Beilinson's construction of nearby cycles and gluing

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The goal of this note is to present Beilinson's construction of the nearby cycles functor, and its application to gluing perverse sheaves (also due to Beilinson). The original reference for all of this is Beilinson's article [1]. See also Reich's article [5] and Sam Lichtenstein's senior thesis [4].

We fix a field k. All the schemes we will consider will be separated and of finite type over k. If X is a scheme, we write $D^b_c(X)$ for the category of bounded constructible ℓ-adic complexes on X (with coefficients in a finite extension of $\mathbb{Q}_\ell$ or in $\overline{\mathbb{Q}}_\ell$) or, if $k = \mathbb{C}$, for bounded constructible complexes on $X(\mathbb{C})$. (The formalism will work in both cases.) In both cases, we'll write $\mathrm{Perv}(X)$ for the heart of the selfdual perverse t-structure on $D^b_c(X)$. When we say "exact" later, it is always understood to mean "exact for the perverse t-structure". The standard reference for this paragraph is the book [2] by Beilinson-Bernstein-Deligne.

Here is a bit of motivation : Let $i : Y \to X$ be a closed embedding; we want to understand the functor i^* as a functor between derived categories of perverse sheaves. The simplest case if when the open embedding $j : X - Y \to X$ is affine, and we can always reduce to this case locally (by an induction on the number of equations defining Y in X), so we will assume that we are in this case. Then, if $K \in \mathrm{Perv}(X)$, i^*K is concentrated in degrees -1 and 0, and we have an exact sequence of perverse sheaves on X :

$$0 \to i_* {}^p\mathrm{H}^{-1} i^* K \to j_! j^* K \to K \to i_* {}^p\mathrm{H}^0 i^* K \to 0,$$

so that $i_* i^* K$ is represented by the complex of perverse sheaves $j_! j^* K \to K$ (with K in degree 0). However, this is a complex of perverse sheaves on X. We would like to find a complex of perverse sheaves on Y (or of perverse sheaves on X with support in Y) representing i^*K. This turns out to be possible, provided we fix an equation $f \in \mathscr{O}(X)$ of Y (which is always possible locally). Then we will see that there exist two functors $\Psi^u_f : \mathrm{Perv}(X - Y) \to \mathrm{Perv}(Y)$ and $\Phi^u_f : \mathrm{Perv}(X) \to \mathrm{Perv}(Y)$ and a functorial exact sequence

$$0 \to {}^p\mathrm{H}^{-1} i^* \to \Psi^u_f j^* \to \Phi^u_f \to {}^p\mathrm{H}^0 i^* \to 0.$$

Remark 0.1 The functors Ψ_f^u and Φ_f^u are the nearby cycles functor and the vanishing cycles functor (shifted by $[-1]$ so they will preserve perverse sheaves), or rather the direct factors of these functors where the monodromy operator acts unipotently. We could remove the "u" and get the same result, but the unipotent versions are a bit simpler to construct. Nearby and vanishing cycles functors have many other properties, some of which we will review later.

We will assume the results of SGA 7 I and XIII and SGA 4 1/2 [Th. finitude]. More precisely, here is what we need : First, fix a topological generator T of the prime-to-p quotient of $\pi_1^{\mathrm{geom}}(\mathbb{G}_{m,k}, 1)$ (where $p = char(k)$). For every scheme X and every morphism $f : X \to \mathbb{A}^1$, if $i : Y := f^{-1}(0) \to X$ and $j : U := X - Y \to X$ are the inclusions, we assume that we know how to construct a nearby cycles functor $\Psi_f : D_c^b(U) \to D_c^b(Y_{\overline{k}})$, a functorial action of $\pi_1(\mathbb{G}_{m,k}, 1)$ on Ψ_f, compatible with the action of $\mathrm{Gal}(\overline{k}/k)$ on $Y_{\overline{k}}$, and a functorial exact triangle $\Psi_f \overset{T-1}{\to} \Psi_f \to i^* j_* \overset{+1}{\to}$. (Technically, the last term should be base changed from Y to $Y_{\overline{k}}$. We'll omit this in the notation.) Note that we have shifted the nearby cycles functor of SGA 7 XIII by $[-1]$, so that it will preserve perverse sheaves (as we will see later).

If $g : X' \to X$ is a morphism of schemes, we have canonical functorial morphisms :

$$\Psi_f g_* \to g_* \Psi_{gf}$$

$$g_! \Psi_{gf} \to \Psi_f g_!$$

$$g^* \Psi_f \to \Psi_{gf} g^*$$

$$\Psi_{gf} g^! \to g^! \Psi_f.$$

If g is proper, then the first two morphisms are isomorphisms and are each other's inverse. If g is smooth, then the last two morphisms are isomorphisms, and are each other's inverse up to some twists and shifts. (We will only use the existence of the third morphism.)

Note also that, for $K, K' \in D_c^b$, we have a canonical morphism

$$\Psi_f K \otimes \Psi_f K' \to \Psi_f (K \otimes K')[-1],$$

see 4.3 of Illusie's *Autour du théorème de monodromie locale* ([3]).

Finally, note that, if $\mathscr{L}$ is a local system on $\mathbb{G}_m$ and L is the representation of $\pi_1(\mathbb{G}_m, 1)$ corresponding to $\mathscr{L}$, then Ψ_{id} sends $\mathscr{L}[1]$ to $L_{\{0\}}$ with its obvious action of $\pi_1(\mathbb{G}_m, 1)$.

Remark 0.2 It is not totally clear from the formula "$Y = f^{-1}(0)$" what scheme structure we are putting on the closed subset Y of X. It doesn't matter in practice, because the category $D_c^b(Y)$ only depends on Y_{red}. So we could put the reduced structure on Y, or we could think of Y as the fiber product of $\{0\}$ (with reduced structure) and X over $\mathbb{A}^1$.

