) = (xg^{-1}, gy)$. So there exists a unique $K_1 \widetilde{\boxtimes} K_2 \in \mathbf{Perv}(G(\!(t)\!) \times^{G[\![t]\!]} \mathrm{Gr}_G)$ with

$$p^*(K_1 \boxtimes K_2) = q^*(K_1 \widetilde{\boxtimes} K_2).$$

Then set the convolution product to be $K_1 * K_2 = m_*(K_1 \widetilde{\boxtimes} K_2)$.

If $K_1 \in \mathbf{Sat}(G)$, then $K_1 \widetilde{\boxtimes} K_2$ is $G[\![t]\!]$-equivariant for $g(x, y) = (gx, y)$, so $K_1 * K_2$ is $G[\![t]\!]$-equivariant. We still need to prove perversity, and more importantly actually define the things we want in the context we have (where things aren't schemes of finite type).

How it actually works: Fix $K_1, K_2 \in \mathbf{Perv}(\mathrm{Gr}_G)$. Choose $Z \subseteq \mathrm{Gr}_G$ closed and $G[\![t]\!]$-stable of finite type such that $Z \supseteq \mathrm{supp}(K_1) \cup \mathrm{supp}(K_2)$. Let $H \subseteq G[\![t]\!]$ be a closed subgroup such that $G' = G[\![t]\!]/H$ is a linear algebraic group and such that H acts trivially on Z. Then have $p : p^{-1}[Z \times Z] \to Z \times Z$, which is a $G[\![t]\!]$-torsor, and this factors through the quotient map $p^{-1}[Z \times Z]$ to $Y = p^{-1}[Z \times Z]/H$, giving a

G'-torsor $p' : Y \to Z \times Z$. Similarly have $q : p^{-1}[Z \times Z] \to q[p^{-1}[Z \times Z]]$ and get $q' : Y \to q[p^{-1}[Z \times Z]]$ which is also a G'-torsor. Then do the exact same thing described above but for this restricted setting rather than the full spaces. So if $K_2 \in \mathbf{Sat}(G)$ there exists a unique $K_1 \widetilde{\boxtimes} K_2 \in \mathbf{Perv}(q[p^{-1}[Z \times Z]])$ such that $(p')^*(K_1 \boxtimes K_2) = (q')^*(K_1 \widetilde{\boxtimes} K_2)$, define $K_1 * K_2 = m_*(K_1 \widetilde{\boxtimes} K_2)$. This is $G[\![t]\!]$-equivariant if K_1 is.

Theorem: (1) For all $K_1 \in \mathbf{Perv}(\mathrm{Gr}_G)$ and $K_2 \in \mathbf{Sat}(G)$ the product $K_1 * K_2$ is perverse.
(2) If $k = \overline{k}$ then $(\mathbf{Sat}(G), *)$ has a unique symmetric monoidal structure such that $\omega : \mathbf{Sat}(G) \to \overline{\mathbb{Q}}_\ell\text{-}\mathbf{vect}$ given by $K \mapsto \bigoplus_{i \in I} H^i(\mathrm{Gr}_G, K)$ is symmetric monoidal and ω is additive, exact, and faithful.

The proof of this theorem will take us a while and a bit of machinery. We'll start by introducing global versions of everything. What do we mean by this? Let X/k be a smooth curve, geometrically connected. Recall that if $x \in X(k)$ and $\widehat{\mathcal{O}}_{X,x} = k[\![t]\!]$ we get $\mathrm{Gr}_G \cong \mathrm{Gr}_G^{\mathrm{glob}}$ via

$$\operatorname{Spec} R \mapsto \{(M, \gamma) : M \in BG(X_R), \gamma : M|_{X_R^\circ} \cong M^0_{X_R^\circ}\}/\sim,$$

where $X^\circ = X \setminus \{x\}$, M^0 is the trivial G-bundle on everything, and BG is the stack of G-bundles.

What about the loop group $G(\!(t)\!)$? For all k-algebras R, let $D_R = \operatorname{Spec} R[\![t]\!] \supseteq D_R^\circ = \operatorname{Spec} R(\!(t)\!)$. Let $G(\!(t)\!)^{\mathrm{glob}}$ be the k-space

$$\operatorname{Spec} R \mapsto \{(M, \gamma, \delta) : (M, \gamma) \in \mathrm{Gr}_G^{\mathrm{glob}}(\operatorname{Spec} R), \delta : M|_{D_R} \cong M^0|_{D_R}\}/\sim .$$

Let $G[\![t]\!]^{\mathrm{glob}}$ be the map

$$\operatorname{Spec} R \mapsto \{(M, \gamma, \delta) \in G(\!(t)\!)^{\mathrm{glob}}(\operatorname{Spec} R) : \delta\gamma^{-1} \in \operatorname{Aut}(M^\circ_{D_R^\emptyset}) \cong G(R(\!(t)\!)) \text{ is actually in } G(R[\![t]\!])\}.$$

What make this work is the Beauville-Laszlo theorem; this tells us $BG(X_R)$ is described by the category of gluing data (M, N, β) for $M \in BG(X_R^\circ)$, $N \in BG(D_R)$, and $\beta : M|_{D_R^\circ} \cong N|_{D_R^\circ}$ is an isomorphism. Get that

$$G(\!(t)\!)^{\mathrm{glob}}(\operatorname{Spec} R) \cong \{(M, N, \delta, \gamma, \beta)\}/\sim$$

where each of the things in this tuple is as above. Then, can define a map $G((t)) \to G((t))^{\mathrm{glob}}$ by $g \mapsto (M^0, X_R^\circ, M^0_{D_R}, 1, 1, g^{-1})$ and this is an isomorphism by BL. Moreover, $G[\![t]\!]$ gets identified with $G[\![t]\!]^{\mathrm{glob}}$ and the projection map $G((t)) \to \mathrm{Gr}_G$ corresponds to the projection map $G((t))^{\mathrm{glob}} \to \mathrm{Gr}_G^{\mathrm{glob}}$ given by $(M, \gamma, \delta) \mapsto (M, \gamma)$.

The Beilinson-Drinfeld affine Grassmannian (G can be any smooth affine group scheme): Define $\mathcal{G}r_X$ by

$$\operatorname{Spec} R \mapsto \{(x, M, \gamma) : x \in X(R), M \in BG(X_R), \gamma : M|_{X_R^\circ} \cong M^0_{X_R^\circ}\}/\sim,$$

where X_R° is X_R minus Γ_x, the graph of X. Then we have a map $\mathcal{G}r_X \to X$ given by $(x, M, \gamma) \mapsto x$. Why stop here? Let $\mathcal{G}r_{X^n}$ be the k-space $\operatorname{Spec} R \mapsto \{(x_1, \ldots, x_n, M, \gamma)\}/\sim$ where $x_i \in X(R)$, $M \in BG(X_R)$, and γ is an isomorphism of M with M^0 over $X_R \setminus \bigcup \Gamma_{x_i}$. Similarly have a map $\mathcal{G}R_{X^n} \to X^n$.

Remark: If $x_1, \ldots, x_n \in X(k)$ are such that $x_i \neq x_j$ for all $i \neq j$, then $\mathcal{G}r_{X^n}|_{(x_1,\ldots,x_n)} \cong \mathrm{Gr}_G \times \cdots \times \mathrm{Gr}_G$. If $x \in X(k)$ then $\mathcal{G}r_{X^n}|_{(x,\ldots,x)} \cong \mathrm{Gr}_G$. If $\Delta_n \subseteq X^n$ is the fat diagonal (the set of all tuple $(x_1, \ldots, x_n)$ with $x_i = x_j$ for some $i \neq j$) then $\mathcal{G}r_{X^n}|_{X^n \setminus D_n} \cong \mathcal{G}r_X \times \cdots \times \mathcal{G}r_X$.

Exercise: $\mathcal{G}r_{\mathbb{A}^1} = \mathbb{A}^1 \times \mathrm{Gr}_G$.

We salso have global versions of $G[\![t]\!]$ and $G(\!(t)\!)$. Define $\mathcal{G}(\!(t)\!)_{X^n}$ by mapping $\operatorname{Spec} R$ to $\{(x_1, \ldots, x_n, M, \gamma, \delta)\}/\sim$ for $(x_1, \ldots, x_n, M, \gamma) \in \mathcal{G}r_{X^n}(\operatorname{Spec} R)$ and $\delta : M|_{D_{(x_i)}} \cong M^0_{D_{(x_i)}}$. Here, if $(X_R|_{\bigcup \Gamma_{x_i}})^\wedge = \operatorname{Spf} A$ we take $D_{(x_i)} = \operatorname{Spec} A$ and $D^\circ_{(x_i)} = D_{(x_i)} \setminus \bigcup \Gamma_{x_i}$. Then $\mathcal{G}[\![t]\!]_{X^n}$ is defined by mapping $\operatorname{Spec} R$ to the subset of classes $(x_1, \ldots, x_n, M, \gamma, \delta)$ such that $\delta_\gamma^{-1} \in \operatorname{Aut}(M^0_{D^\circ_{(x_i)}}) = G(\mathcal{O}_{D^\circ_{(x_i)}})$ is actually in $G(\mathcal{O}_{D_{(x_i)}})$.

Multiplication: $\mathcal{G}(\!(t)\!)_{X^n}$ is a group k-space over X^n with product $(x_1, \ldots, x_n, M, \gamma, \delta) \cdot (x_1, \ldots, x_n, M', \gamma', \delta') = (x_1, \ldots, x_n, N, \alpha, \beta)$ where $N \in BG(X_R)$ corresponds to the gluing data $(M|_{X_R \setminus \bigcup \Gamma_{x_i}}, M'|_{D(x_i)}, (\gamma')^{-1}\delta)$, $\alpha = \gamma$, and $\beta = \delta$. We also have a map $\mathcal{G}(\!(t)\!)_{X^n} \to \mathcal{G}r_{X^n}$ bu $(x_1, \ldots, x_n, M, \gamma, \delta) \mapsto (x_1, \ldots, x_n, M, \gamma)$ which induces an isomorphism $\mathcal{G}(\!(t)\!)_{X^n}/\mathcal{G}[\![t]\!]_{X^n} \cong \mathcal{G}r_{X^n}$.

Lemma: $\mathcal{G}[\![t]\!]_{X^n}$ is a scheme. $\mathcal{G}(\!(t)\!)_{X^n}$ is an ind-scheme. $\mathcal{G}r_{X^n}$ is an ind-scheme of ind-finite type, which is ind-projective over X^n iff G is reductive. (Proof: Reduce to $G = \mathrm{GL}_n$ and use the Grassmannian; follow the proof for the original affine Grassmannian).

11 Lecture - 04/02/2014

Beilinson-Drinfeld affine Grassmannians, continuing from last time: had $\mathcal{G}r_{X^n}$ given by $\operatorname{Spec} R \mapsto [x_1, \ldots, x_n, M, \gamma]$, $\mathcal{G}(\!(t)\!)_{X^n}$ given by $\operatorname{Spec} R \mapsto [x_1, \ldots, x_n, M, \gamma, \delta]$, with subgroup $\mathcal{G}[\![t]\!]_{X^n}$.

Today (for doing geometric Satake) will only need cases $n = 1, 2$. However, introduce notation $\mathcal{G}r_n = \mathcal{G}r_{X^n}$, and so on, and write $\mathcal{G}r = \mathcal{G}r_1$.

Remark: If $X = \mathbb{A}^1$ then $\mathcal{G}r \cong \mathrm{Gr}_G \times \mathbb{A}^1$. (Proof: Comes from the simply transitive action $\mathbb{G}_a$ on $\mathbb{A}^1$ by translations; for $a \in \mathbb{G}_a$ let u_a be the action map. We can lift this to an action ν_a on $\mathcal{G}r$ by $(x, M, \gamma) \mapsto (x + a, \mu_{-a}^* M, \mu_{-a}^* \gamma)$. The actions ν_a and μ_a are compatible with the projection $\mathcal{G}r \to \mathbb{A}^1$. Since $\mathrm{Gr}_G = \mathcal{G}r_0$ get an isomorphism $\mathrm{Gr}_G \times \mathbb{G}_a \to \mathcal{G}r$ by $(p, a) \mapsto v_a(p)$).

Remark 2: Virasoro action. The group that acts on Gr_G is not $\mathbb{G}_m[\![t]\!]$ but $\mathrm{Aut}_{k\text{-}\mathbf{alg}}(k[\![t]\!])$. Since $\mathcal{G}r \to X$ is an étale locally trivial Gr_G-bundle, the transition isomorphisms are given by the Virasoro action.

Remark 3: Let $\Delta \subseteq X \times X$ be the diagonal. Then $\mathcal{G}r_2|_{X^2 \setminus \Delta} = (\mathcal{G}r \times \mathcal{G}r)|_{X^2 \setminus \Delta}$ and $\mathcal{G}r_2|_\Delta = \mathcal{G}r$.

The local convolution diagram: Wanted to have convolution map $\mathrm{Gr}_G \times \mathrm{Gr}_g \to \mathrm{Gr}_G$. Started by lifting via $p = \pi \times \mathrm{id}$ to $G(\!(t)\!) \times \mathrm{Gr}_G$, which has the action map to Gr_G. But this is a bundle so want to descend it via a map $q : G(\!(t)\!) \times \mathrm{Gr}_G \to G(\!(t)\!) \times^{G[\![t]\!]} \mathrm{Gr}_G$. Then for $K_1, K_2 \in \mathbf{Sat}(G)$ there exists a unique $K_1 \widetilde{\boxtimes} K_2$ on $G(\!(t)\!) \times^{G[\![t]\!]} \mathrm{Gr}_G$ such that "$q^*(K_1 \widetilde{\boxtimes} K_2) = p^*(K_1 \boxtimes K_2)$. By definition took $K_1 * K_2 = m_*(K_1 \widetilde{\boxtimes} K_2)$.

Global convolution diagram. Let $\widetilde{\mathcal{G}}$ be the thing sending $\operatorname{Spec} R$ to $(x_1, x_2, N_1, N_2, \delta_1, \delta_2)$ modulo equivalence, where $x_1, x_2 \in X(R)$, $N_1, N_2 \in BG(X_R)$, and δ_i are isomorphisms of N_i with M^0 on $X_R \setminus \Gamma_{x_i}$. Now, as before start with $\mathcal{G}r \times \mathcal{G}r$, have map p^{glob} from $\mathcal{G}(\!(t)\!) \times \mathcal{G}r$ to this, have map $q^{\mathrm{glob}} : \mathcal{G}(\!(t)\!) \times \mathcal{G}r \to \widetilde{\mathcal{G}}$, and map $m^{\mathrm{glob}} : \widetilde{\mathcal{G}} \to \mathcal{G}r$; and all of these live over X^2.

Note: If we take the fiber over (x, x), get back the local diagram. As before can define the convolution product using this diagram. If $K_1 \in \mathbf{Perv}_{\mathcal{G}[\![t]\!]_1}(\mathcal{G}_r)$ and $K_2 \in \mathbf{Perv}(\mathcal{G}_r)$ then there exists a unique $K_1 \widetilde{\boxtimes} K_2 \in \mathbf{Perv}(\widetilde{\mathcal{G}})$ such that " $q^{\mathrm{glob}*}(K_1 \widetilde{\boxtimes} K_2) = p^{\mathrm{glob}*}(K_1 \boxtimes K_2)$. We set $K_1 *^{\mathrm{glob}} K_2 = m_*^{\mathrm{glob}}(K_1 \widetilde{\boxtimes} K_2)$. This is $\mathcal{G}[\![t]\!]$-equivariant if K_2 is.