1 Unipotent nearby cycles functor

We fix a scheme X and a morphism $f : X \to \mathbb{A}^1$. As before, we denote by $i : Y := f^{-1}(0) \to X$ and $j : U := X - Y \to X$ the inclusions.

Proposition 1.1 *There exists a functorial T-equivariant direct sum decomposition $\Psi_f = \Psi_f^u \oplus \Psi_f^{nu}$ such that, for every $K \in D_c^b(U)$, $T-1$ acts nilpotently on $\Psi_f^u(K)$ and invertibly on $\Psi_f^{nu}(K)$.*

In particular, the functorial exact triangle $\Psi_f \stackrel{T-1}{\rightarrow} \Psi_f \rightarrow i^*j_* \stackrel{+1}{\rightarrow}$ induces a functorial exact triangle $\Psi_f^u \stackrel{T-1}{\rightarrow} \Psi_f^u \rightarrow i^*j_* \stackrel{+1}{\rightarrow}$.

The functor Ψ_f^u is called the unipotent nearby cycles functor.

Proof. It suffices to prove that, for every $K \in D_c^b(U)$, there exists a nonzero polynomial P (with coefficients in the coefficient field F that we are using for the categories D_c^b) such that $P(T)$ acts by 0 on $\Psi_f(K)$. (The rest is standard linear algebra.) As we know that Ψ_f sends $D_c^b(X)$ to $D_c^b(Y_{\overline{k}})$ (ie preserves constructibility), this follows from the fact that, for every $L \in D_c^b(Y_{\overline{k}})$, the ring of endomorphisms of L is finite-dimensional (over the same coefficient field F). To prove this fact, we use induction on the dimension of X to reduce to the case where the cohomology sheaves of L are local systems, and then it is trivial.

□

Remark 1.2 We can recover the full nearby cycles functor from Ψ_f^u, at least if we extend the coefficient field to make it algebraically closed.

Proposition 1.3 *The functor $\Psi_f^u : D_c^b(X) \rightarrow D_c^b(Y_{\overline{k}})$ is t-exact.*

Proof. Let $K \in \mathrm{Perv}(U)$. We want to prove that $\Psi_f^u K$ is perverse. Remember that we have an exact triangle $\Psi_f^u K \stackrel{T-1}{\rightarrow} \Psi_f^u K \rightarrow i^*j_*K \stackrel{+1}{\rightarrow}$. As i^*j_*K is concentrated in perverse degrees -1 and 0, $\Psi_f^u K$ is concentrated in perverse degrees -1, 0 and 1. Moreover, looking at the long exact sequence in perverse cohomology coming from the exact triangle above, we see that the maps $T-1 : {}^p\mathrm{H}^{-1}\Psi_f^u K \rightarrow {}^p\mathrm{H}^{-1}\Psi_f^u K$ and $T-1 : {}^p\mathrm{H}^1\Psi_f^u K \rightarrow {}^p\mathrm{H}^1\Psi_f^u K$ are respectively injective and surjective. But $T-1$ is nilpotent on all the ${}^p\mathrm{H}^i\Psi_f^u K$ by construction, so we get ${}^p\mathrm{H}^{-1}\Psi_f^u K = {}^p\mathrm{H}^1\Psi_f^u K = 0$, and $\Psi_f^u K$ is concentrated in perverse degree 0.

□

Our next task will be to show that Ψ_f^u can actually be seen as a functor from $\mathrm{Perv}(U)$ to $\mathrm{Perv}(Y)$ (instead of $\mathrm{Perv}(Y_{\overline{k}})$). We will need a more canonical version of the endomorphism $T-1$, as $T-1$ will not in general descend to an endomorphism in $D_c^b(Y)$.

Let t be the usual surjective map from $\pi_1^{\mathrm{geom}}(\mathbb{G}_{m,k}, 1)$ to $\mathbb{Z}_\ell(1)$ (in the ℓ-adic case) or $\widehat{\mathbb{Z}}(1) := (2i\pi)\widehat{\mathbb{Z}}$ (in the complex case), cf SGA 7 I (0.3). Let $K \in D_c^b(U)$. Then $T : \Psi_f^u K \rightarrow \Psi_f^u K$ is unipotent, so there exists a unique nilpotent $N : \Psi_f^u K \rightarrow \Psi_f^u K(-1)$ such that $T = \exp(t(T)N)$ on $\Psi_f^u K$. The operator N is usually called the "logarithm of the unipotent part of the monodromy". (Here it is the logarithm of the monodromy, because we are looking only at the part where the monodromy is unipotent.) We get a functorial exact triangle $\Psi_f^u \xrightarrow{N} \Psi_f^u \to i^* j_* \xrightarrow{+1}$.

2 Some local systems on $\mathbb{G}_{m,k}$

If $\mathrm{char}(k) > 0$, we write $p = \mathrm{char}(k)$; otherwise we take $p = 1$.

Note that the category of local systems on $\mathbb{G}_{m,k}$ is equivalent to the category of (continuous) representations of $\pi_1(\mathbb{G}_{m,k}, 1)$. We have an exact sequence

$$1 \to \pi_1^{\mathrm{geom}}(\mathbb{G}_{m,k}, 1) \to \pi_1(\mathbb{G}_{m,k}, 1) \to \mathrm{Gal}(\overline{k}/k) \to 1.$$

Any k-rational point of $\mathbb{G}_{m,k}$ gives a section of the last map. Using the section given by the point 1, we get an isomorphism $\pi_1(\mathbb{G}_{m,k}, 1) \simeq \pi_1^{\mathrm{geom}}(\mathbb{G}_{m,k}, 1) \rtimes \mathrm{Gal}(\overline{k}/k)$, where $\mathrm{Gal}(\overline{k}/k)$ acts on $\pi_1^{\mathrm{geom},(p)}(\mathbb{G}_{m,k}, 1) = \widehat{\mathbb{Z}}^{(p)}(1)$ (the exponents (p) mean that we are taking the prime-to-p quotients) by multiplication by the cyclotomic character.

As before, we denote by F the coefficient field that we use in the categories D_c^b. Fix $a \in \mathbb{N}$. Let $L_a = F \oplus F(-1) \oplus \cdots \oplus F(-a)$, and let $N : L_a \to L_a(-1)$ be the (nilpotent) morphism given by

$$N = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}.$$

We define an action of the group $\pi_1(\mathbb{G}_{m,k}, 1) = \pi_1^{\mathrm{geom}}(\mathbb{G}_{m,k}, 1) \rtimes \mathrm{Gal}(\overline{k}/k)$ on L_a in the following way : an element $u \rtimes \sigma$ acts by the matrix $\exp(t(u)N) \begin{pmatrix} 1 & & & \\ & \chi(\sigma)^{-1} & & \\ & & \ddots & \\ & & & \chi(\sigma)^{-a} \end{pmatrix}$, where $\chi : \mathrm{Gal}(\overline{k}/k) \to \widehat{\mathbb{Z}}(1)$ is the cyclotomic character. We denote by $\mathscr{L}_a$ the local system on $\mathbb{G}_{m,k}$ associated to L_a.

For $a \leq b$, there is an obvious injection $\alpha_{a,b} : L_a \to L_b$ and an obvious surjection $L_b \to L_a(a-b)$, and these maps are $\pi_1(\mathbb{G}_{m,k}, 1)$-equivariant, hence they define morphisms of local systems $\alpha_{a,b} : \mathscr{L}_a \hookrightarrow \mathscr{L}_b$ and $\beta_{a,b} : \mathscr{L}_b \twoheadrightarrow \mathscr{L}_a(a-b)$. Moreover, the dual of L_a is isomorphic to $L_a(a)$, and again this isomorphism is $\pi_1(\mathbb{G}_{m,k}, 1)$-equivariant, hence we get an isomorphism $D(\mathscr{L}_a) \simeq \mathscr{L}_a(a+1)[2]$. Via these isomorphisms, $D(\alpha_{a,b})$ corresponds to $\beta_{a,b}(b+1)[2]$. Finally, note that the composition $L_a \xrightarrow{\alpha_{a,a+1}} L_{a+1} \xrightarrow{\beta_{a+1,a}} L_a(-1)$ is equal to N.

3 Beilinson's construction of Ψ_f^u (cf [1])

Let X, f etc be as in section 1. Note that, for every perverse sheaf K on U, $K \otimes^L f^*\mathscr{L}_a$ is also perverse; we often denote this perverse sheaf by $K \otimes \mathscr{L}_a$.

Proposition 3.1 *Let $K \in \mathrm{Perv}(U)$. Then, for every $a \in \mathbb{N}$, there is a natural isomorphism* $\mathrm{Ker}(N^{a+1}, \Psi_f^u K) \simeq \mathrm{Ker}(N, \Psi_f^u(K \otimes \mathscr{L}_a))$.

Corollary 3.2 *For any $a \in \mathbb{N}$ such that $N^{a+1}(\Psi_f^u K) = 0$ (in particular, for any a big enough), there is a natural isomorphism*

$$i_*\Psi_f^u K \simeq \mathrm{Ker}(j_!(K \otimes \mathscr{L}_a) \to j_*(K \otimes \mathscr{L}_a)) = {}^p\mathrm{H}^{-1} i^* j_*(K \otimes \mathscr{L}_a) = i^* j_{!*}(K \otimes \mathscr{L}_a)[-1].$$

(The last two equalities are corollary 4.1.12 of [2].)

Proof. By the exact triangle $\Psi_f^u \xrightarrow{N} \Psi_f^u \to i^* j_* \xrightarrow{+1}$, we have

$$\begin{aligned} i_* \mathrm{Ker}(N, \Psi_f^u(K \otimes \mathscr{L}_a)) &= i_* {}^p\mathrm{H}^{-1} i^* j_*(K \otimes \mathscr{L}_a) \\ &= \mathrm{Ker}(j_!(K \otimes \mathscr{L}_a) \to j_*(K \otimes \mathscr{L}_a)) \end{aligned}.$$

□

Proof of the proposition. By the lemma below, for every $a \in \mathbb{N}$, we have $\Psi_f^u(K \otimes f^*\mathscr{L}_a) \simeq \Psi_f^u K \otimes L_a$, and the action of N on the tensor product in the second term is $N \otimes \mathrm{id} + \mathrm{id} \otimes N$. Define a map

$$\gamma : \Psi_f^u K \to \Psi_f^u K \otimes L_a = \Psi_f^u K \oplus \Psi_f^u K(-1) \oplus \dots \Psi_f^u K(-a)$$

by $\gamma(x) = (x, -Nx, \dots, (-N)^a x)$. Then $N\gamma(x) = (0, \dots, 0, (-1)^a N^{a+1} x)$, so γ induces an isomorphism between $\mathrm{Ker}(N^{a+1}, \Psi_f^u K)$ and $\mathrm{Ker}(N, \Psi_f^u(K \otimes f^*\mathscr{L}_a))$.