Case $X = \mathbb{A}^1$: $\mathcal{G}r = \mathrm{Gr}_G \times \mathbb{A}^1$, and project to Gr_G via π_1. If $L_1 \in \mathbf{Sat}(G)$ and $L_2 \in \mathbf{Perv}(\mathrm{Gr}_G)$ then set $K_i = \pi_1^* L_i[1] = L_i \boxtimes \overline{\mathbb{Q}}_{\ell, \mathbb{A}^1}[1]$. Then we have

$$(K_1 *^{\mathrm{glob}} K_2)|_\Delta = (L_1 * L_2) \boxtimes \overline{\mathbb{Q}}_{\ell, \mathbb{A}^1}[1].$$

Proof: Over Δ, the global diagram is the local diagram crossed with $\mathbb{A}^1$.

The fusion product: Have inclusions $j : \mathcal{G}r_2|_{X^2 \setminus \Delta} \hookrightarrow \mathcal{G}r_2$ and $i : \mathcal{G}r \cong \mathcal{G}r_2|_\Delta \hookrightarrow \mathcal{G}r_2$. Also, know $\mathcal{G}r_2|_{X^2 \setminus \Delta} \cong (\mathcal{G}r \times \mathcal{G}r)|_{X^2 \setminus \Delta}$ and can take inclusion $j' : (\mathcal{G}r \times \mathcal{G}r)|_{X^2 \setminus \Delta} \to \mathcal{G}r \times \mathcal{G}r$. Definition: If $K_1, K_2 \in \mathbf{Perv}(\mathcal{G}r)$, their fusion product is

$$K_1 \star K_2 = j_{!*}((j')^* K_1 \boxtimes K_2) \in \mathbf{Prev}(\mathcal{G}r_2).$$

Remark: If K_1 is $\mathcal{G}[\![t]\!]$-equivariant then $j^*(K_1 \star K_2) = j^*(K_1 *^{\mathrm{glob}} K_2)$. (Exercise).

Theorem: If K_1, K_2 are universally locally acyclic (ULA) with respect to $\mathcal{G}r \to X$, then $i^*(K_1 \star K_2)$ is perverse. Moreover, if K_1 is $\mathcal{G}[\![t]\!]$-equivariant then $K_1 \star K_2 = K_1 *^{\mathrm{glob}} K_2$.

In particular, if $X = \mathbb{A}^1$ and $K_i = L_i \boxtimes \overline{\mathbb{Q}}_{\ell, \mathbb{A}^1}[1]$ then $L_1 * L_2 = i^*(K_1 \star K_2)[-1]$ so $L_1 * L_2$ is perverse and we get commutativity isomorphism $L_1 * L_2 \cong L_2 * L_1$. (However we'll have to modify this commutativity constraint by a sign later on, because we need compatibility with global sections...)

Nearby and vanishing cycles. Will take $k = \overline{k}$ for convenience. Consider the following situation (all new notation). Let X be a k-scheme of finite type, $f : X \to \mathbb{A}^1$ a function. Let U be the inverse image of $\mathbb{A}^1 \setminus \{0\}$ and $j : U \to X$ the inclusion; also let Y be the fiber over 0 and $i : Y \to X$ the inclusion.

Let $S = \operatorname{Spec} \mathcal{O}^h_{\mathbb{A}^1, (0)}$ (the Henselianization). Have $s \hookrightarrow S$ (the special point) and $\eta \hookrightarrow S$ (generic point); if we take $\overline{S}$ to be the normalization (integral closure); get $s \hookrightarrow \overline{S}$ still and $\overline{\eta} \hookrightarrow \overline{S}$. Can base change our entire original picture to get $\overline{i} : Y \to \overline{X} = X_{\overline{S}}$ and $\overline{j} : X_{\overline{\eta}} \to X_{\overline{S}}$. Also have $\pi : X_{\overline{\eta}} \to U$.

What are nearby cycles? If $K \in D_c^b(U)$, set $\psi K = \overline{i}^* \overline{j}_* \pi^* K \in D_c^b(Y)$ which has an action of $\pi_1^{\text{ét}}(\eta, \overline{\eta})$. (This action is quasi-unipotent). Let T be a generator of the prime-to-p part of $\pi_1^{\text{ét}}(\eta, \overline{\eta})$ (where p is the characteristic of k). We have an exact triangle

$$i^*j_*K \to \psi_f K \to \psi_f K \to \cdots .$$

We then write N for the logarithm of the unipotent part of T acting on $\psi_f K$.

Vanishing cycles: If $K \in D^b_c(X)$ then have $i^*K \to i^*j_*j^*K$ by adjunction and then a map from this to $\psi_f j^*K$. The cone of that map is the complex of vanishing cycles $\Phi_f K$. There is a way to make this functorial. We get an exact triangle

$$i^*K \to \psi_f j^*K \to \Phi_f K \to \cdots .$$

Theorem: (i): If $K \in \mathbf{Perv}(U)$ then $\psi_f K[-1] \in \mathbf{Perv}(Y)$.
(ii) If $K \in D^b_c(X)$ and $K|_U \in \mathbf{Perv}(U)$ then TFAE:
(a) $K = j_{!*}j^*K$ and $i^*K[-1] \in \mathbf{Perv}(Y)$.
(b) $\Phi_f K = 0$ and the unipotent part of T is 1.

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Situation: $k = \overline{k}$ with characteristic p, have $f : X \to \mathbb{A}^1$ and fibers Y, U over $\{0\}, \mathbb{A}^1 \setminus \{0\}$. We let $S = \operatorname{Spec} \mathcal{O}^h_{\mathbb{A}^1 \setminus \{0\}}$ (where the h means henselization), let η be the generic point of S, and $I = \pi_1^{\text{ét}}(\eta, \overline{\eta})$ which we think of as an inertia group. Had a nearby cycle functor $\psi_f : D^b_c(U) \to D^b_C(Y)$ and vanishing cycles functor $\Phi_f : D^b_c(X) \to D^b_c(Y)$ (both with actions of inertia).

Recall: Have $1 \to P \to I \to I_t \to 1$ with I_t the tame inertia (isomorphic to $\widehat{\mathbb{Z}}^p(1)$) and P the wild inertia (a pro-p group). Facts (correcting what we had last time):
(1) $i^* j_* K = R\Gamma(I, \psi_f K) = R\Gamma(I_t, (\psi_f K)^P)$. We all $(\psi_f K)^P$the tame nearby cycles $\psi^t_f K$. If T is a pro-generator of I_t then we have an exact triangle $i^* j_* K \to \psi_f^t K \to \psi^t_f K \to \cdots$ (where the second map is defined by $T - 1$).
(2) We have an exact triangle $i^* K \to \psi_f j^* K \to \Phi_f K \to \cdots$.

Some properties of ψ_f and Φ_f:
(A) ψ_f and Φ_f commute with duality (i.e. $\psi_f(DK) \cong D(\psi_f K)$, etc.).
(B) Base change: If you have $g : X' \to X$ and U' lying over U, have $\psi_f g_* \to g_* \psi_{fg}$ and $g^* \psi_f \to \psi_{gf} g^*$ (and similarly for Φ). Proper base change says the first map is an isomorphism if g is proper, and smooth base change says the second is an isomorphism if g is smooth.

Example: (a): $f = \mathrm{id} : X = \mathbb{A}^1 \to \mathbb{A}^1$. Then $\psi_{\mathrm{id}} \overline{\mathbb{Q}}_{\ell, \mathbb{G}_m} = \overline{\mathbb{Q}}_\ell$ and $\Phi_{\mathrm{id}} \overline{\mathbb{Q}}_{\ell, \mathbb{A}^1} = 0$. (b): If $f : X \to \mathbb{A}^1$ is smooth then $\psi_f \overline{\mathbb{Q}}_{\ell, U} = \overline{\mathbb{Q}}_{\ell, Y}$ and $\Phi_f \overline{\mathbb{Q}}_{\ell, X} = 0$.

Theorem (from last time):
(1) $\psi_f[-1]$ sends $\mathbf{Perv}(U)$ to $\mathbf{Perv}(Y)$.
(2) Let $K \in D^b_c(X)$ be such that $j^* K \in \mathbf{Perv}(U)$. Then TFAE:
(a) $K = j_{!*} j^* K$ (which means that if K is in $\mathbf{Perv}(X)$ then $i^* K[-1] \in \mathbf{Perv}(Y)$).
(b) $\Phi_f K = 0$.

Proof: (i) is in BBD. For (ii), start by showing (b) $\implies$ (a). If $\Phi_f K = 0$ then $i^* K = \psi_f j^* K$. By (i), $i^* K[-1] \in \mathbf{Perv}(Y)$ so $K \in^p D^{\leq 0}$. We also have $\Phi_f(DK) = 0$ and $j^* DK \in \mathbf{Pev}(U)$ so $DK \in^p D^{\leq 0}$ so $K \in \mathbf{PervX}$. If $K \twoheadrightarrow L$ is in $\mathbf{Perv}(X)$ with $L = i_* i^* L$ then ${}^p H^0 i^* K \twoheadrightarrow i^* L$ because i^* is right-exact; but ${}^p H^0 i^* K = 0$ and thus $i^* L = 0$ and thus $L = 0$. So K has no quotient supported on Y and neither does DK (and thus K has no subobjects there) so $K = j_{!*} j^* K$ by one of the characterizations of the middle extension.

Back to the convolution diagram (which we screwed up last time) - can't use $\mathcal{G}(\!(t)\!) \times \mathcal{G}r$ in the middle. Instead we need another space $\mathcal{G}(\!(t)\!) \widetilde{\times} \mathcal{G}r$ which has maps p^{glob} and q^{glob} to $\mathcal{G}r \times \mathcal{G}r$ and to $\widetilde{\mathcal{G}r}$, and a map $m^{\text{glob}} \widetilde{\mathcal{G}r} \to \mathcal{G}r_2$. (Everything lies over $X \times X$. Recall X/k is a smooth curve, G/k is connected reductive, and $k = \overline{k}$).

What's this new space? Define it by

$$\mathcal{G}(\!(t)\!) \widetilde{\times} \mathcal{G}r(\operatorname{Spec} R) = \{(x_1, x_2, M_1, M_2, \gamma_1, \gamma_2, \beta_1)\}/\sim$$

where $(x_i, M_i, \gamma_i) \in \mathcal{G}r(\operatorname{Spec} R)$ and β_1 is an isomorphism between M_1 and M^0 on D_{x_2}. The map p^{glob} is the obvious one (forgetting β_1) and q^{glob} is the one going to $(x_1, x_2, M_1, M_2', \gamma_1, \gamma_2')$ (using the definition of $\widetilde{\mathcal{G}}$ from last time) where M_2' corresponds via B-L to the gluing data $(M_2|_{X_R \setminus \Gamma_{X^2}}, M_1|_{D_{X^2}}, \beta_1^{-1}\gamma_2)$ and γ_2' is the gluing of $(\gamma_1^{-1}\gamma_2, \gamma_1^{-1}\beta_1)$.

In the case $X = \mathbb{A}^1$: Have $X = \Delta \subseteq X^2$ the diagonal, and $\mathcal{G}r_2|_\Delta = \mathcal{G}r$ and $\mathcal{G}r_2|_{X^2 \setminus \Delta} = (\mathcal{G}r \times \mathcal{G}r)|_{X^2 \setminus \Delta}$. Over Δ remember that $\mathcal{G}r \cong \mathrm{Gr} \times \mathbb{A}^1$ and the global diagram becomes isomorphic to the local diagram times $\mathbb{A}^1$. Over $X^2 \setminus \Delta$ the $\mathcal{G}r \times \mathcal{G}r$ is isomorphic to $\mathcal{G}r_2$; claim that if we put in this isomorphism the diagram commutes (so that $(K_1, K_2) \in \mathcal{G}r \times \mathcal{G}r$ corresponds to $K_1 \boxtimes K_2$ in $\mathcal{G}r_2$ under the isomorphism).

Proof of commutativity: Fix x_1, x_2 with $\Gamma_{x_1} \cap \Gamma_{x_2} = \emptyset$. Then fix an element of $\mathcal{G}(\!(t)\!) \widetilde{\times} \mathcal{G}r$ over (x_1, x_2); this is tuple $(M_1, M_2, \gamma_1, \gamma_2, \beta_1)$. We know p^{glob} takes this to the pair (M_1, γ_1) and (M_2, γ_2) and the isomorphism takes it to some (N, γ). Also, q^{glob} takes it to $(N_1, N_2, \delta_1, \delta_2)$ and then m^{glob} goes to some (N', γ'); we need to show $(N, \gamma) = (N', \gamma')$.

What are all of these things? Well, by B-L we know N corresponds to the gluing data of $M^0|_{X_R \setminus \Gamma_{x_1, x_2}}$,

$\gamma_2 : M_2|_{D_{x_1}} \cong M^0$, and $\gamma_1 : M_1|_{D_{x_2}} \cong M^0$, and $\gamma = \mathrm{id}$. Also $(N_1, \delta_1) = (M_1, \gamma_1)$ and N_2 corresponds to $(M_1|_{X_R \setminus \Gamma_{x_2}}, \beta_1^{-1}\gamma_2)$ and $\delta_2 = \mathrm{id}$. Finally, $N' = N_2$ and $\gamma' = \delta_1\delta_2 = \gamma_1$.

Or does the diagram maybe not commute? Not sure... But can replace it by defining a section $s : \mathcal{G}r \times \mathcal{G}r \to \widetilde{\mathcal{G}(\!(t)\!)} \times \mathcal{G}r$ (by taking $\beta_1 = \gamma_1|_{D_{x_2}}$) and can show that going from $\mathcal{G}r \times \mathcal{G}r$ all of the way over to $\mathcal{G}r_2$ via this section and q^{glob} and m is the same as the usual isomorphism.

Hence: $(K_1 *^{\mathrm{glob}} K_2)|_{X^2\setminus\Delta} = (K_1 \boxtimes K_2)|_{X^2\setminus\Delta}$. Recall that we wanted to prove $K_1 *^{\mathrm{glob}} K_2 = m_*^{\mathrm{glob}}(K_1 \widetilde{\boxtimes} K_2)$ is equal to $j_{!*}((K_1 \boxtimes K_2)|_{X^2\setminus\Delta})$. But we were only going to prove it when $K_i = L_i \boxtimes \overline{\mathbb{Q}}_{\ell,\mathbb{A}^1}[1]$ for $L_i \in \mathbf{Perv}(\mathrm{Gr}_G)$. How will we prove this? By showing the vanishing cycles are zero; need that $\Phi_f(K_1 *^{\mathrm{glob}} K_2) = 0$, where $f : \mathcal{G}r_2 \to X^2 \to X$ is the structure map for $\mathcal{G}r_2$ composed with the map $(x_1, x_2) \mapsto x_1 - x_2$. As m^{glob} is ind-projective, it's enough to show $\Phi(K_1 \widetilde{\boxtimes} K_2) = 0$. (Why is m^{glob} ind-projective? Its fibers are twisted Gr_G's and can conclude it from that).

Why is $\Phi(K_1 \widetilde{\boxtimes} K_2) = 0$? Recall it's defined so that $p^{-1}[K_1 \widetilde{\boxtimes} K_2] \cong q^{-1}[K_1 \widetilde{\boxtimes} K_2]$. But this was a bit of a lie since we haven't defined this inverse image of such things; actually we had p and q factor through $\mathcal{Z}$, giving p', q' that are torsors under a finite type quotient of $\mathcal{G}[\![t]\!]$ and required $(p')^*[K_1 \boxtimes K_2] = (q')^*[K_1 \widetilde{\boxtimes} K_2]$. But these p', q' are smooth so by smooth base change we need to show $\Phi(K_1 \boxtimes K_2) = 0$.