□

Lemma 3.3 *Let $K \in D_c^b(U)$. Then, for every $a \in \mathbb{N}$, the canonical morphism*

$$\Psi_f^u(K) \otimes f^*\Psi_{\mathrm{id}}^u(\mathscr{L}_a)[1] \to \Psi_f^u(K) \otimes \Psi_f^u(f^*\mathscr{L}_a)[1] \to \Psi_f^u(K \otimes f^*\mathscr{L}_a)$$

is an isomorphism.

Note that $\Psi_{\mathrm{id}}^u(\mathscr{L}_a)[1]$ is $L_{a,\{0\}}$ with the obvious action of $\pi_1(\mathbb{G}_m, 1)$.

Proof. We prove the result by induction on a. If $a = 0$, then $\mathscr{L}_a = F_{\mathbb{G}_m}$, so all complexes in the formula above are isomorphic to $\Psi_f^u K$, and it is clear that the maps are the identity maps.

Assume that we know the result for $a \geq 0$. Then the exact sequence

$$0 \to \mathscr{L}_a \to \mathscr{L}_{a+1} \to \mathscr{L}_0(-a-1) \to 0$$

gives a commutative diagram

$$\begin{array}{ccccc}
\Psi_f^u(K) \otimes f^*\Psi_{\mathrm{id}}^u(\mathscr{L}_a)[1] & \longrightarrow & \Psi_f^u(K) \otimes \Psi_f^u(f^*\mathscr{L}_a)[1] & \longrightarrow & \Psi_f^u(K \otimes f^*\mathscr{L}_a) \\
\downarrow & & \downarrow & & \downarrow \\
\Psi_f^u(K) \otimes f^*\Psi_{\mathrm{id}}^u(\mathscr{L}_{a+1})[1] & \longrightarrow & \Psi_f^u(K) \otimes \Psi_f^u(f^*\mathscr{L}_{a+1})[1] & \longrightarrow & \Psi_f^u(K \otimes f^*\mathscr{L}_{a+1}) \\
\downarrow & & \downarrow & & \downarrow \\
\Psi_f^u(K)(-a-1) & = & \Psi_f^u(K)(-a-1) & = & \Psi_f^u(K)(-a-1) \\
\downarrow{\scriptstyle +1} & & \downarrow{\scriptstyle +1} & & \downarrow{\scriptstyle +1}
\end{array}$$

whose columns are exact triangles. We know that the morphisms on the first and the third lines are isomorphisms, so the morphisms on the second line are also isomorphisms.

□

The corollary above gives a construction of $\Psi_f^u K$, for $K \in \mathrm{Perv}(U)$. We now explain how to see the map $N : \Psi_f^u K \to \Psi_f^u K(-1)$ on this construction. The following proposition is obvious from the explicit formula for the map $\mathrm{Ker}(N^{a+1}, \Psi_f^u K) \xrightarrow{\sim} \mathrm{Ker}(N, \Psi_f^u(K \otimes f^*\mathscr{L}_a))$.

Proposition 3.4 *Let $K \in \mathrm{Perv}(U)$. Let $a \geq 0$ such that $N^{a+1} = 0$ on $\Psi_f^u K$. By the corollary above, we have an exact sequence $0 \to i_*\Psi_f^u K \to j_!(K \otimes f^*\mathscr{L}_b) \to j_*(K \otimes f^*\mathscr{L}_b)$ for every $b \geq a$. Then the following diagram is commutative :*

$$\begin{array}{ccccccc}
0 & \longrightarrow & i_*\Psi_f^u K & \longrightarrow & j_!(K \otimes f^*\mathscr{L}_a) & \longrightarrow & j_*(K \otimes f^*\mathscr{L}_a) \\
 & & \| & & \downarrow{\scriptstyle \alpha_{a,a+1}} & & \downarrow{\scriptstyle \alpha_{a,a+1}} \\
0 & \longrightarrow & i_*\Psi_f^u K & \longrightarrow & j_!(K \otimes f^*\mathscr{L}_{a+1}) & \longrightarrow & j_*(K \otimes f^*\mathscr{L}_{a+1}) \\
 & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle \beta_{a,a+1}} & & \downarrow{\scriptstyle \beta_{a,a+1}} \\
0 & \longrightarrow & i_*\Psi_f^u K(-1) & \longrightarrow & j_!(K \otimes f^*\mathscr{L}_a)(-1) & \longrightarrow & j_*(K \otimes f^*\mathscr{L}_a)(-1)
\end{array}$$

Remark 3.5 By the results of this section, we can redefine $\Psi_f^u K$, for $K \in \mathrm{Perv}(U)$, as the direct limit of the $i^*j_{!*}(K \otimes f^*\mathscr{L}_a)[-1]$, where the transition maps are given by the $\alpha_{a,a+1}$. (In fact, we will see in corollary 4.3 that $\Psi_f^u K$ is also isomorphic to the direct limit of the $i^*j_*(K \otimes f^*\mathscr{L}_a)[-1]$.) So we can see Ψ_f^u as a functor from $\mathrm{Perv}(U)$ to $\mathrm{Perv}(Y)$ (instead of $\mathrm{Perv}(Y_{\overline{k}})$, admitting a functorial morphism $N : \Psi_f^u \to \Psi_f^u(-1)$.

4 Duality

We keep the notation of the previous section.

In section 4 of *Autour du théorème de monodromie locale* ([3]), Illusie shows that the functor Ψ_f commutes with Verdier duality (up to a twist). We will show how to deduce this result (at least for Ψ_f^u) from Beilinson's construction.