This is where we need some hypothesis since it's not true that $\Phi(K_1 \boxtimes K_2) = 0$ in general. But in our case, $K_i = L_i \boxtimes \overline{\mathbb{Q}}_{\ell,\mathbb{A}^1}[1]$. So

$$\Phi(K_1 \boxtimes K_2) = (L_1 \boxtimes L_2) \boxtimes \Phi(\overline{\mathbb{Q}}_\ell \boxtimes \overline{\mathbb{Q}}_\ell)[2]$$

by identifying $(\mathcal{G}r \times \mathcal{G}r)|_\Delta \cong \mathrm{Gr}_G \times \mathrm{Gr}_G \times \Delta$. But then we have a map $\mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbb{A}^1 \to \mathcal{G}r \times \mathcal{G}r \to \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ defined by $(x_1, x_2) \mapsto x_1 - x_2$ on the last map, which is smooth, and thus get $\Phi(\overline{\mathbb{Q}}_\ell \boxtimes \overline{\mathbb{Q}}_\ell) = 0$.

This finishes the proof of the theorem that the fusion product equals the convolution product.

Corollary:

$$(L_1 * L_2) \boxtimes \overline{\mathbb{Q}}_{\ell,\mathbb{A}^1}[1] = i^*(K_1 *^{\mathrm{glob}} K_2)[-1] = \psi_f(K_1 \boxtimes K_2|_{X^2\setminus\Delta}[-1])$$

is perverse. (Proof: From the exact triangle $i^* \to \psi_f \to \Phi_f$).

Corollary: Let $a : \mathrm{Gr}_G \to \operatorname{Spec} k$ be the structural map. Then if $L_1 \in \mathbf{Sat}(G)$ and $L_2 \in \mathbf{Perv}(\mathrm{Gr}_G)$, we have $a_*(L_1 * L_2) = (a_* L_1) \otimes^L (a_* L_2)$ and this is compatible with the commutativity isomorphisms of $*$ and $\otimes$. (Proof: Proper base change).

Now: We want our fiber functor to be $\omega : \mathbf{Sat}(G) \to \mathbf{Vect}(\overline{\mathbb{Q}}_\ell)$ given by $K \mapsto \bigoplus_{i\in\mathbb{Z}} H^i(\mathrm{Gr}_G, K)$. But this $\bigoplus H^i : D^b(\mathbf{Vect}(\overline{\mathbb{Q}}_\ell)) \to \mathbf{Vect}(\overline{\mathbb{Q}}_\ell)$ is not compatible with the commutativity constraints (there's a sign problem). We'll fix this by modifying the commutativity constraint on $*$.

13 Lecture - 04/09/2014

Parity vanishing. Recap: $k = \overline{k}$, G connected reductive. Have defined $* : \mathbf{Sat}(G) \times \mathbf{Sat}(G) \to \mathbf{Sat}(G)$, and have seen that this makes $\mathbf{Sat}(G)$ a symmetric monoidal category. Have also said that $a_* : \mathbf{Sat}(G) \to D^b(\mathbf{Vect}_{\overline{\mathbb{Q}}_\ell})$ is a tensor functor, for $a : \mathrm{Gr}_G \to \operatorname{Spec} k$ the obvious map. Why? For $X = \mathbb{A}^1$, have diagram

$$\begin{tikzcd}
\mathcal{G}r_2|_{X^2 \setminus \Delta} \arrow[r, "j"] \arrow[d] & \mathcal{G}r_2 \arrow[d, "h"] & \mathcal{G}r_2|_\Delta \arrow[l, "i"'] \arrow[d, "a"] \\
X^2 \setminus \Delta \arrow[r] & X^2 \arrow[d, "s"] & \Delta \arrow[l] \\
& X &
\end{tikzcd}$$

If $L_1, L_\in \mathbf{Sat}(G)$ and $K_i = L_i \boxtimes \overline{\mathbb{Q}}_{\ell,X}[1]$ then

$$(L_1 * L_2) \boxtimes \overline{\mathbb{Q}}_{\ell,X}[1] = \Psi_{s \circ h}(K_1 \boxtimes K_2|_{X^2 \setminus \Delta})[-1].$$

So

$$a_*(L_1 * L_2) = a_* \Psi_{s\circ h}(K_1 \boxtimes K_2|_{X^2\setminus\Delta})[-2]_0 = \Psi_s(a \times a)_*(K_1 \boxtimes K_2|_{X^2\setminus\Delta})[-2]_0 = a_* L_1 \otimes a_* L_2 \otimes (\Psi_s \overline{\mathbb{Q}}_\ell)_0,$$

and the last thing is just $\overline{\mathbb{Q}}_\ell$.

Problem: we want $\omega : \mathbf{Sat}(G) \to \mathbf{Vect}_{\overline{\mathbb{Q}}_\ell}$ given by $\omega(K) = \bigoplus_{i \in \mathbb{Z}} H^i(\mathrm{Gr}_G, K)$ to be symmetric monoidal. But it isn't because $H^* : D^b(\mathbf{Vect}_{\overline{\mathbb{Q}}_\ell}) \to \mathbf{Vect}_{\overline{\mathbb{Q}}_\ell}$ isn't; the way we defined our commutativity constraints in the derived category gives us some sign problems.

Solution: We modify the isomorphism $IC_\lambda * IC_\mu \cong IC_\mu * IC_\lambda$ by multiplying it by $(-1)^{\langle 2\rho, \lambda+\mu\rangle}$. This works thanks to the parity vanishing theorem:

Theorem (Lusztig): For all $\lambda \in X_*(T)^+$, $H^*(IC_\lambda)$ is concentrated in degree $\equiv \langle 2\rho, \lambda \rangle \pmod 2$.

Proof (Ngo-Polo): Write $O_\lambda = Orb(t^\lambda)$. Start by considering minimal elements in $X_*(T)^+ \setminus \{0\}$.

Lemma: Let $\lambda \in X_*(T)^+ \setminus \{0\}$ be minimal. Then there are two possibilities:
(i) Either λ is minuscule (i.e. $\langle \lambda, \alpha \rangle \leq 1$ for all $\alpha \in \Phi^+$ and then λ is minimal in $X_*(T)^+$; or
(ii) $\lambda = \gamma^\vee$ is a coroot for γ a maximal root. (Call this λ being quasi-minuscule).

In case (i), $\overline{O}_\lambda = O_\lambda$ is smooth so

$$H^*(IC_\lambda) = H^*(O_\lambda, \overline{\mathbb{Q}}_\ell)[\langle 2\rho, \lambda \rangle].$$

But we've seen that O_λ is paved by affine spaces, so $H^*(O_\lambda, \overline{\mathbb{Q}}_\ell)$ is concentrated in even degree.

In case ii), $\overline{O}_\lambda = O_\lambda \cup O_0$ with O_0 a point. Let

$$P = \{g \in G : \lim t \to \infty t^\lambda g t^{-\lambda} \text{ exists}\}.$$

This is a parabolic subgroup and there exists $\mathcal{L} \to G/P$ an $\mathbb{A}^1$-bundle with $\mathcal{L} \cong O_\lambda$. Can show that there exists a $\mathbb{P}^1$-bundle $\overline{\mathcal{L}} \supseteq \mathcal{L}$ such that $\mathcal{L} \cong O_\lambda$ extends to $\overline{\mathcal{L}} \twoheadrightarrow \overline{O}_\lambda$. By the decomposition theorem, $H^*(IC_\lambda)$ is a direct factor of $H^*(\overline{\mathcal{L}}, \overline{\mathbb{Q}}_\ell)[\langle 2\rho, \lambda \rangle]$. Since $\overline{\mathcal{L}}$ is a bundle over G/P which is paved by affine spaces, get that this cohomology is concentrated in even degrees.

Next: The n-fold convolution diagram. Remember the (local) convolution diagram

$$\begin{tikzcd}
& G(\!(t)\!) \times \mathrm{Gr}_G \arrow[dl, "p = \pi \times \mathrm{id}"'] \arrow[dr, "q"] & & \\
\mathrm{Gr}_G \times \mathrm{Gr}_G & & G(\!(t)\!) \times^{G[\![t]\!]} \mathrm{Gr}_G \arrow[r, "m"] & \mathrm{Gr}_G
\end{tikzcd}.$$

In fact, we can reinterpret this by noting that $G(\!(t)\!) \times^{G[\![t]\!]} \mathrm{Gr}_G \cong \mathrm{Gr}_G \times \mathrm{Gr}_G$ again; we get this isomorphism by taking $q' : G((t)) \times \mathrm{Gr}_G \to \mathrm{Gr}_G \times \mathrm{Gr}_G$ defined by $(g, x) \mapsto (\pi(g), gx)$, and find this factors through $G(\!(t)\!) \times^{G[\![t]\!]} \mathrm{Gr}_G$ and the resulting map is an isomorphism. Via this isomorphism m just becomes projection onto the second factor.

We use this point of view to get an n-fold convolution diagram:

$$\begin{tikzcd} & G(\!(t)\!)^{n-1} \times \mathrm{Gr}_G \arrow[dl, "p_n = \pi^{n-1} \times \mathrm{id}"'] \arrow[dr, "q_n"] & & \\ \mathrm{Gr}_G^n & & \mathrm{Gr}_G^n \arrow[r, "m_n"] & \mathrm{Gr}_G \end{tikzcd}$$

where m_n is projection onto the last coordinate, and q is given by

$$q(g_1, \ldots, g_{n-1}, x) = (\pi(g_1), \pi(g_1 g_2), \ldots, \pi(g_1 \cdots g_{n-1}), g_1 \cdots g_{n-1} x).$$

Now, if $L_1, \ldots, L_n \in \mathbf{Sat}(G)$ then there exists a unique $L_1 \widetilde{\boxtimes} \cdots \widetilde{\boxtimes} L_n$ such that

$$q_n^*(L_1 \widetilde{\boxtimes} \cdots \widetilde{\boxtimes} L_n) = p_n^*(L_1 \boxtimes \cdots \boxtimes L_n),$$

where again we interpret this as actually being an equality for pullbacks on finite-type subobjects/quotients... Define $L_1 * \cdots * L_n = M_{n*}(L_1 \widetilde{\boxtimes} \cdots \widetilde{\boxtimes} L_n)$. Using a global diagram as before, we get the following interpretation (for $X = \mathbb{A}^1$): if U is X^n minus the fat diagonal and $\Delta = X$ is the thin diagonal, then $\mathcal{G}r^n|_U = \mathcal{G}r_n|_U$ embeds in $\mathcal{G}r_n$ via j, and $\mathcal{G}r_n|_\Delta = \mathcal{G}r \cong \mathrm{Gr}_G \times X$ embeds in it by i. If $K_i = L_i \boxtimes \overline{\mathbb{Q}}_{\ell,X}[1] \in \mathbf{Perv}(\mathcal{G}r)$, then

$$L_1 * \cdots * L_n \boxtimes \overline{\mathbb{Q}}_{\ell,X}[1] = i^* j_{!*}(L_1 \boxtimes \cdots \boxtimes L_n|_U)[1-n],$$

this is perverse, and you can calculate it by using iterated Ψ. In particular, $a_*(L_1 * \cdots * L_n) = (a_1 * L_1) \otimes \cdots \otimes (a_* L_n)$.

Now, fix $\lambda_1, \ldots, \lambda - n \in X_*(T)^+$. Let

$$\overline{O}_{\lambda_1} \widetilde{\times} \cdots \widetilde{\times} \overline{O}_{\lambda_n} = q_n[p_n^{-1}[\overline{O}_{\lambda_1} \times \cdots \times \overline{O}_{\lambda_n}]],$$

which is the support of $IC_{\lambda_1} \widetilde{\boxtimes} \cdots \widetilde{\boxtimes} IC_{\lambda_n}$. This is closed in Gr_G^n. Notes: this product of intersection complexes is the intersection complex of this set we've just defined ("being the intersection complex" is local in the smooth topology). Moreover,

$$m_n[\overline{O}_{\lambda_1} \widetilde{\times} \cdots \widetilde{\times} \overline{O}_{\lambda_n}] = \overline{O}_{\lambda_1 + \cdots + \lambda_n}.$$

By the decomposition theorem, we see that $IC_{\lambda_1 + \cdots + \lambda_n}$ is a direct factor of an appropriate shift of $IC_{\lambda_1} * \cdots * IC_{\lambda_n}$. So $a_* IC_{\lambda_1 + \cdots + \lambda_n}$ is a direct factor of an appropriate shift of a tensor product of the $a_* IC_{\lambda_i}$. The theorem will then follow from:

Lemma: Every $\lambda \in X_*(T)^+$ is a sum of minuscule and quasi-minuscule characters in $X_*(T) \setminus \{0\}$.

We conclude that any $a_* IC_\lambda$ is concentrated in either even or odd degree.

Recalling what the decomposition theorem was: Let $F : X \to Y$ be a proper map between $\mathbb{F}_q$-sheaves of finite type. Let K be a pure ℓ-adic complex on X (e.g. the intersection complex). Then

$$(f_* K)_{Y_{\overline{\mathbb{F}}_q}} \cong \bigoplus_{i \in \mathbb{Z}} {}^p H^i(f_* K)[-i],$$

and each ${}^p H^i(f_* K)|_{Y_{\overline{\mathbb{F}}_q}}$ is a semisimple perverse sheaf.

So, if we know that there exists $U \hookrightarrow Y$ open sense such that some $H^i(f_* K)|_{Y_{\overline{\mathbb{F}}_q}}$ is $\overline{\mathbb{Q}}_{\ell,U}$, then $IC_\lambda[\cdots]$ is a direct factor of $(f_* K)|_{Y_{\overline{\mathbb{F}}_q}}$.

Now: We wanted $\mathbf{Sat}(G)$ with $*$ to be symmetric monoidal and ω a symmetric monoidal exact faithful additive functor. Have almost everything; fixed everything up so that the symmetric monoidal stuff works, and it's immediate that ω is additive and exact (the latter because $\mathbf{Sat}(G)$ is semisimple). Why is ω faithful? Since ω is exact, suffices to show that $K \neq 0$ implies $\omega(K) \neq 0$. So we just need to check that every $\omega(IC_\lambda)$ is nonzero, but this is $\bigoplus_{i\in\mathbb{Z}} H^i(\overline{O}_\lambda, IC_\lambda)$, and we know that $H^{-\langle 2\rho,\lambda\rangle}(\overline{O}_\lambda, IC_\lambda)$ is $\overline{\mathbb{Q}}_\ell$.

Note that intersection cohomology satisfies hard Lefschetz, hence $H^i(\overline{O}_\lambda, IC_\lambda) \neq 0$ for integers

$$-\langle 2\rho,\ell\rangle, -\langle 2\rho,\ell\rangle+2, -\langle 2\rho,\ell\rangle+4, \ldots, \langle 2\rho,\ell\rangle-2, \langle 2\rho,\ell\rangle.$$

So, $\dim\omega(IC_\lambda) = 1$ implies $\langle 2\rho,\lambda\rangle = 0$, i.e. t^λ is central in $G(k(\!(t)\!))$.

Also, note that the unit object in $\mathbf{Sat}(G)$ is IC_0.

The point of all that we've done is to say that $\mathbf{Sat}(G)$ is a Tannakian category; the one part of the definition we're missing is "rigid". Remember that if F is a field, a neutralized Tannakian category over F is a triple $(\mathbf{C},\otimes,\omega)$ where $(\mathbf{C},\otimes)$ is a F-linear rigid abelian tensor category and $\omega : \mathbf{C} \to \mathbf{Vect}_F$ is a faithful exact tensor functor, and $\mathrm{End}(1) = F$. (F-linear means that all hom-sets are F-vector spaces and composition is F-bilinear). Abelian tensor category means that we have an abelian category and a tensor category such that tensor products are additive. What does rigid mean? Two things:

(i) Internal Homs exist: For every $K, L \in \mathbf{C}$, the functor $\mathbf{C} \to \mathbf{Set}$ given by $T \mapsto \mathrm{Hom}(T\otimes K, L)$ is representable; the representing object is the internal hom $\underline{\mathrm{Hom}}(K,L)$. It comes with an evaluation map $ev_{K,L} : \underline{\mathrm{Hom}}(K,L)\otimes K \to L$. Also, internal Homs must be compatible with tensor products.