Proposition 4.1 *Let $K \in \mathrm{Perv}(U)$. If a and b are big enough (more precisely, if $N^{a+1} = N^{b+1} = 0$ on $\Psi_f^u K$), then there is a canonical isomorphism*

$$\mathrm{Ker}(j_!(K \otimes f^*\mathscr{L}_b) \to j_*(K \otimes f^*\mathscr{L}_b))(-a-1) \tilde{\to} \mathrm{Coker}(j_!(K \otimes f^*\mathscr{L}_a) \to j_*(K \otimes f^*\mathscr{L}_a)),$$

and the morphism

$$^p\mathrm{H}^0 i^* j_*(K \otimes \mathscr{L}_a) \to {}^p\mathrm{H}^0 i^* j_*(K \otimes \mathscr{L}_{a+b+1})$$

induced by $\alpha_{a,a+b+1}$ is zero.

Corollary 4.2 *For every $K \in \mathrm{Perv}(U)$, we have a canonical isomorphism $D(\Psi_f^u K) \simeq \Psi_f^u(DK)(-1)$.*

Proof. Let a be big enough. Then $\Psi_f^u K = \mathrm{Ker}(j_!(K \otimes f^*\mathscr{L}_a) \to j_*(K \otimes f^*\mathscr{L}_a))$, so

$$D(\Psi_f^u K) \simeq \mathrm{Coker}(j_! D(K \otimes f^*\mathscr{L}_a) \to j_* D(K \otimes f^*\mathscr{L}_a)).$$

We see easily that $D(K \otimes f^*\mathscr{L}_a) \simeq D(K) \otimes f^*\mathscr{L}_a(a)$, and the proposition now gives the result.

□

Corollary 4.3 *The maps $\Psi_f^u K \to i^* j_*(K \otimes \mathscr{L}_a)[-1]$ of corollary 3.2 induce an isomorphisme*

$$\Psi_f^u K \tilde{\to} \varinjlim_a i^* j_*(K \otimes \mathscr{L}_a)[-1]$$

(where the transition morphisms are given by the $\alpha_{a,a+1}$).

Proof. We have already seen (in corollary 3.2 the map

$$\Psi_f^u K \tilde{\to} \varinjlim_a {}^p\mathrm{H}^{-1} i^* j_*(K \otimes \mathscr{L}_a)$$

is an isomorphism, so it remains to show that $\varinjlim_a {}^p\mathrm{H}^0 i^* j_*(K \otimes \mathscr{L}_a) = 0$. But this follows immediately from the second statement of the proposition.

□

Proof of the proposition. We will write $\mathrm{Ker}_a(K)$ (resp $\mathrm{Coker}_a(K)$) for the kernel (resp the cokernel) of the map $j_!(K \otimes f^*\mathscr{L}_a) \to j_*(K \otimes f^*\mathscr{L}_a)$. As in the proof of the corollary above, we see that there is a canonical isomorphism $D(\mathrm{Ker}_a(K)) \simeq \mathrm{Coker}_a(D(K))(a)$.

Consider the following commutative diagram, where the rows are exact sequences :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & j_!(K \otimes f^*\mathscr{L}_a) & \xrightarrow{\alpha_{a,a+b+1}} & j_!(K \otimes f^*\mathscr{L}_{a+b+1}) & \xrightarrow{\beta_{b,a+b+1}} & j_!(K \otimes f^*\mathscr{L}_b)(-a-1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & j_*(K \otimes f^*\mathscr{L}_a) & \xrightarrow{\alpha_{a,a+b+1}} & j_*(K \otimes f^*\mathscr{L}_{a+b+1}) & \xrightarrow{\beta_{a,a+b+1}} & j_*(K \otimes f^*\mathscr{L}_b)(-a-1) & \longrightarrow & 0 \end{array}$$

The snake lemma gives an exact sequence

$$0 \to \mathrm{Ker}_a(K) \to \mathrm{Ker}_{a+b+1}(K) \to \mathrm{Ker}_b(K)(-a-1) \to$$

$$\to \mathrm{Coker}_a(K) \to \mathrm{Coker}_{a+b+1}(K) \to \mathrm{Coker}_b(K)(-a-1) \to 0.$$

By the results of the previous section, we have isomorphisms $\mathrm{Ker}_a(K) \simeq \Psi_f^u K$, $\mathrm{Ker}_{a+b+1}(K) \simeq \Psi_f^u(K)$ and $\mathrm{Ker}_b(K)(-a-1) \simeq \Psi_f^u K(-a-1)$, that identify the first map with the identity and the second map with $N^{a+1} = 0$. By duality, we also have isomorphisms $\mathrm{Coker}_a(K) \simeq D(\Psi_f^u D(K))(-a)$, $\mathrm{Coker}_{a+b+1}(K) \simeq D(\Psi_f^u D(K))(-a-b-1))$ and $\mathrm{Coker}_b(K)(-a-1) \simeq D(\Psi_f^u D(K))(-a-b-1)$ that identify the last arrow with the identity and the next to last arrow with $D(N^{b+1})(-a) = 0$. The last part gives the second statement of the proposition, and the middle of the sequence gives the isomorphism of the first statement.

□

5 The maximal extension functor

The maximal extension functor is a functor $\Xi_f : \mathrm{Perv}(U) \to \mathrm{Perv}(X)$. It will be useful to construct Φ_f^u and for gluing, though it will not appear in the statements.

Fix $K \in \mathrm{Perv}(U)$. For each $a \geq 1$, we have a commutative diagram :

$$\begin{array}{ccc} j_!(K \otimes f^*\mathscr{L}_a) & \longrightarrow & j_*(K \otimes f^*\mathscr{L}_a) \\ \downarrow{\scriptstyle \beta_{a,a+1}} & & \downarrow{\scriptstyle \beta_{a,a+1}} \\ j_!(K \otimes f^*\mathscr{L}_{a-1})(-1) & \longrightarrow & j_*(K \otimes f^*\mathscr{L}_{a-1})(-1) \end{array}$$

We write $\gamma_{a,a-1} : j_!(K \otimes f^*\mathscr{L}_a) \to j_*(K \otimes f^*\mathscr{L}_{a-1})(-1)$ for the diagonal map in this diagram.