(ii) Every object is reflexive: For any K, let $K^\vee = \underline{\mathrm{Hom}}(K,1)$; then $ev : K^\vee \otimes K \to 1$ plus the commutativity constraint gives map $K \otimes K^\vee \to 1$. Comparing this with the evaluation map $K^\vee \otimes K^{\vee\vee} \to 1$ get a map $i_K : K \to K^{\vee\vee}$. We say K is reflexive if this i_K is an isomorphism.

Example: Let G be an affine group scheme over F. Let $\mathbf{Rep}_G$ be the category of representations of G on finite-dimensional F vector spaces, $\otimes$ the usual tensor product, and ω the forgetful functor. Then this defines a neutralized Tannakian category.

In fact this is the only example! If $(\mathbf{C},\otimes,\omega)$ is a neutralized Tannakian category, let $\mathrm{Aut}^\otimes(\omega)$ be the group k-space sending a k-algebra R to $\mathrm{Aut}^\otimes(\omega\otimes_F R)$. Theorem: (1) If $(\mathcal{C},\otimes,\omega) = \mathbf{Rep}_G$ then the obvious morphism $G \to \mathrm{Aut}^\otimes(\omega)$ is an isomorphism. (2) For any $(\mathcal{C},\otimes,\omega)$, $\mathrm{Aut}^\otimes(\omega)$ is representable by an affine group scheme G over F, and the obvious functor $\mathcal{C} \to \mathbf{Rep}_G$ is an equivalence of categories. (Grothendieck, Deligne, Milne, Rivano).

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Theorem: For $k = \overline{k}$ and G connected reductive, $(\mathbf{Sat}(G), *, \omega)$ is a neutralized Tannakian category. (The algebraically closed assumption isn't strictly necessary).

Proof: We know everything except that $\mathbf{Sat}(G)$ is rigid. Actually, it's enough to prove that objects with $\dim \omega(K) = 1$ have $*$-inverses. (Prove that for 1-dimensional objects this inverse is the dual, and for a general object with dimension, d, can define the dual as $K^\vee = (\bigwedge^{d-1} K) * (\bigwedge^d K)^{-1}$. Then prove $\underline{\mathrm{Hom}}(K, L) = K^\vee * L$).

So how do we prove that $K \in \mathbf{Sat}(G)$ with $\dim \omega(K) = 1$ has an inverse? Well, such a K must be IC_λ where λ such that $\langle \lambda, \alpha \rangle = 0$ for every root λ. But this means $IC_\lambda = \overline{\mathbb{Q}}_{\ell, t^\lambda}$, so $K^{-1} = \overline{\mathbb{Q}}_{\ell, t^{-\lambda}}$.

In fact, we can prove a formula for the dual in general; if $\pi : G((t)) \to \mathrm{Gr}_G$ is the projection and $inv : G((t)) \to G((t))$ is the inversion map, then $K^\vee$ is the unique element of $\mathbf{Sat}(G)$ such that $\pi^* K^\vee = D(inv^* \pi^* K)$.

Remark: Last time said that if $\lambda_1, \ldots, \lambda_n \in X_*(T)^+$ then IC_λ is a direct factor if $IC_{\lambda_1} * \cdots * IC_{\lambda_n}$ up to a shift. But since they are both perverse sheaves, the shift must be zero.

Remark: What about $k \neq \overline{k}$? Then $*$ is still defined and still related to $\star$ in the same way. However, $\mathbf{Sat}(G)$ is no longer necessarily semisimple. We can fix the commutativity constraint because if $K \in \mathbf{Sat}(G)$ is indecomposable, then it's supported on a connected component of Gr_G so its simple constituents IC_λ (up to twist) all have the same parity of $\langle 2\rho, \lambda \rangle$.

Let $k = \overline{k}$ and G connected reductive. Let $G' = \mathrm{Aut}^{\otimes}(\omega)$; this is an affine group scheme over $\overline{\mathbb{Q}}_\ell$. Main theorem: $G' = \widehat{G}_{\overline{\mathbb{Q}}_\ell}$. In other words, there's a $\otimes$-equivalence of categories between $\mathbf{Sat}(G)$ and $\mathbf{Rep}_{\widehat{G}_{\overline{\mathbb{Q}}_\ell}}$ such that ω corresponds to the forgetful functor. (Here $\widehat{G}$ is the dual group).

Lemma: G' is a connected reductive group over $\overline{\mathbb{Q}}_\ell$.

Proof: First check that G' is of finite type. This is equivalent to there existing $K \in \mathbf{Sat}(G)$ such that every $L \in \mathbf{Sat}(G)$ is a subquotient of some K^{*n}. Choose $\lambda_1, \ldots, \lambda_n$ generating the semigroup $X_*(T)^+$ and take $K = IC_\lambda \oplus \cdots \oplus IC_{\lambda_n}$.

Then, G' is reductive because $\mathbf{Sat}(G)$ is semisimple. It's connected because connectedness is equivalent to saying that if $K \neq 0, 1$, then $\langle K \rangle$ (the smallest thick subcategory of $\mathbf{Sat}(G)$ containing all $K^{\oplus n}$) is not stable by $*$ (if $G' \neq (G')^\circ$ then $G' \twoheadrightarrow \Gamma$ with Γ nontrivial finite, so $\mathbf{Rep}_\Gamma \hookrightarrow \mathbf{Rep}_{G'}$, and the regular representation of K' would be stable). So, let $K = IC_{\lambda_1} \oplus \cdots \oplus IC_{\lambda_n}$; then $IC_{2\lambda_1 + \cdots + 2\lambda_n} \notin \langle K \rangle$.

So we know G' is connected reductive; how do we prove it's $\widehat{G}_{\overline{\mathbb{Q}}_\ell}$ (the geometric Satake isomorphism)? Strategy: Pick $G' \supseteq B' \supseteq T'$, and take $X^*(T')^+$. So far we know $X_*(T')^+$ is isomorphic to the simple objects of $\mathbf{Sat}(G)$, which is isomorphic to $\mathbf{Rep}_{G'}$ and this to $X_*(T)^+ = X^*(\widehat{T})$, via $\lambda \mapsto [IC_\lambda]$ (in the reverse direction).

The next step is to prove the following statements:

(A) A connected reductive group H over an algebraically closed F is uniquely determined by $(X^*(T_H)^\perp, +, \leq)$.

(B) The bijection $X^*(T') \cong X_*(T)^+$ is compatible with $\prec$, where if λ, μ are two elements (in either set) then $\lambda \prec \mu$ iff $\mu - \lambda = \sum a_i \alpha_i$ with α_i positive roots (or coroots) and $a_i \in \mathbb{R}_{\geq 0}$.

(C) $X^*(T')^+ \cong X_*(T)_+$ is compatible with $+$, hence extends to an isomorphism of groups $\tau : X^*(T') \cong X_*(T)$.

(D) τ takes the root lattice to the coroot lattice and respects $\leq$.

To prove these, we'll need two big inputs, one from geometry and one from representation theory:

Theorem A: Let $\mu_1, \ldots, \mu_n \in X_*(T)^+$. Then

$$IC_{\mu_1} * \cdots * IC_{\mu_n} = IC_{\mu_1 + \cdots + \mu_n} \oplus \bigoplus_{v < \mu_1 + \cdots + \mu_n} IC_v^{\oplus a_v}.$$

Moreover, if $\lambda = \nu_1 + \cdots + \nu_n \in X_*(T)^+$ with $\nu_i \in W\mu_i$ for all i (where W is the Weyl group) then IC_λ is a subquotient of $IC_{\mu_1} * \cdots * IC_{\mu_n}$. (The first equality is not so hard; the "moreover" is more involved).

Theorem B: Let H be a connected reductive group over F such that $F = \overline{F}$ (no assumptions on the characteristic). Fix $H \supseteq B_H \supseteq T_H$ and $W_H = W(T_H, H)$. Let $\mu_1, \ldots, \mu_n \in X^*(T_H)^+$ and let $\nu_i \in W_H \mu_i$ such that $\lambda = \nu_1 + \cdots + \nu_n \in X^*(T_H)^+$. Then V_λ (the highest weight representation associated to λ) is a subquotient of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_n}$.

Proof of Step (B): Let $\lambda \in X^*(T_H)^+$, and let V_λ be the highest weight representation associated to λ. Set

$$Dom_{\prec\lambda} = \{\mu \in X^*(T_H)^+ : \mu \prec \lambda\}.$$

Proposition: Let $\lambda, \mu \in X^*(T_H)^+$. Then TFAE:
(i) $\lambda \prec \mu$.
(ii) There exists $F \subseteq X^*(T_H)^+$ finite such that for all $k \in \mathbb{N}$, $Dom_{\prec k\lambda} \subseteq W_H F + \sum_{i=1}^k W_H \mu$.
(iii) There exists a representation $U \in \mathbf{Rep}_H$ such that for all $k \in \mathbb{N}$, every simple subquotient of $V_\lambda^{\otimes k}$ is a subquotient of $U \otimes V_\mu^{\otimes k}$.

Corollary: $\iota : X^*(T')^+ \cong X_*(T)^+$ preserves $\prec$ (this is Step (B)).

Proof of Corollary: Let $\lambda, \mu \in X_*(T)^+$. If $\lambda \prec \mu$, let $F \subseteq X_*(T)^+$ be a finite set as in (ii) and $L = \bigoplus_{\nu \in F} IC_\nu$. By (i) $\iff$ (iii), $\iota(\lambda) \leq \iota(\mu)$ iff for all k there exists K such that every simple subquotient of IC_λ^{*k} is a subquotient of $K * IC_\mu^{*l}$. Let $k \in \mathbb{N}$ and let IC_ν a simple subquotient of IC_λ^{*k}. Then $\nu \leq k\lambda$ so $\nu \prec k\lambda$ so $\nu \in WF + \sum W\mu_i$ so by Theorem A, IC_ν is a subquotient of $L * IC_\mu^{*k}$. So

Conversely, suppose $\iota(\lambda) \prec \iota(\mu)$. By the proposition there's $K \in \mathbf{Sat}(G)$ such that IC_λ^{*k} is a subquotient of $K * IC_\mu^{*k}$. Then the supports satisfy

$$\overline{O}_{k\lambda} = \mathrm{supp}(IC_\lambda^{*k}) \subseteq \mathrm{supp}(K * IC_\mu^{*k}) = \bigcup_{i=1}^n \overline{O}_{\nu_i + k\mu} \subseteq \mathcal{O}_{\nu + k\mu}$$

if $\nu \geq \nu_1, \ldots, \nu_n$. Then Theorem A implies that $k\lambda \leq \nu + k\mu$ for all k, so $\lambda \prec \mu$.

Proof of Proposition: (iii) $\implies$ (i): If (iii) holds then there exists $\nu \in X^*(T_H)^+$ such that for all k, $k\lambda \leq v + k\mu$. So $\lambda \prec \mu$.
(ii) $\implies$ (iii): Assume (ii) holds; take $U = \bigoplus_{v \in F} V_v$. If $k \in \mathbb{N}$ and V_χ is a subquotient of $V_\lambda^{\otimes k}$ then $\chi \leq \lambda k$ implies $\chi \prec \lambda k$ so $\chi \in WF + \sum W\mu$. By Theorem B, V_χ is a subquotient of $U \otimes V_\mu^k$.
(i) $\implies$ (ii): This is the hard part; we'll prove it in the special case of $H = \mathrm{GL}_2$. What happens if we take $X^*(T_H) = X^*$? It's $\mathbb{Z}^2$. Then $(X^*)^+ = \{(\lambda_1, \lambda_2) : \lambda_1 \geq \lambda_2\}$, and $(\lambda_1, \lambda_2) \leq (\mu_1, \mu_2)$ iff $(\lambda_1, \lambda_2) \prec (\mu_1, \mu_2)$ iff $\lambda_1 + \lambda_2 = \mu_1 + \mu_2$ and $\lambda_1 \leq \mu_1$. Fix $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2)$ with $\lambda \leq \mu$, and let $k \in \mathbb{N}$. Then

$$Dom_{\prec k\lambda} = \{(a_1, a_2) \in \mathbb{Z}^2 : a_1 + a_2 = k\lambda_1 + \lambda_2, k\lambda_1 \geq a_1 \geq \frac{1}{2}k(\lambda_1 + \lambda_2)\}.$$

Moreover, $\frac{1}{2}k(\lambda_1 + \lambda_2) = \frac{1}{2}k(\mu_1 + \mu_2) \geq k\mu_2$. Then

$$\sum W\mu = \{(\ell \impliedby {}_1 + (k-\ell)\mu_2, (k-\ell)\mu_1 + \ell\mu_2) : 0 \leq \ell \leq k\},$$

and we can take $F = \{(0,0), (1,-1), \ldots, (\mu_1 - \mu_2, \mu_2 - \mu_1)\}$.

Step (C): $\iota : X^*(T)^+ \cong X^*(T')^+$ respects addition (so extends to an isomorphism of $X^*(T) \cong X^*(T')$). Proof: For all $\alpha, \beta \in X^)(T')^+$, then $\alpha + \beta$ is the biggest element $\gamma \in X^*(T')^+$ such that V_γ is a direct summand of $V_\alpha \otimes V_\beta$. Also, for all $\lambda\mu \in X_*(T)^+$, $\lambda + \mu$ is the biggest $\nu \in X_*(T)^+$ such that IC_ν is the direct summand of $IC_\lambda * IC_\mu$.

Step (D): ι preserves $\leq$ (not just $\prec$): Let $Q' \subseteq X^*(T')$ and $Q^v \subseteq X_*(T)$ be the root (coroot) lattice.

Lemma: Let $\lambda, \mu \in X^*(T_H)$. Then TFAE:
(i) $\lambda \leq \mu$
(2) $\lambda \prec \mu$ and $\mu - \lambda \in Q_H$.

Given this lemma, proving (D) is clear; need to show that ι identifies the root and coroot lattice.

15 Lecture - 04/16/2014

Proof of Geometric Satake: Had G connected reductive over $k = \overline{k}$. Showed $\mathbf{Sat}(G)$ with $*$ and ω was a Tannakian category, and let G' be its Tannakian group over $\overline{\mathbb{Q}}_\ell$. Our goal is to prove $G' = \widehat{G}$.

From what we proved last time: We know G' is a linear algebraic group that's connected reductive. Moreover if $G' \supseteq B' \supseteq T'$, saw we had an isomorphism $\iota : X_*(T)^+ \to X^*(T)^+$ which was compatible with addition and preserved the "weird Bruhat order $\prec$". Thus it extended to an isomorphism $X_*(T) \cong X^*(T')$.

What was left from our setup last time was proving Step (D), that $\iota(Q^\vee) = Q'$ where $Q^\vee$ is the coroot lattice of $X_*(T)$ and Q' the root lattice of $X^*(T)$, and Step (A) that a connected reductive group over an ACF is uniquely determined by $X^*(T)^+$ together with $+, \prec$.

Proof of Step (D): Start with a Lemma 1: H connected reductive over $F = \overline{F}$. Let $H \supseteq B_H \supseteq T_H$ and $X^*(T_H) \supseteq Q_H$ as usual, and let $Q_H^+ = Q_H \cap X^*(T_H)^+$. Then if $\alpha \in X^*(T_H)$, have $\alpha \in Q_H^+$ iff there is $\mu \in X^*(T_H)^+$ such that $2\mu - \alpha \in X^*(T_H)^+$ and $V_{2\mu-\alpha}$ is a direct factor $V_\mu \otimes V_\mu$.

Remark: This lemma lets us characterize $(Q^\vee)^+$ and Q^+ in terms of things that we already know are compatible under ι; thus: Corollary: $\iota(Q^\vee) = Q'$. (Note that the condition of the lemma is stable by sums: if we have (α_1, μ_1) and (α_2, μ_2) satisfying it and we set $\mu = \mu_1 + \mu_2$ and $\alpha = \alpha_1 + \alpha_2$ then $2\mu - \alpha \in X^*(T_H)^+$ and ... ?)

Corollary (of the last corollary): ι respects $\leq$. This follows from the following: Lemma 2: Let $\lambda, \mu \in X^*(T_H)$. TFAE: (i) $\lambda \leq \mu$, (ii) $\lambda \prec \mu$ and $\mu - \lambda \in Q_H^+$.

Proof of Lemma 1: $\Longrightarrow$: If α is a positive root, take μ such that $\langle \mu, \alpha \rangle = 2$. Then $2\mu - \alpha = \alpha + s_\alpha(\mu)$, so $V_{2\mu-\alpha}$ is a direct summand of $V_\mu \otimes V_\mu$ by Theorem B (from last time). The $\Longleftarrow$ direction is trivial.

Step (A): Goal is to prove that if H, H' are two connected reductive groups over an algebraically closed field (of characteristic zero?) $F = \overline{F}$, with root data of H being $(X, \Phi, \Delta, X^\vee, \Phi^\vee, \Delta^\vee)$ and for H' being $(X', \Phi', \Delta', X'\vee, \Phi'^\vee, \Delta'^\vee)$. If we have an isomorphism $\iota : (X^+, \leq, +) \to (X'^+, \leq, +)$ then $H \cong H'$.

Proof: (From paper of T. Richarz): Since such an H is determined uniquely by its root data, we just need to show that ι gives an isomorphism of the root data. First, ι extends to $\iota : X \cong X'$ and dualizing gives $\iota^\vee : (X')^\vee \cong X^\vee$.

Next, $\iota(\Delta) = \Delta'$ (i.e. ι preserves simple roots) because Δ is the set of minimal elements of the set $\{\alpha \in X : \alpha \geq 0, \alpha \neq 0\}$, same for Δ'. Then, $\iota^\vee[\Delta'^\vee] = \Delta^\vee$ because if $\alpha \in \Delta$ get $\alpha^\vee \in \Delta^\vee$; want to show $\iota^\alpha(\alpha^\vee) = \iota(\alpha)^\vee$. For this, note that for all $\mu \in X'^+$, $\langle \mu', \iota(\alpha)^\vee \rangle$ is the unique $m \in \mathbb{N}$ such that $2\mu - m\iota(\alpha) \in X'^+$ but $2\mu - (m+1)i(\alpha) \notin X'^+$ (exercise), and similar statement for pairing with $\langle -, \alpha^\vee \rangle$. So get $\langle \mu, \iota(\alpha)^\vee \rangle = \langle \iota^{-1}(\mu), \alpha^\vee \rangle$.

Finally: $\iota[\Phi] = \Phi'$: The Weyl groups W, W' are generated by s_α for $\alpha \in \Delta$ and $\alpha \in \Delta'$, respectively. So ι intertwines the actions of W and W', so $\Phi' = W'\Delta' = \iota[W\Delta] = \iota[\Phi]$ and similarly for $\iota^\vee[\Phi'^\vee]$

So that deals with the easy parts of geometric Satake, proving it modulo the two big theorems stated last time. Now let's start working on those.

Theorem A: Let G be connected reductive with $k = \overline{k}$. Let $\mu_1, \ldots, \mu_n \in X_*(T)^+$. Then

$$IC_{\mu_1} * \cdots * IC_{\mu_n} = \bigoplus_{\lambda \leq \mu_1 + \cdots + \mu_n} V^\lambda_{\mu_1,\ldots,\mu_n} \otimes IC_\lambda$$

where $V^\lambda_{\mu_1,\ldots,\mu_n}$ is a $\overline{\mathbb{Q}}_\ell$-vector space determining the multiplicities, such that
(i) If $\lambda = \mu_1 + \cdots + \mu_n$, $V^\lambda_{\mu_1,\ldots,\mu_n}$ is 1-dimensional.
(ii) If $\lambda = w_1\mu_1 + \cdots + w_n\mu_n$ with $w_i \in W$ then $\dim V^\lambda_{\mu_1,\ldots,\mu_n} \geq 1$.

Proof of this uses the n-fold convolution diagram. For all $\lambda \in X_*(T)^+$, let O_λ be the orbit of t^λ in Gr_G. For all $\lambda_1, \ldots, \lambda_n X_*(T)^+$, we have q_n^{-1} of a twisted product is the same as p_n^{-1} of an untwisted product. Notation: Write

$$O_{\lambda_1,\ldots,\lambda_n} = O_{\lambda_1} \widetilde{\times} \cdots \widetilde{\times} O_{\lambda_n}$$

and

$$\overline{O}_{\lambda_1,\ldots,\lambda_n} = \overline{O}_{\lambda_1}\widetilde{\times}\cdots\widetilde{\times}\overline{O}_{\lambda_n}.$$

Also let $IC_{\lambda_1,\ldots,\lambda_n} = IC_{\lambda_1}\widetilde{\boxtimes}\cdots\widetilde{\boxtimes}IC_{\lambda_n}$, which we showed was the intersection complex of $\overline{O}_{\lambda_1,\ldots,\lambda_n}$. Note that $\overline{O}_{\lambda_1,\ldots,\lambda_n}$ is stratified by the $O_{\nu_1,\ldots,\nu_n}$ for $\nu_u \leq \lambda_i$.

We need the following notion of Goresky-MacPherson: Let $f : X = \bigcup U_\alpha \to Y = \bigcup Y_\beta$ be a map of stratified schemes of finite type over k. Assume that f is proper surjective birational and for all α, $f[X_\alpha]$ is a union of strata Y_β. We say f is *stratified semi-small* if, for all α, β such that $f[X_\alpha] \supseteq Y_\beta$ and all $y \in Y_\beta(k)$, we have

$$\dim f^{-1}[y] \cap X_\alpha \leq \frac{1}{2}(\dim X_\alpha - \dim Y_\beta)$$

Theorem (G-M): If f is stratified semi-small then $f_* IC_X \in \mathbf{Perv}(Y)$.
Some part of this statement might be wrong????

Definition: f is stratified locally trivial if for all α, β with $f[\alpha] \supseteq Y_\beta$, the map

$$f|_{f^{-1}[Y_\beta]\cap X_\alpha} : f^{-1}[Y_\beta] \cap X_\alpha \to Y_\beta$$

is a Zariski-locally trivial fibration.

Assume the theorem applies. Assume that $f_* IC_X = \bigoplus_\beta V_\beta \otimes IC_{Y_\beta}$. Where IC_{Y_β} is the intermediate extension of the intersection complex on Y_β.

Proposition (Haines): Suppose that X, Y are proper. Fix β and $y \in Y_\beta(k)$. Then $\dim V_\beta$ is the number of irreducible components of $f^{-1}[y]$ of dimension $\frac{1}{2}(\dim X - \dim Y_\beta)$, which is the dimension of $H^{\dim Y_B}(f^{-1}[y], IC_X)$.

Go back to our affine Grassmannian, fix $\underline{\mu} = (\mu_1, \ldots, \mu_n) \in X_*(T)^+$, and let $m_{\underline{\mu}} = m_n|_{\overline{O}_{\mu_1,\ldots,\mu_n}}$. Have map $\overline{O}_{\mu_1,\ldots,\mu_n} \to \overline{O}_\mu$ for $\mu = \mu_1 + \cdots + \mu_n$.

Theorem (Ngo-Polo, Haines): $m_{\underline{\mu}}$ is stratified semi-small and stratified locally trivial (with the stratifications given before).

The main tool in the proof of theorem A will be "semi-infinite strata". Let $B \subseteq G$ be the Borel, so we get $\mathrm{Gr}_B \to \mathrm{Gr}_G$ which is an isomorphism on k-points. Have $\cong$: $\pi_0(\mathrm{Gr}_B) \to \pi_0(\mathrm{Gr}_T) = X_*(T)$. For all $\nu \in X_*(T)$ let S_ν be the image of the corresponding connected component in Gr_G, so

$$S_\nu(k) = B(k[\![t]\!])t^\nu G(k[\![t]\!])/G(k[\![t]\!]).$$

Notation: For $\lambda \in X_*(T)^+$, let $\Omega(\lambda)$ be the weights of $T^\vee$ on V_λ, so $\{\nu \in X_*(T) : \forall w \in W, w\nu \leq \lambda\}$.

Theorem (Ngo-Polo, Gortz-Haines-Kottwitz-Revmon) For all $\nu \in X_*(T)$ and $\lambda \in X_*(T)^+$, $S_\nu \cap O_\lambda \neq \emptyset$ only if $\nu \in \Omega(\lambda)$, and in that case $S_\nu \cap O_\lambda$ (a locally closed subscheme of O_λ with k-points $S_\nu(k) \cap O_\lambda(k)$) is of pure dimension $\langle \rho, \nu + \lambda \rangle$.

16 Lecture - 04/18/2014

Fix the lemma from last time (about the characterization of Q_H^+). What it should be:

Lemma: Q_H^+ is generated (as a semigroup) by the set of $\alpha \in X^*(T_H)$ such that there exists $\mu \in X^*(T_H)^+$ with $2\mu - \alpha \in X^*(T_H)^+$ and $V_{2\mu-\alpha} \hookrightarrow V_\mu \otimes V_\mu$. (Last time said "equal" but couldn't figure out why this was stable by sums... Answer is it's not!)

Now: Look at $\mathrm{Gr}_G \supseteq O_\lambda = \mathbb{G}[\![t]\!]t^\lambda$, and also take $S_\nu \subseteq \mathrm{Gr}_G$ given by $S_\nu(k) = B(k[\![t]\!])t^\nu$ for $\nu \in X_*(T)$.

Theorem: For all $\nu \in X_*(T)$ and for all $\lambda \in X_*(T)^+$ then $S_\nu \cap \overline{O}_\lambda$ is $\emptyset$ unless $\nu \in \Omega(\lambda)$ (the weights of V_λ), and in that case it's of pure dimension $\langle \rho, \lambda + \nu \rangle$.

Proof: Let $2\rho^\vee = \sum_{\alpha^\vee \in \Phi^{\vee +}} \alpha^\vee : \mathbb{G}_m \to T$. Via $2\rho^\vee$, $\mathbb{G}_m$ acts on Gr_G (by left multiplication), and:
(a) The O_λ are stable.
(b) The fixed points are the t^λ for $\lambda \in X_*(T)$.
(c) For all $x \in S_\nu(k)$, we have $\lim_{u\to 0} 2\rho^\vee(u)x = t^\nu$.
If $x \in S_\nu(k) \cap \mathrm{O}_\lambda(k)$ then t^ν is this limit which is in $\overline{O}_\lambda(k)$ so $\nu \in \Omega(\lambda)$.

The second part follows easily from the statement that (when $\nu \in \Omega(\lambda)$) we have that $H_c^*(S_\nu \cap \overline{O}_\lambda, IC_\lambda)$ is concentrated in degree $\langle 2\rho, \nu \rangle$. To prove this statement, use following steps:

(A) Prove everything in the case where λ is minuscule or quasi-minuscule; can explicitly compute $S_\nu \cap \overline{O}_\lambda$ there. (Remark: Messed up the definition of minuscule last time because we're not working with a semisimple group; if λ is a set of $X_*(T)^+ \setminus X_*(Z)$ where Z is the center, have two possibilities: Either λ is minimal in $X_*(T)^+$ and $\langle \lambda, \alpha \rangle \le 1$ for all $\alpha \in \Phi^+$ (in which case λ is minimal) or there is $\lambda \in \Phi^+$ such that $\langle \lambda, \alpha \rangle \ge 2$ then $\lambda \in \beta^\nu + X_*(Z)$ for β a maximal root (in which case λ is quasi-minuscule)).

(B) In general write $\lambda = \lambda_1 + \cdots + \lambda_n$ with each λ_i minuscule or quasi-minuscule. Using setup from last time, have $m_{\underline{\lambda}} : \overline{O}_{\underline{\lambda}} \to \overline{O}_\lambda$ and know IC_λ is a direct factor of $m_{\underline{\lambda}*} IC_{\underline{\lambda}} = IC_{\lambda_1} * \cdots * IC_{\lambda_n}$. So

$$H^*(\overline{O}_\lambda \cap S_\nu, IC_\lambda) \hookrightarrow H^*(\overline{O}_\lambda \cap S_\nu, m_{\underline{\lambda}*} IC_{\underline{\mu}}),$$

and by proper base change this latter thing is $H_c^*(m_{\underline{\mu}}^{-1}[S_\nu], IC_{\underline{\mu}})$. But $m_{\underline{\mu}}^{-1}$ decomposes as $\bigcup_{\underline{\nu}} S_{\underline{\nu}} \cap \overline{O}_{\underline{\lambda}}$ where $\underline{\nu}$ runs over tuples with $\nu_1 + \cdots + \nu_n =$
nu, and

$$S_{\underline{\nu}} = S_{\nu_1} \times S_{\nu_1+\nu_2} \times \cdots \times S_{\nu_1+\cdots+\nu_n} \hookrightarrow \mathrm{Gr}_G^n.$$

Also,

$$S_{\underline{\nu}} \cap \overline{O}_{\underline{\lambda}} \cong (S_{\nu_1} \cap \overline{O}_{\lambda_1}) \times \cdots \times (S_{\nu_n} \cap \overline{O}_{\lambda_n})$$

and this induces

$$H_c^*(S_{\underline{\nu}} \cap \overline{O}_{\underline{\lambda}}, IC_{\underline{\lambda}}) \cong \bigotimes_{i=1}^n H_c^*(S_{\nu_i} \cap \overline{O}_{\lambda_i}, IC_{\lambda_i}).$$

Use the stratification spectral sequence to deduce that $H^*(S_{\overline{\nu}} \cap \overline{O}_{\underline{\lambda}}, IC_{\underline{\lambda}})$ is concentrated in degree $\langle 2\rho, \nu \rangle$.

Now, recall the notation that for for $\underline{\mu} = (\mu_1, \ldots, \mu_n)$ with $\mu_i \in X_*(T)^+$, set $\mu = \mu_1 + \cdots + \mu_n$ and then the natural map $m_{\underline{\mu}} : \overline{O}_{\underline{\mu}} \to \overline{O}_\mu$ is semi-small (?).

Corollary: Take $\lambda \in X_*(T)^+$ such that $\lambda \le \mu$. Then for all $x \in O_\lambda(k)$, $\dim m_{\underline{\mu}}^{-1}(x) \le \langle \rho, \mu - \lambda \rangle$.
Proof: The fibers are all isomorphic, so it's enough to do it for one x. Now, $S_\lambda \cap O_\lambda$ is open dense in O_λ by the preceding theorem. So it's enough to show $\dim m_{\underline{\mu}}^{-1}(S_\lambda \cap O_\lambda) \le \langle \rho, \mu - \lambda \rangle$. But we have a stratification

$$m_{\underline{\mu}}^{-1}(S_\lambda \cap O_\lambda) \subseteq m_{\underline{\mu}}^{-1}(S_\lambda \cap \overline{O}_\lambda) = \bigcup_{\underline{\nu}:\nu=\lambda} S_{\underline{\nu}} \cap \overline{O}_{\underline{\mu}}$$

and each $S_{\underline{\nu}} \cap \overline{O}_{\underline{\mu}}$ is a product of $S_{\nu_i} \cap \overline{O}_{\mu_i}$. But the dimension of $S_{\nu_i} \cap \overline{O}_{\mu_i}$ is $\langle \rho, \nu_i + \mu_i \rangle$, and the sum of this over all i is $\leq \langle \rho, \lambda + \mu \rangle$.