Proposition 5.1 *For a big enough, the (injective) map $\mathrm{Ker}(\gamma_{a,a-1}) \to \mathrm{Ker}(\gamma_{a+1,a})$ induced by $\alpha_{a,a+1} : j_!(K \otimes f^*\mathscr{L}_a) \to j_!(K \otimes f^*\mathscr{L}_{a+1})$ is an isomorphism. We write $\Xi_f K$ for the direct limit of the $\mathrm{Ker}(\gamma_{a,a-1})$. This defines a left exact functor from $\mathrm{Perv}(U)$ to $\mathrm{Perv}(X)$, and we have a functorial exact sequence*

$$0 \to j_! \to \Xi_f \to i_*\Psi_f^u(-1) \to 0.$$

Moreover, if a and b are big enough, then the map $\mathrm{Coker}(\gamma_{a,a-1}) \to \mathrm{Coker}(\gamma_{a+b,a+b-1})$ *induced by* $\alpha_{a-1,a+b-1}(-1)$ *is zero. In particular, we have*

$$\varinjlim_a \mathrm{Coker}(\gamma_{a-1,a}) = 0.$$

Proof. Let $a \geq 1$. Remember that the map $\alpha_{a,a-1} : \mathscr{L}_a \to \mathscr{L}_{a-1}(-1)$ is surjective, and that its kernel is $F_{\mathbb{G}_m}$ (in the abelian category of local systems on $\mathbb{G}_m$). So we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & j_!K & \longrightarrow & j_!(K \otimes f^*\mathscr{L}_a) & \longrightarrow & j_!(K \otimes f^*\mathscr{L}_{a-1})(-1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \gamma_{a,a-1}} & & \downarrow & & \\ & & 0 & \longrightarrow & j_*(K \otimes f^*\mathscr{L}_{a-1})(-1) & \longrightarrow & j_*(K \otimes f^*\mathscr{L}_{a-1})(-1) & \longrightarrow & 0 \end{array}$$

Applying the snake lemma, we get an exact sequence

$$0 \to j_!K \to \mathrm{Ker}(\gamma_{a,a-1}) \to i_*i^*j_{!*}(K \otimes f^*\mathscr{L}_{a-1})(-1)[-1] \to 0$$

and an isomorphism

$$\mathrm{Coker}(\gamma_{a-1,a}) = \mathrm{Coker}(j_!(K \otimes f^*\mathscr{L}_{a-1})(-1) \to j_*(K \otimes f^*\mathscr{L}_{a-1})(-1)) = {}^p\mathrm{H}^0 j_*(K \otimes \mathscr{L}_{a-1})(-1).$$

It is clear that the following diagram is commutative (where the last two vertical maps are induced by $\alpha_{a,a+1}$) :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & j_!K & \longrightarrow & \mathrm{Ker}(\gamma_{a,a-1}) & \longrightarrow & i_*i^*j_{!*}(K \otimes f^*\mathscr{L}_{a-1})(-1)[-1] & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & j_!K & \longrightarrow & \mathrm{Ker}(\gamma_{a+1,a}) & \longrightarrow & i_*i^*j_{!*}(K \otimes f^*\mathscr{L}_a)(-1)[-1] & \longrightarrow & 0 \end{array}$$

But, if a is big enough, the last terms on the two rows are isomorphic to $\Psi_f^u K(-1)$, and the last vertical arrow is an isomorphism. This proves the first assertion and the existence of the exact sequence. The construction of $\Xi_f K$ is obviously functorial in K, and the left exactness follows from the snake lemma.

Finally, the last statement follows from the second statement of proposition 4.1.

□

Remark 5.2 For every $a \geq 1$, we have a commutative diagram

$$\begin{array}{ccc} j_!(K \otimes f^*\mathscr{L}_{a+1}) & \xrightarrow{\gamma_{a+1,a}} & j_*(K \otimes f^*\mathscr{L}_a)(-1) \\ \downarrow{\scriptstyle \beta_{a,a+1}} & & \downarrow{\scriptstyle \beta_{a-1,a}(-1)} \\ j_!(K \otimes f^*\mathscr{L}_a)(-1) & \xrightarrow{\gamma_{a,a-1}(-1)} & j_*(K \otimes f^*\mathscr{L}_{a-1})(-2) \end{array}$$

This induces a map $\mathrm{Ker}(\gamma_{a+1,a}) \to \mathrm{Ker}(\gamma_{a,a-1})(-1)$. By taking a big enough, we get a functorial map $\Xi_f \to \Xi_f(-1)$, which we denote by N, and we see easily from the definition that the following diagram is commutative :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & j_!K & \longrightarrow & \Xi_f K & \longrightarrow & i_*\Psi_f^u K(-1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N(-1)} & & \\
0 & \longrightarrow & j_!K(-1) & \longrightarrow & \Xi_f K(-1) & \longrightarrow & i_*\Psi_f^u K(-2) & \longrightarrow & 0
\end{array}$$

Remark 5.3 Using the last statement of the proposition above, we see easily that we could have defined $\Xi_f K$ as the inductive limit of the complexes $j_!(K \otimes \mathscr{L}_a) \overset{\gamma_{a,a-1}}{\to} j_*(K \otimes f^*\mathscr{L}_{a-1})(-1)$, where the first term is in degree 0 and the transition maps are given by the $\alpha_{a,a+1}$.

We will now show that the functor Ξ_f commutes with duality.