Back to Theorem A: Write $m_{\underline{\mu}*} IC_{\underline{\mu}} = \bigoplus_{\lambda \leq \mu} V_{\underline{\mu}}^{\lambda} \otimes IC_{\lambda}$. We wanted to show that:
(i) $\dim V_{\underline{\mu}}^{\mu} = 1$,
(ii) If $\lambda = w_1\mu_1 + \cdots + w_n\mu_n \in X_*(T)^+$ with $w_i \in W$, we have $\dim V_{\underline{\mu}}^{\lambda} \geq 1$.

Proof: (i) is obvious because $m_{\underline{\mu}*}$ is birational. For (ii), reduce to $n = 2$, and then it's an easy direct calculation. This proves Theorem A!

Theorem B: If H is connected reductive over $F = \overline{F}$, and we take $\mu_1, \ldots, \mu_n \in X^*(T_H)^+$ and $w_1, \ldots, w_n \in W_H$ such that $\lambda = w_1\mu_1 + \cdots + \wedge_n\mu_n \in X^*(T_H)^+$, then V_λ is a direct summand of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_n}$. (Due to Kumar, Matheiu; known as the PRV conjecture).

Idea of proof (for characteristic zero): Take $\lambda, \mu \in X^*(T_H)^+$, $W \in W_H$, and $\nu \in W_H \cdot (\lambda + w\mu)$ dominant. Goal is $V_\nu \hookrightarrow V_\lambda \otimes V_\mu$. Let $X = H/B_H = \bigcup_{v \in W_H} X_v$ for $X_v = B_H v B_H / B_H$. Then have the Schubert variety $\overline{X}_w$ for any w; the Bott-Samelson resolution is a resolution of singularities $\widetilde{X}_w \to \overline{X}_w$ (with $\widetilde{X}_w$ smooth). This $\widetilde{X}_w$ lives in $X \times X$. Now, λ, μ give line bundles $\mathcal{L}_\lambda, \mathcal{L}_\mu$ over X.

Kumar proves: $H^0(\widetilde{X}_w, \mathcal{L}_w(\lambda, \mu)) \hookrightarrow V_\lambda \otimes V_\mu$, and H^i of this is zero for $i > 0$. But this H^0 is the $U(\mathfrak{h})$-submodule generated by $e_\lambda \otimes e_{w\mu}$ (the weight vectors for λ and $w\mu$). On the other hand, this H^0 also decomposes as

$$\bigoplus_{\theta \in X^*(T_H)} V_\theta^\vee \otimes \mathrm{Hom}_{\mathfrak{b}_H}(\mathbb{C}_\lambda \otimes V_{w\mu}, V_\theta)$$

where $\mathfrak{b}_H$ is the Borel, $\mathbb{C}_\lambda$ is $\mathbb{C}$ with $\mathfrak{b}_H$ acting through λ, and $V_{w\mu}$ is the sub-$U(\mathfrak{b}_H)$-module of V_μ generated by $e_{w\mu}$. This $\mathrm{Hom}_{\mathfrak{b}_H}$ serves as a space of multiplicities, and it's 1-dimensional in the case $\theta = \nu$.

So this (sketchily) proves Theorem B, finishing the proof of geometric Satake! At least, the case of it we've stated, where k is algebraically closed. Now we move on to the case where k isn't necessarily algebraically closed. Fix an algebraic closure $\overline{k}$, and set $\Gamma = \mathrm{Gal}(\overline{k}/k)$ and ${}^L G = \widehat{G}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma$.

Theorem (Timo Richarz): The category $(\mathbf{Sat}(G), *)$ (which we can still form, and fix the commutativity constraint, etc.) is equivalent to $(\mathbf{Rep}^c_{{}^L G}, \otimes)$ where $\mathbf{Rep}^c_{{}^L G}$ is the category of finite-dimensional continuous $\overline{\mathbb{Q}}_\ell$-representations ρ of ${}^L G$ such that $\rho|_{\widehat{G}(\overline{\mathbb{Q}}_\ell)}$ is algebraic.

We will need: Corollary: There is an exact tensor functor $\mathbf{Rep}_{\widehat{G}} \to \mathbf{Sat}(G)$ that's a section of $\omega : \mathbf{Sat}(G) \to \mathbf{Vec}_{\overline{\mathbb{Q}}_\ell}$ given by $\omega(K) = \bigoplus H^i(\mathrm{Gr}_{G,\overline{k}}, K_{\overline{k}})$ (which has an action of $\widehat{G}(\overline{\mathbb{Q}}_\ell)$). This follows from $\mathbf{Sat}(G_{\overline{k}}) \hookrightarrow \mathbf{Sat}(G)$.

17 Lecture - 04/23/2014

Pseudo-representations: Let Γ be a profinite group, ℓ prime. Trying to construct continuous homomorphisms $\Gamma \mapsto H(\overline{\mathbb{Q}}_\ell)$ for H connected reductive (where $H = \widehat{G}$). If $H = \mathrm{GL}_n$ then ρ is just an n-dimensional representation of Γ.

Know that if $\rho, \rho' : \Gamma \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ are semisimple then they're equivalent iff $\mathrm{tr}(\rho(\gamma)) = \mathrm{tr}(\rho'(\gamma))$ for all $\gamma \in \Gamma$. A *pseudo-representation* of dimension n is a continuous function $T : \Gamma \to \overline{\mathbb{Q}}_\ell$ satisfying the same properties as $\mathrm{tr}\,\rho$ (due to Wiles, Taylor, ...):

1. $T(1) = n$.
2. $T(\gamma\gamma') = T(\gamma'\gamma)$ for all $\gamma, \gamma \in \Gamma$.
3. $\sum_\sigma \varepsilon(\sigma) T_\sigma(g_1, \dots, g_{n+1}) = 0$ for all $\gamma_1, \dots, \gamma_{n+1} \in \Gamma$.

For condition 3, let $\varepsilon : S_{n+1} \to \{\pm 1\}$ be the sign function. If $\sigma \in S_{n+1}$ has cycle decomposition $\prod (i_1^{(j)} \ \cdots \ i_{r_j}^{(j)})$ define $T_\sigma : \Gamma^{n+1} \to \overline{\mathbb{Q}}_\ell$ by

$$T_\sigma(\gamma_1, \dots, \gamma_{n+1}) = \prod T(\gamma_{i_1}^{(j)} \cdots \gamma_{i_{r_j}}^{(j)}).$$

Can prove trace satisfies 3. Theorem (Taylor): Each such T is of the form $\mathrm{tr}\,\rho$ for $\rho : \Gamma \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ continuous semisimple.

Problem: What if $H \neq \mathrm{GL}_n$? Try a Tannakian approach: For every $\psi : H \to \mathrm{GL}_N$, give compatible T_φ satisfying (1), (2), (3). This gives family of maps $\rho_\varphi : \Gamma \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$. How do we know this comes from $\rho : \Gamma \to H(\overline{\mathbb{Q}}_\ell)$. This approach doesn't seem very doable.

Lafforgue's Solution: Instead of just $\mathrm{tr}\,\rho$, use all conjugacy-invariant functions on $H(\overline{\mathbb{Q}}_\ell)$ and in fact all simultaneous-conjugacy invariant functions on $H(\overline{\mathbb{Q}}_\ell)^n$'s. (Notation: Let $H(\overline{\mathbb{Q}}_\ell)$ act on $H^n(\overline{\mathbb{Q}}_\ell)^n$ by conjugacy, denoted by $\gamma \cdot (\gamma_1, \dots, \gamma_n) = (\gamma\gamma_i\gamma^{-1})$). If $\rho : \Gamma \to H(\overline{\mathbb{Q}}_\ell)$, for all $f : H(\overline{\mathbb{Q}}_\ell)^n \to \overline{\mathbb{Q}}_\ell$ regular invariant by conjugacy (????) have that $T_{f,\rho} : \Gamma^n \to \overline{\mathbb{Q}}_\ell$ given by $(\gamma_1, \dots, \gamma_n) \mapsto f(\rho(\gamma_1), \dots, \rho(\gamma_n))$ depends only on he conjugacy class of ρ. Let $H^n//H$ be the coarse quotient $\mathrm{Spec}\,\mathcal{O}(H^n)^H$, and take $E \subseteq \overline{\mathbb{Q}}_\ell$ finite over $\mathbb{Q}_\ell$ such that H is defined and spit over E.

Theorem (V. Lafforgue): The map $\varphi \mapsto (T_{f,\rho})_{n, f \in \mathcal{O}(H^n//H)}$ induces a bijection between conjugacy classes of continuous $\rho : \Gamma \to H(\overline{\mathbb{Q}}_\ell)$ that are semisimple (i.e. $\overline{\rho(\Gamma)}^{Zar}$ is reductive) with families of E-algebra maps $\Xi_n : \mathcal{O}(H^n//H) \to \mathcal{C}(\Gamma^n, \overline{\mathbb{Q}}_\ell)$ such that:
(0) There exists E'/E finite with $\mathrm{img}\,\Xi_n \subseteq \mathcal{C}(\Gamma^n, E')$.
(1) For all $m, n > 0$, for all $\zeta : \{1, \dots, m\} \to \{1, \dots, n\}$, for all $f \in \mathcal{O}(H^n//H)$, and for all $(\gamma_1, \dots, \gamma_n) \in \Gamma^n$, we have

$$\Xi_n(f^\zeta)(\gamma_i) = \Xi_m(f)(\gamma_{\zeta(i)})$$

where $f^\zeta(g_i) = f(g_{\zeta(i)})$.
(2) For all $n > 0$, for all $f \in \mathcal{O}(H^n//H)$, and for all $(\gamma_1, \dots, \gamma_{n+1}) \in \Gamma^{n+1}$, have

$$\Xi_{n+1}(\widehat{f})(\gamma_1, \dots, \gamma_n) = \Xi_n(f)(\gamma_1, \dots, \gamma_{n-1}, \gamma_n\gamma_{n+1})$$

where $\widehat{f}(g_1, \dots, g_n) = f(g_1, \dots, g_{n-1}, g_n g_{n+1})$.
Moreover, if our correspondence identifies ρ with (Ξ_n) (given by $\Xi_n(f) = T_{f,\rho}$) then there exists m such that for all profinite quotients $\Gamma \twoheadrightarrow \overline{\Gamma}$, then if $\Xi_m(H^m//H) \subseteq \mathcal{C}(\overline{\Gamma}^m, \overline{\mathbb{Q}}_\ell)$ then ρ factors through $\overline{\Gamma}$.

Remark: If $H = \mathrm{GL}_N$, by the work of Procesi, $\mathcal{O}(H^n//H)$ is generated (as an algebra) by functions $(g_1, \dots, g_n) \mapsto \mathrm{tr}(g_{i_1} \cdots g_{i_r})$. So (Ξ_n) satisfying (1) and (2) are uniquely determined by $\Xi_1(\mathrm{tr})$. Then find that (Ξ_n) satisfies (1) and (2) iff $\Xi_1(\mathrm{tr})$ is a pseudo-representation. So this is an actual generalization!

Invariant theory. Let E be a field ($E = \overline{E}$, $\operatorname{char} E = 0$ unless said otherwise), H connected reductive over E, X an affine variety over E with a (left) action by H. Notation: $\mathcal{O}(X)^H$ is the H-invariant in $\mathcal{O}(X)$, if $x \in X(E)$ then H_x is the stabilizer and Hx is the orbit.

The Reynolds operator (characteristic zero): For all $V \in \mathbf{Rep}_H$, $V = V^H \oplus V_H$ where V_H is the sum of all nontrivial irreducible subrepresentations. The Reynolds operator is the projection $R_V : V \twoheadrightarrow V^H$ corresponding to this decomposition. Then R_v is H-equivariant, functorial in V. If V is in $\mathbf{IndRep}_H$ (possibly infinite-dimensional but every vector in a finite-dimensional subrepresentation, e.g. $V = \mathcal{O}(X)$) define $R_v : V \twoheadrightarrow V^H$ as the limit of the finite-dimensional ones. In particular get $R_X : \mathcal{O}(X) \twoheadrightarrow \mathcal{O}(X)^H$, which is $\mathcal{O}(X)^H$-linear because if $a \in \mathcal{O}(X)^H$ then $b \mapsto ab$ is H-equivariant.

Theorem: (i) $\mathcal{O}(X)^H$ is finitely-generated. (True in positive characteristic too, but proof harder).
(ii) Let $X//H = \operatorname{Spec} \mathcal{O}(X)^H$ be the coarse quotient, $\pi_X : X \twoheadrightarrow X//H$. Then π_X satisfies the following property for all affine H-varieties Y:

$$\operatorname{Hom}_H(X, Y) \cong \operatorname{Hom}(X//H, Y)$$

where we go from the RHS to the LHS via pullbacks.
(iii) If $Y, Y' \subseteq X$ are closed and H-invariant then $\pi_X|_Y$ is isomorphic to $\pi_Y : Y \twoheadrightarrow Y//H$, and $\pi_X(Y \cap Y') = \pi_X(Y) \cap \pi_X(Y')$.
(iv) For every $x \in (X//H)(E)$ there exists a unique closed H-orbit in $\pi_X^{-1}[x]$.
Proof: (i) First note $\mathcal{O}(X)^H$ is Noetherian; if $J \subseteq \mathcal{O}(X)^H$ is an ideal and $I = J\mathcal{O}(X)$, then $I^H = R_X(I) = J \cdot R_X(\mathcal{O}(X)) = J$. Case when $X = V \in \mathbf{Rep}_H$: Write $\mathcal{O}(V) = \bigoplus_{n \in \mathbb{N}} \mathcal{O}(V)_n$, with

$$\mathcal{O}(V)_n = \{f \in \mathcal{O}(V) : f(\lambda v) = \lambda^n f(v) \forall \lambda \in \mathbb{G}_m\}.$$

This is preserved by H and induces $\mathcal{O}(V)^H = \bigoplus_{n \in \mathbb{N}} \mathcal{O}(V)_n^H$. But $\oplus_{n \geq 1} \mathcal{O}(V)_n^H$ is finitely generated as an ideal, so $\mathcal{O}(V)^H$ is finitely generated.

General case: There exists a H-equivariant closed immersion $X \hookrightarrow V \in \mathbf{Rep}_H$; then $\mathcal{O}(V)^H \twoheadrightarrow \mathcal{O}(X)^H$. Why does $X \hookrightarrow V$ exist? Choose $W \subseteq \mathcal{O}(X)$ a finitely-generated H-invariant subspace generated $\mathcal{O}(X)$ as an algebra. Then take $X \to W^*$ by $x \mapsto (v \mapsto v(x))$.

(ii) obvious. (iii) follows from the arguments in the first part of (i). (e.g. if $J = \mathcal{O}(Y)$ and $J' = \mathcal{O}(Y')$ take $I = J\mathcal{O}(X)$ and $I' = J'\mathcal{O}(X)$ and want $(I + I')^H = I^H + (I')^H$, but both equal $J + J'$).

(iv): uniqueness follows from (iii). Existence: take an H-orbit in $\pi^{-1}[X]$ of minimal dimension.

Theorem (Kempf, strong Hilbert-Mumford theorem): Let E be a perfect field of any characteristic. Let $x \in X(E)$, let $\mathcal{O}$ be a closed orbit contained in $\overline{Hx}$. Then there exists $\lambda : \mathbb{G}_m \to H$ such that $\lim_{t \to 0} \lambda(t)x$ exists and is in $\mathcal{O}$.