Fix $K \in \mathrm{Perv}(U)$. For each $a \geq 0$, we have a commutative diagram :

$$\begin{array}{ccc}
j_!(K \otimes f^*\mathscr{L}_a) & \longrightarrow & j_*(K \otimes f^*\mathscr{L}_a) \\
\downarrow{\scriptstyle \alpha_{a,a+1}} & & \downarrow{\scriptstyle \alpha_{a,a+1}} \\
j_!(K \otimes f^*\mathscr{L}_{a+1}) & \longrightarrow & j_*(K \otimes f^*\mathscr{L}_{a+1})
\end{array}$$

We write $\gamma_{a,a+1} : j_!(K \otimes f^*\mathscr{L}_a) \to j_*(K \otimes f^*\mathscr{L}_{a+1})$ for the diagonal map in this diagram.

Proposition 5.4 *For a big enough, the (surjective) map $\mathrm{Coker}(\gamma_{a+1,a+2}) \to \mathrm{Coker}(\gamma_{a,a+1})(-1)$ induced by $\beta_{a,a+1} : j_*(K \otimes f^*\mathscr{L}_{a+1}) \to j_*(K \otimes f^*\mathscr{L}_a)(-1)$ is an isomorphism, and we have a canonical isomorphism $\mathrm{Coker}(\gamma_{a,a+1}) \simeq \mathrm{Ker}(\gamma_{b,b-1})(-a-1)(= \Xi_f K(-a-1))$ for a and b big enough.*

Moreover, for a and b big enough, the map $\mathrm{Ker}(\gamma_{a+b,a+b+1}) \to \mathrm{Ker}(\gamma_{a,a+1})(-b)$ induced by $\beta_{a,a+b}$ is zero.

Corollary 5.5 *We have a canonical functorial isomorphism $D \circ \Xi_f \simeq \Xi_f \circ D$. The functor Ξ_f is right exact (so it is exact), and there is a functorial exact sequence :*

$$0 \to i_*\Psi_f^u \to \Xi_f \to j_* \to 0,$$

that is dual to the exact sequence in the previous proposition.

Proof of the proposition. All the proofs are the duals of what we already did, except for the isomorphism $\mathrm{Coker}(\gamma_{a,a+1}) \simeq \mathrm{Ker}(\gamma_{b,b-1})(-a-1)$.

Fix $K \in \mathrm{Perv}(U)$, and $a, b \in \mathbb{N}$ with $b \geq 1$. Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & j_!(K\otimes f^*\mathscr{L}_a) & \xrightarrow{\alpha_{a,a+b+1}} & j_!(K\otimes f^*\mathscr{L}_{a+b+1}) & \xrightarrow{\beta_{b,a+b+1}} & j_!(K\otimes f^*\mathscr{L}_b)(-a-1) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma_{a,a+1}} & & \downarrow & & \downarrow{\scriptstyle \gamma_{b,b-1}(-a-1)} & & \\ 0 & \longrightarrow & j_*(K\otimes f^*\mathscr{L}_{a+1}) & \xrightarrow{\alpha_{a+1,a+b+1}} & j_*(K\otimes f^*\mathscr{L}_{a+b+1}) & \xrightarrow{\beta_{b-1,a+b+1}} & j_*(K\otimes f^*\mathscr{L}_{b-1})(-a-2) & \longrightarrow & 0 \end{array}$$

Note that $\alpha_{a,a+1} : j_*(K\otimes f^*\mathscr{L}_a)\to j_*(K\otimes f^*\mathscr{L}_{a+1})$ is injective, so

$$\operatorname{Ker}(j_!(K\otimes f^*\mathscr{L}_a)\to j_*(K\otimes f^*\mathscr{L}_{a+1}) = \operatorname{Ker}(j_!(K\otimes f^*\mathscr{L}_a)\to j_*(K\otimes f^*\mathscr{L}_a)).$$

Similarly (or dually),

$$\operatorname{Coker}(j_!(K\otimes f^*\mathscr{L}_b))\to j_*(K\otimes f^*\mathscr{L}_{b-1})) = \operatorname{Coker}(j_!(K\otimes f^*\mathscr{L}_{b-1})\to j_*(K\otimes f^*\mathscr{L}_{b-1})).$$

So, applying the snake lemma to the diagram above and taking a and b big enough, we get an exact sequence

$$0\to \Psi_f^u K = \Psi_f^u K \to \operatorname{Ker}(\gamma_{b,b-1})(-a-1)\to \operatorname{Coker}(\gamma_{a,a+1})\to$$

$$\to \Psi_f^u K(-a-b-1) = \Psi_f^u K(-a-2)(-b+1)\to 0.$$

This gives the isomorphism that we were looking for.

□

Remark 5.6 As before, we have a commutative diagram :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & i_*\Psi_f^u K & \longrightarrow & \Xi_f K & \longrightarrow & j_*K & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & i_*\Psi_f^u K(-1) & \longrightarrow & \Xi_f K(-1) & \longrightarrow & j_*K(-1) & \longrightarrow & 0 \end{array}$$

Remark 5.7 It is clear from the definitions that the composition of the two functorial morphisms $i_*\Psi_f^u\to\Xi_f$ and $\Xi_f\to i_*\Psi_f^u(-1)$ is just $N : i_*\Psi_f^u\to i_*\Psi_f^u(-1)$, and that the composition of the two functorial morphisms $j_!K\to\Xi_f K$ and $\Xi_f K\to j_*K$ is the canonical morphism $j_!K\to j_*K$.

6 The unipotent vanishing cycles functor

We keep the situation of the preceding section.