Remark: what does $\lim_{t \to 0} \lambda(t)x$ exists? It means λ extends to a map $\mathbb{A}^1 \to X$, and the value at 0 is in $\mathcal{O}$.

Application by Richardson: Let $E = \overline{E}$ be of characteristic zero, and H acts on H^n as before. Say an element $\underline{g} = (g_1, \ldots, g_n) \in H^n$ is *semisimple* if $A(\mathfrak{g})$, the Zariski closure of $\langle g_1, \ldots, g_n \rangle$ in H, is reductive.

Corollary: Let $(g_1, \ldots, g_n) \in H$. TFAE:
(i) $(g_1, \ldots, g_n)$ is semisimple.
(ii) $H \cdot (g_1, \ldots, g_n)$ is closed.

Definition: If $\underline{g} \in H^n(E)$ then a *Levi decomposition* of $\underline{g}$ is a decomposition $\underline{g} = \underline{s}\underline{n}$ such that $A(\underline{s})$ is a Levi subgroup of $A(\underline{g})$ and $A(\underline{n}) \subseteq R(A(\underline{g}))$. If $n = 1$ this is just the Jordan decomposition.

Corollary: Levi decompositions always exist (but are not always unique in general). If $\underline{g} = \underline{s}\underline{n}$ is a Levi decomposition then $H_{\underline{g}} = H_{\underline{s}} \cap H_{\underline{n}}$, and there exists $\lambda : \mathbb{G}_m \to H$ with $\lim \lambda(t)\underline{g} = \underline{s}$.

Sketch of proof of Kempf's theorem (for $E = \overline{E}$ case): Take $x \in X$, and let $\mathcal{O}$ be a closed orbit in $\overline{Hx}$. Then there exists a curve C in Hx such that $x \in C(E)$ such that $\overline{C} \cap \mathcal{O} \neq \emptyset$. Then there exists a rational map $g(t) : \operatorname{Spec} E[\![t]\!] \to G$ with $\lim g(t)x \in \mathcal{O}$. So $g(t) \in H(E((t)))$. Use Cartan decomposition to get $h_1(t), h_2(t) \in H(E[\![t]\!])$ and $\mu : \mathbb{G}_m \to H$ such that $h_1(t)g(t) = t^\mu h_2(t)$. Set $h_i = h_i(0)$. Let $y = \lim g(t)x$.

Then $h_1 y = \lim h_1(t) g(t) x = \lim h_2(t) t^\mu x \in \mathcal{O}$. But it is not always true that $h_1 y = \lim h_2(0) t^\mu x$. Claim: (a) $\lim t^\mu h_2 x$ always exists, and (b) if $X = V \in \mathbf{Rep}_H$ and $y = 0$, then $h_1 y = \lim h_2 t^\mu x$ does hold. Claim implies theorem: there exists $f : X \to V$ that's H-equivariant such that $\mathcal{O} = f^{-1}[0]$ (easy) then $\lim t^\mu h_2 x \in \mathcal{O}$ so $\lim h_2^{-1} t^\mu h_2 x \in \mathcal{O}$ (a 1-parameter subgroup of H). Proof of claim: Assume $X = V$. Then $h_2(t) x = h_2 x + \varepsilon(t)$ for $\varepsilon \in V[\![t]\!]$ and $\varepsilon(0) = 0$. Decompose V into $\mathbb{G}_m$-eigenspaces for the action via $\mu : \mathbb{G}_m \to H$, $V = \bigoplus V_i$; then

$$h_2(t) x = \sum ((h_2 x)_i + \varepsilon(t)_i) \implies t^\mu \sum ((h_2 x)_i + \varepsilon(t)_i) = \sum t^i ((h_2 x)_i + \varepsilon(t)_i).$$

Then the limit of $t^\mu h_2(t) x$ existing means $(h_2 x)_i = 0$ for $i < 0$. So the limit $t^\mu h_2 x$ exists.

18 Lecture - 04/25/2014

Theorem: Let $E/\mathbb{Q}_\ell$ be a finite extension, H connected reductive split, Γ profinite group. Let H act on H^n by diagonal conjugation. Suppose for all $n \in \mathbb{N}$ you have an E-algebra morphism $\Xi_n : \mathcal{O}(H^n//H) \to \mathcal{C}(\Gamma^n, E)$ such that:
(1) Ξ_n is functorial in $\{1, \ldots, n\}$.
(2) $\Xi_{n+1}(f)(\gamma_1, \ldots, \gamma_{n+1}) = \Xi_n(\widehat{f})(\gamma_1, \ldots, \gamma_n\gamma_{n+1})$.
Then there exists E'/E finite and $\rho : \Gamma \to H(E')$ unique up to $H(\overline{\mathbb{Q}}_\ell)$-conjugacy such that $\Xi_n(f)(\gamma_1, \ldots, \gamma_n) = f(\rho(\gamma_1), \ldots, \rho(\gamma_n))$ and $\overline{\rho(\Gamma)}^{Zar}$ is reductive. Moreover there's $m > 0$ such that for all $\Gamma \twoheadrightarrow \overline{\Gamma}$, if $\Xi_m : \mathcal{O}(H^m//H) \to \mathcal{C}(\overline{\Gamma}^m, E)$ then ρ factors through $\overline{\Gamma}$.

Remark: Invariants for other groups (assuming algebraically closed characteristic zero base field). Procesi: The algebra of conjugacy invariants on $O(n)^N$ or $\mathrm{Sp}(N)^N$ is generated by

$$(g_1, \ldots, g_N) \mapsto \mathrm{tr}(\varepsilon_{i_1}(g_{i_1}), \ldots, \varepsilon_{i_r}(g_{i_r}))$$

for ε_i being either id or $^\top$ (in the orthogonal case) or id or $g \mapsto J^\top g^\top J$ (in the symplectic case).

Proof of theorem: Remember that we say that $(g_1, \ldots, g_n) \in H(\overline{\mathbb{Q}}_\ell)^n$ is *semisimple* if $\overline{\langle g_1, \ldots, g_n\rangle}^{Zar}$ is a reductive subgroup and that $(g_1, \ldots, g_n)$ is semisimple iff $H(g_1, \ldots, g_n)$ is closed (??). We also have seen that $(H^n//H)(\overline{\mathbb{Q}}_\ell)$ is isomorphic to the closed H-orbits in $H^n(\overline{\mathbb{Q}}_\ell)$, which is the same as the set of semisimple $(g_1, \ldots, g_n)$ modulo conjugacy.

Let $(\gamma_1, \ldots, \gamma_n) \in \Gamma^n$. we have a character $\mathcal{O}(H^n//H) \to \overline{\mathbb{Q}}_\ell$ given by $f \mapsto \Xi_n(f)(\gamma_1, \ldots, \gamma_n)$; this gives a point of $(H^n//H)(\overline{\mathbb{Q}}_\ell)$. We write $\xi_n^{ss}(\gamma_1, \ldots, \gamma_n)$ for the corresponding semisimple conjugacy class in $H^n(\overline{\mathbb{Q}}_\ell)$. We also write $\xi_n(\gamma_1, \ldots, \gamma_n)$ for the fiber over this point.

If $(g_1, \ldots, g_n) \in \xi_n^{ss}(\gamma_1, \ldots, \gamma_n)$, we set

$$C(g_1, \ldots, g_n) = Z_H(\langle g_1, \ldots, g_n\rangle) \qquad D(g_1, \ldots, g_n) = Z_H(C(g_1, \ldots, g_n)),$$

both reductive group. Let

$$\mathcal{N} = \{(n, \gamma_1, \ldots, \gamma_n) : n > 0, \gamma_1, \ldots, \gamma_n \in \Gamma\},$$
$$\mathcal{N}^1 = \{(n, \gamma_1, \ldots, \gamma_n) \in \mathcal{N} : \dim \overline{\langle g_1, \ldots, g_n\rangle} \text{ maximal }\}$$
$$\mathcal{N}^2 = \{(n, \gamma_1, \ldots, \gamma_n) \in \mathcal{N}^1 : \dim C(g_1, \ldots, g_n) \text{ minimal }\},$$
$$\mathcal{N}^3 = \{(n, \gamma_1, \ldots, \gamma_n) \in \mathcal{N}^1 : |\pi_0(C(g_1, \ldots, g_n)) \text{ minimal }\},$$

All of these are taken for fixed $(g_1, \ldots, g_n) \in \xi_n^{ss}(\gamma_1, \ldots, \gamma_n)$. Anyway, fix $(n, \gamma_1, \ldots, \gamma_n) \in \mathcal{N}^3$ and a corresponding $(g_1, \ldots, g_n)$.

Lemma: Let $m > 0$, let $\delta_1, \ldots, \delta_n \in \Gamma$. Then there exists a unique $h_1, \ldots, h_m \in H(\overline{\mathbb{Q}}_\ell)$ such that $(g_1, \ldots, g_n, h_1, \ldots, h_m) \in \xi_{n+m}^{ss}(\gamma_i, \delta_j)$. We have $C(g_1, \ldots, g_n, h_1, \ldots, h_m) = C(g_1, \ldots, g_n)$ and $h_1, \ldots, h_m \in D(g_1, \ldots, g_n)$. Moreover:
(i) If $p > 0$ and $\zeta : \{1, \ldots, p\} \to \{1, \ldots, n\}$ then

$$(g_1, \ldots, g_n, h_{\zeta(1)}, \ldots, h_{\zeta(p)}) \in \xi_{n+p}^{ss}(\gamma_i, \delta_{\zeta(j)}).$$

(ii) Take $m = 2$. Then $(g_1, \ldots, g_n, h_1h_2) \in \xi_{n+1}^{ss}(\gamma_i, \delta_1\delta_2)$.

Proof: Pick $(x_1, \ldots, x_n, y_1, \ldots, y_m) \in \xi_{n+m}^{ss}(\gamma_i, \delta_j)$. Is $(x_1, \ldots, x_n)$ semisimple? By condition (1) of the theorem applied to the inclusion $\{1, \ldots, n\} \hookrightarrow \{1, \ldots, n+m\}$, have $(x_1, \ldots, x_n) \in \xi_n(\gamma_1, \ldots, \gamma_n)$. Hence $\overline{\langle g_1, \ldots, g_n\rangle}^{Zar}$ is conjugate to (a subgroup of) a Levi subgroup of $\overline{\langle x_1, \ldots, x_n\rangle}^{Zar}$. (We know there exists a cocharacter $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_\ell} \to H_{\overline{\mathbb{Q}}_\ell}$ such that $\lim \mu(t)(x_1, \ldots, x_n)$ exists and is conjugate to $(g_1, \ldots, g_m)$. Then $p_\mu = \{g \in H : \lim \mu(t)g\mu(t)^{-1} \text{ exists}\}$ is a parabolic subgroup of $H_{\overline{\mathbb{Q}}_\ell}$ with Levi $Z_H(\mu)$. Then $\overline{\langle x_1, \ldots, x_n\rangle}^{Zar} \subseteq P_\mu$ and $\overline{\langle g_1, \ldots, g_n\rangle}^{Zar}$ is contained in a conjugate of $Z_H(\mu)$). In particular,

$$\dim \overline{\langle g_1, \ldots, g_n\rangle}^{Zar} \leq \dim \overline{\langle x_1, \ldots, x_n\rangle}^{Zar} \leq \dim \overline{\langle x_1, \ldots, x_n, y_1, \ldots, y_m\rangle}^{Zar}.$$

As $(n,\gamma_1,\dots,\gamma_n)$ is in $\mathcal{N}^1$, these are equalities so $\overline{\langle g_1,\dots,g_n\rangle}^{Zar}$ and $\overline{\langle x_1,\dots,x_n\rangle}^{Zar}$ are conjugate.

So $(x_1,\dots,x_n)\in\xi_n^{ss}(\gamma_1,\dots,\gamma_n)$ so we may assume $x_i=g_i$. Take $h_i=y_i$. We have $(g_i,h_j)\in\gamma_{n+m}^{ss}(\gamma_i,\delta_j)$ and $(h_1,\dots,h_n)$ is unique up to $C(g_1,\dots,g_n)$-conjugacy. We have $C(g_1,\dots,g_n,h_1,\dots,h_m)\subseteq C(g_1,\dots,g_n)$ and $\overline{\langle x_1,\dots,x_n\rangle}^{Zar}=\overline{\langle x_1,\dots,x_n,y_1,\dots,y_m\rangle}^{Zar}$. So $(n{+}m,\gamma_i,\delta_j)\in\mathcal{N}^1$. As $(n,\gamma_i)\in\mathcal{N}^3$ we get $C(g_i,h_j)=C(g_i)$ so $h_1,\dots,h_m\in D(g_1,\dots,g_m)$ so are uniquely determined. Then (i) and (ii) follow from (1) and (2) of the statement plus uniqueness.

End of proof of theorem: Let $\rho:\Gamma\to H(\overline{\mathbb{Q}}_\ell)$ be defined by: for each $\gamma\in\Gamma$, $\rho(\gamma)$ is the unique element of $H(\overline{\mathbb{Q}}_\ell)$ such that $(g_1,\dots,g_n,\rho(\gamma))\in\xi_{n+1}^{ss}(\gamma_1,\dots,\gamma_n,\gamma)$. Then we need to check:
(A) There's E'/E finite such that $\rho[\gamma]\subseteq H(E')$ (immediate if we can prove continuity without knowing this?): Just choose E' with $g_1,\dots,g_n\in H(E')$.
(B) By (i) in the lemma, for all $\delta_1,\dots,\delta_m\in\Gamma$ we have

$$(g_1,\dots,g_n,\rho(\delta_1),\dots,\rho(\delta_n))\in\xi_{n+m}^{ss}(\gamma_1,\dots,\gamma_n,\delta_1,\dots,\delta_n).$$

(C) By (B) and (ii) of the lemma, for all $\delta_1,\dots,\delta_2\in\Gamma$ we have

$$(g_1,\dots,g_n,\rho(\delta_1)\rho(\delta_2))\in\xi_{n+1}^{ss}(\gamma_1,\dots,\gamma_n,\delta_1\delta_2),$$

so $\rho(\delta_1)\rho(\delta_2)=\rho(\delta_1\delta_2)$.
(D) Let $m\in\mathbb{N}$, $\delta_1,\dots,\delta_m\in\Gamma$, and $f\in\mathcal{O}(H^m//H)$. Then

$$(g_1,\dots,g_n,\rho(\delta_1),\dots,\rho(\delta_m))\in\xi_{n+m}^{ss}(\gamma_1,\dots,\gamma_m,\delta_1,\dots,\delta_n).$$

By applying (1) of the theorem to $\{1,\dots,m\}\to\{1,\dots,n{+}m\}$ given by $j\mapsto j{+}n$, we get $(\rho(\delta_1),\dots,\rho(\delta_m))\in\xi_m(\delta_1,\dots,\delta_m)$, i.e. $f(\rho(\delta_1),\dots,\rho(\delta_m))=\Xi_m(f)(\delta_1,\dots,\delta_m)$.
(E) ρ is continuous: Note $\rho[\Gamma]\subseteq D(g_1,\dots,g_n)$; we want to show that, for all $f\in\mathcal{O}(D(g_1,\dots,g_n))$, $f\circ\rho:\Gamma\to\overline{\mathbb{Q}}_\ell$ is continuous. Claim: $q:\mathcal{O}(H_{E'}^{n+1}//H_{E'})\to\mathcal{O}(D(g_1,\dots,g_n))$ given by $f\mapsto(g\mapsto f(g_1,\dots,g_n,g))$ is surjective. Granting this, for $f\in\mathcal{O}(D(g_1,\dots,g_n))$ let $f'\in\mathcal{O}(H_{E'}^{n+1}//H_{E'})$ be such that $q(f')=f$. Then $f(\rho(\gamma))=f'(g_1,\dots,g_n,\rho(\gamma))=\Xi_{n+1}(f')(\gamma_1,\dots,\gamma_n,\gamma)$ is continuous.

Proof of claim: q is the composition of $q_1:\mathcal{O}(H_{E'}^{N+1}//H_{E'})\to\mathcal{O}(H_{E'}//C(g_1,\dots,g_n))$ given by $f\mapsto(g\mapsto f(g_1,\dots,g_n,g))$ and the restriction map $q_2:\mathcal{O}(H_{E'}//C(g_1,\dots,g_n))\to\mathcal{O}(D(g_1,\dots,g_n))$.

Proof that q_2 is surjective: $D(g_1,\dots,g_n)=C(g_1,\dots,g_n)\cdot D(g_*)$ is a closed $C(g_1,\dots,g_n)$-invariant subvariety of $H_{E'}$ so $D(g_1,\dots,g_n)//C(g_1,\dots,g_n)$ has a closed embedding to $H_{E'}//C(g_1,\dots,g_n)$. Proof that q_1 is surjective: The set $Y=H_{E'}((g_1,\dots,g_n)\times H_{E'})$ is closed in $H_{E'}^{n+1}$ and $H_{E'}$-invariant. Then $Y//H_{E'}$ has a closed embedding to $H_{E'}^{n+1}//H_{E'}$, and is isomorphic to $H_{E'}//C(g_1,\dots,g_n)$ via $g\mapsto(g_1,\dots,g_n,g)$.
(F) ρ is semisimple because $\overline{\rho(\Gamma)}^{Zar}=\overline{\langle g_1,\dots,g_n\rangle}^{Zar}$.
Proof of the last statement of the theorem: $\rho(\gamma)$ is uniquely characterized by all $f\in\mathcal{O}(H^{n+1}//H)$. But this is because $f(g_1,\dots,g_n,\rho(\gamma))=\Xi_{n+1}(f)(\gamma_1,\dots,\gamma_n,\gamma)$.

Remark: What happens if we work with $\overline{\mathbb{F}}_\ell$ rather than $\overline{\mathbb{Q}}_\ell$? Everything works, if you use the right definitions. Definition: k a field, G/k reductive; we say a subgroup Γ of $G(k)$ is G-completely reducible if for all parabolics $P\subseteq G$ such that $\Gamma\subseteq P(K)$ there exists a Levi with $\Gamma\subseteq L(k)$. The theorem goes through if we replace 'semisimple" with this everywhere. (Remark: If $G=\mathrm{GL}(V)$ this says that the representation V of Γ is semisimple). Proposition: if L is a Levi of G with $\Gamma\subseteq L(k)$, then Γ is G-CR iff it's L-CR.

Theorem (Bate-Martin-Rhorle): Suppose $k=\overline{k}$, let $\Gamma\subseteq G(k)$, let T be a maximal torus in $Z_G(\Gamma)$ and let $L=Z_G(T)$ (so $\Gamma\subseteq L(k)$). Then Γ is completely reducible iff Γ is not contained in any proper parabolic of L. Cor (BMR): Let $g_1,\dots,g_n\in G(k)$. Then $\langle g_1,\dots,g_n\rangle$ is completely reducible iff $G(k)\cdot(g_1,\dots,g_n)$ is closed.

19 Lecture - 04/30/2014

Aim today: repeat the first lecture in more detail. Let $X/\mathbb{F}_q$ be a curve (smooth proper geometrically connected). Let $F = \mathbb{F}_q(X)$, $|X|$ the closed points of X, for all $v \in |X|$ let $F_v \supseteq \mathcal{O}_v$ be the completion, and $\mathbb{A}$ the restricted direct product $\prod' F_v$ with respect to the $\mathcal{O}_v$'s. Let $G/\mathbb{F}_q$ be a connected reductive split algebraic group.

To deal with levels: Let $N \subseteq X$ be a finite subscheme, and let $\mathcal{O}_N = \mathcal{O}(N)$ and $K_N = \ker(G(\mathcal{O}) \to G(\mathcal{O}_N))$. Fix a lattice Ξ in $Z(F)\backslash Z(\mathbb{A})$. Then can define the space of cusp forms

$$C_c^{\text{cusp}} = C_c^{\text{cusp}}(G(F)\backslash G(\mathbb{A})/K_N\Xi, \overline{\mathbb{Q}}_\ell);$$

a smooth function f is a cusp form iff for all $P \subseteq G$ parabolic proper subgroups and for all $g \in G(\mathbb{A})$,

$$\int_{N_P(F)\backslash N_P(\mathbb{A})} f(ng)dn = 0$$

where $N_P = R_u(P)$. Then C_c^{cusp} has an action of the Hecke algebra $\mathcal{H}_N = C_c(K_N\backslash G(\mathbb{A})/K_N, \overline{\mathbb{Q}}_\ell)$.

Theorem (V. Lafforgue): There exists a canonical decomposition of $\mathcal{H}_N$-modules

$$C_c^{\text{cusp}} = \bigoplus_\sigma H_\sigma$$

(where σ runs over all isomorphism classes of continuous semisimple (i.e. $\overline{\sigma(\Gamma_F)}^{Zar}$ is reductive) unramified outside of N homomorphism $\sigma : \text{Gal}(\overline{F}/F) = \Gamma_F \to \widehat{G}(\overline{\mathbb{Q}}_\ell))$, which is compatible with the Satake isomorphism at places $v \notin N$ (since $\mathcal{H}_N = {\bigotimes'}_{v\in|X|}\mathcal{H}_{N,v}$).

Idea: For number fields we'd use Shimura varieties to construct global correspondences. For function fields, have analogues of these for every group G and every cocharacter. These are moduli stacks of shtukas. For all finite I, for all level levels N, and for all irreducible representations W of $\widehat{G}^I$, we have a Deligne-Mumford stack $Cht_{I,W,N}$ (G-bundle with additional structure). These are all substacks of a big ind-stack shtuka $Cht_{I,N}$ all living over $(X \setminus N)^I$. Now, W corresponds to a cocharacter $\lambda : \mathbb{G}_m \to G^I$. If λ is not minuscule, then $Cht_{I,W,N} \to (X \setminus N)^I$ is not always smooth. We consider the (intersection) cohomology (with compact support) of $Cht_{I,N,w}$, in the middle degree, seen as an ind-constructible sheaf over $(X/N)^I$. Namely, for $Cht_{I,W,N}/\Xi$ have an open dense smooth substack of dimension d; call it U and the embedding j. Then set

$$IC_{Cht_{N,I,W}} = (j_{!*}\overline{\mathbb{Q}}_{\ell,U}[d])[-|I|]$$

Look at

$$R^0\pi_!(Cht_{I,W,N}/\Xi, IC_{Cht_{N,I,W}}).$$

This has an action of $\mathcal{H}_N$, and expect it to contain all of the cuspidal representations.

But there are some annoying technical problems to get around. First of all, we can restrict to the generic point η of $(X \setminus N)^I$ to get a representation of $\pi_1^{\text{ét}}(\eta, \overline{\eta})$ which is not Γ_F^I (the thing we want a representation of). By Drinfeld, if the sheaf were constructible then it would be lisse on some open of the form U^I for $U \neq \emptyset$ in $X \setminus N$, and we would then get a representation of Γ_F^I in that case. Lafforgue: Define a subspace of "Hecke-finite" elements (elements that are in a finite-dimensional $\mathcal{H}_N$-invariant subspace) and show that it is stable by enough elements to make Drinfeld's lemma work.

Result: Get $H_{I,N,W}$, an inductive limit(?) of finite-dimensional representations of $\mathcal{H}_N \times \Gamma_F^I$. Heuristic (for $G = \text{GL}_n$): This $H_{I,N,w}$ should be $\bigoplus_\sigma A_\sigma \otimes W_{\sigma^I}$ where A_σ is a representation of $\mathcal{H}_N$ and W_{σ^I} is a representation of GL_F^I (which is just W with the action of Γ_F^I coming from $\Gamma_F^I \to \widehat{G}(\overline{\mathbb{Q}}_\ell)^I$ and then the original action W). Expect $A_\sigma = (\pi_\sigma)^{K_N}$ where π_σ is the thing corresponding to σ under local Langlands. Moreover: we have smooth maps from $Cht_{I,N,W}$ and from the closure of the orbit corresponding to W in

$\mathcal{G}r_{X^I}$ to a fixed thing $*$. (Closure of the orbit: Note W corresponds to $\lambda : \mathbb{G}_m \to G^I$, and recall that $\mathcal{G}r_{X^I} \to X^I$ is locally isomorphic to $(\mathrm{Gr}_G)^I \times X^I$ outside of the orbits; can consider $\overline{O}_\lambda$ in $(\mathrm{Gr}_G)^I$ and then we take $\overline{O}_\lambda \times X^I$ outside the diagonals and take its closure in $\mathcal{G}r_{X^I}$).

Also have geometric Satake: $\mathbf{Rep}_{\widehat{G}^I} \to \mathbf{Perv}_{\mathcal{G}[\![t]\!]}(\mathcal{G}r_{X^I})$ given by taking W irreducible to $IC_{\mathcal{O}}$ where $\mathcal{O}$ is the closure of the orbit corresponding to W. Will then get a map from $\mathbf{Rep}_{\widehat{G}^I} \to \mathbf{Perv}(Cht_{I,N})$ by $W \mapsto IC_{Cht_{I,N,W}}$. Modulo generalizing the stuff about Drinfeld's lemma, we get a functor $\mathcal{H}$ from $\mathbf{Rep}_{\widehat{G}^I}$ to the category of inductive limits of finite-dimensional representations of $\mathcal{H}_N \times \Gamma_F^I$, sending W to something we denote $H_{I,W,N}$.

Proposition: (a) For all $\zeta : I \to J$ we have $X_\zeta : H_{I,W,N} \cong H_{J,W^\zeta,N}$ functorial in W, where W^ζ is W given the action of $\widehat{G}^J$ coming from $\zeta^* : \widehat{G}^J \to \widehat{G}^I$. This is Γ_F^J-equivariant and $\mathcal{H}_N$-equivariant. Also $\chi_{\zeta\circ\zeta'} = \chi_\zeta \circ \chi_{\zeta'}$.
(b) If $I = \emptyset$ and $W = \mathbb{1}$, then $H_{\emptyset,\mathbb{1},N} = C_c^{\mathrm{cusp}}$ with the action of $\mathcal{H}_N$.

Write $Bun_{G,N}(\mathbb{F}_q) = G(F)\backslash G(\mathbb{A})/K_N$. Recall our heuristic for $G = \mathrm{GL}_n$ that $H_{I,N,W}$ should be $\bigoplus_\sigma A_\sigma \otimes W_{\sigma^I}$. If this is true then let $\zeta_I : I \to \{0\}$ be the only map, and for all $x : \mathbb{1} \to W^{\zeta_I}$ (i.e. $x \in W^{\text{diagonal } \widehat{G}}$) and $\xi : W^{\zeta_I} \to \mathbb{1}$ (i.e. $\zeta \in (W^*)^{\text{diagonal } \widehat{G}}$) and all $(\gamma_i)_{i\in I}$ with $\gamma_i \in \Gamma_F^I$, consider $S_{I,x,\xi,W,(\gamma_i)}$, an operator on C_c^{cusp} defined by a chain of maps

$$\begin{array}{ccccccccc}
C_c^{\mathrm{cusp}} & \xrightarrow{=} & H_{\emptyset,\mathbb{1}} & \xrightarrow{\cong} & H_{\{0\},\mathbb{1}} & \xrightarrow{\mathcal{H}(x)} & H_{\{0\},W^{\zeta_i}} & \xrightarrow{\chi_{\zeta_I}^{-1}} & H_{I,W} \\
 & & & & & & & & \big\downarrow{\scriptstyle (\gamma_i)} \\
C_c^{\mathrm{cusp}} & \xleftarrow[=]{} & H_{\emptyset,\mathbb{1}} & \xleftarrow[\cong]{} & H_{\{0\},\mathbb{1}} & \xleftarrow[\mathcal{H}(\xi)]{} & H_{\{0\},W^{\zeta_i}} & \xleftarrow[\chi_{\zeta_I}]{} & H_{I,W}
\end{array}$$

On $A_\sigma \otimes W_{\sigma^I}$ this is equal to multiplication by $\langle x, (\sigma(\gamma_i))\xi\rangle$; but $S_{I,x,\xi,(\gamma_i)}$ makes sense in general (despite its very complicated definition). Idea: define then diagonalize these to get the decomposition.

Note: The functions $(g_i)_{i\in I} \mapsto \langle x, (g_i)\xi\rangle$ (with $g_i \in G^I$) for W, x, ξ varying are all functions in $\mathcal{O}(\widehat{G}\backslash\backslash\widehat{G}^I//\widehat{G})$.

Proposition: (a) $S_{I,x,\xi,W,(\gamma_i)}$ only depends on the corresponding function $f \in \mathcal{O}(\widehat{G}\backslash\backslash\widehat{G}^I//\widehat{G})$. So simplify notation and call it $S_{I,f,(\gamma_i)}$
(b) $f \mapsto S_{I,f,(\gamma_i)}$ is a map of $\overline{\mathbb{Q}}_\ell$-algebras

$$\mathcal{O}(\widehat{G}\backslash\backslash\widehat{G}^I//\widehat{G}) \to \mathrm{End}_{\mathcal{H}_N}(C_c^{\mathrm{cusp}}).$$

(c) As the parameters vary, the operators $S_{f,I,(\gamma_i)}$ generate a commutative subalgebra $\mathcal{B}$ of $\mathrm{End}_{CH_N}(C_c^{\mathrm{cusp}})$.

Let $n > 0$ and $\nu : \mathcal{B} \to \overline{\mathbb{Q}}_\ell$ a character. Note we have $\widehat{G}^n//\widehat{G} \cong \widehat{G}\backslash\backslash\widehat{G}^{\{0,\dots,n\}}//\widehat{G}$ (where the action for the domain is given by conjugacy) by $(g_1,\dots,g_n) \mapsto (1, g_1, \dots, g_n)$. So $\theta_{\{0,\dots,n\}}$ gives a $\overline{\mathbb{Q}}_\ell$-algebra map $\Xi_{n,\nu} : \mathcal{O}(\widehat{G}^n//\widehat{G}) \to \mathcal{B} \to \overline{\mathbb{Q}}_\ell$.

Proposition: The $(\Xi_{n,\nu})_n$ are a pseudo-representation in the sense of last week. We get $\sigma_\nu : \Gamma_F \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$ continuous semisimple such that the on ν-eigenspaces of $\mathcal{B}$ on C_c^{cusp}, $S_{I,f,(\gamma_i)}$ acts by multiplication by $f(\sigma_v(\gamma_i))$.

So we have a decomposition. Somewhat harder is that it's a decomposition that actually means something to us:

Key proposition (more difficult): If $v \in |X \setminus N|$ and if V is an irreducible representation of $\widehat{G}$, then let $h_{V,v}$ be the corresponding element of $\mathcal{H}_{N,v} = C_c(G(\mathcal{O}_v)\backslash G(F_v)/G(\mathcal{O}_v), \overline{\mathbb{Q}}_\ell)$. Then $h_{V,v}$ acts on C_c^{cusp} as $S_{\{1,2\},f,(\mathrm{Frob},1)}$ where $f(g_1,g_2) = \mathrm{tr}(g_1 g_2^{-1}, V)$ and $\mathrm{Frob} = \mathrm{Frob}_v$ is any lift of the (geometric) Frobenius at v.