Exposés I and XIII of SGA 7 also explain the construction of a vanishing cycles functor $\Phi_f : D^b_c(X) \to D^b_c(Y_{\overline{k}})$ such that there is a functorial exact triangle

$$\Psi_f j^* \overset{\mathrm{can}}{\to} \Phi_f \to i^* \overset{+1}{\to} .$$

(We have shifted Φ_f by -1, like we did for Ψ_f.) This functor actually takes $K \in D^b_c(X)$ to a complex of sheaves with an action of $\pi_1(\mathbb{G}_m, 1)$ compatible with its action on $Y_{\overline{k}}$, so in particular we get an action of T on it. Let $I = \pi_1^{\mathrm{geom}}(\mathbb{G}_m, 1)$ (the inertia subgroup of $\pi_1(\mathbb{G}_m, 1)$). In addition to the functorial morphism $\mathrm{can} : \Psi_f \to \Phi_f$ in the triangle above, we can construct, for each $\sigma \in I$, a functorial morphism $\mathrm{Var}(\sigma) : \Phi_f \to \Psi_f$ such that $\mathrm{can} \circ \mathrm{Var}(\sigma) = \sigma - 1$ and $\mathrm{Var}(\sigma) \circ \mathrm{can} = \sigma - 1$ (this works because I acts trivially on i^*K, for every $K \in D^b_c(X)$). The finiteness theorem for Ψ_f (cf SGA 4 1/2 [Th. finitude]) gives a finiteness theorem for Φ_f, so we can define the direct factor Φ^u_f where T acts unipotently like we did for Ψ^u_f, and we can also define a nilpotent functorial morphism $N : \Phi^u_f \to \Phi^u_f(-1)$ by "taking the logarithm of T".

The goal of this section is to define Φ^u_f directly as a functor $\mathrm{Perv}(X) \to \mathrm{Perv}(Y)$ and to establish some of its basic properties. We will not need the results recalled in the preceding paragraph.

Recall that we have functorial morphisms $\delta : j_! \to \Xi_f$ and $\varepsilon : \Xi_f \to j_*$, with δ injective and ε surjective. We will denote by adj the adjunction morphisms $j_!j^* \to \mathrm{id}$ and $\mathrm{id} \to j_*j^*$.

For $K \in \mathrm{Perv}(X)$, we define a complex $C^\bullet(K)$ of objects of $\mathrm{Perv}(X)$ by

$$C^\bullet(K) = (j_!j^*K \overset{\delta \oplus \mathrm{adj}}{\to} \Xi_f j^*K \oplus K \overset{\varepsilon - \mathrm{adj}}{\to} j_*j^*K),$$

where $j_!j^*K$ is in degree -1. This construction is obviously functorial in K. Note that the only nonzero cohomology object of $C^\bullet(K)$ is in degree 0, and that it is a perverse sheaf with support in Y.

We define a functor $\Phi^u_f : \mathrm{Perv}(X) \to \mathrm{Perv}(Y)$ by $\Phi^u_f(K) = i^*\mathrm{H}^0(C^\bullet(K))$. By the remarks above (and the long exact sequence of cohomology for $C^\bullet(K)$), Φ^u_f does send $\mathrm{Perv}(X)$ to $\mathrm{Perv}(Y)$, and it is an exact functor.

The map $i_*\Psi^u_f j^*K \to \Xi_f j^*K$ sends $i_*\Psi^u_f j^*K$ to $Z^0(C^\bullet(K))$, so it defines a functorial map $\mathrm{can} : \Psi^u_f j^*K \to \Phi^u_f K$. Similarly, the map $\Xi_f j^*K \to \Psi^u_f K(-1)$ defines a functorial map $\mathrm{var} : \Phi^u_f K \to \Psi^u_f j^*K(-1)$. We obviously have $\mathrm{var} \circ \mathrm{can} = N$. On the other hand, the morphism $\begin{pmatrix} N & 0 \\ 0 & 0 \end{pmatrix} : \Xi_f j^*K \oplus K \to \Xi_f K(-1) \oplus K(-1)$ sends $Z^0(C^\bullet(K))$ (resp. $B^0(C^\bullet(K))$) to $Z^0(C^\bullet(K))(-1)$ (resp. $B^0(C^\bullet(K))(-1)$), because $\varepsilon(-1) \circ N = 0$ and $N \circ \delta = 0$, so it induces a functorial morphism $N : \Phi^u_f K \to \Phi^u_f K(-1)$, and it is an easy exercise to show that $\mathrm{can}(-1) \circ \mathrm{var} = N$.

Remark 6.1 The automorphism $\begin{pmatrix} \mathrm{id} & 0 \\ 0 & -1 \end{pmatrix}$ of $\Xi_f j^*K \oplus K$ induces a functorial isomorphism between $\Phi^u_f K$ and $\mathrm{H}^0(C'^\bullet(K))$, where

$$C'^\bullet(K) = (j_!j^*K \overset{\delta \oplus (-\mathrm{adj})}{\to} \Xi_f j^*K \oplus K \overset{\varepsilon + \mathrm{adj}}{\to} j_*j^*K),$$

with $j_!j^*K$ in degree -1. As $D(C^\bullet(K))$ is canonically isomorphic to $C'^\bullet(D(K))$, we get a functorial isomorphism $D \circ \Phi_f^u \simeq \Phi_f^u \circ D$, and the duality exchanges can and var.

Proposition 6.2 *There are canonical isomorphisms* $\mathrm{Ker}(\mathrm{can}) = {}^p\mathrm{H}^{-1}i^*K$ *and* $\mathrm{Coker}(\mathrm{can}) = {}^p\mathrm{H}^0 i^*K$.

Dually, we have canonical isomorphisms $\mathrm{Ker}(\mathrm{var}) = {}^p\mathrm{H}^0 i^!K$ *and* $\mathrm{Coker}(\mathrm{var}) = {}^p\mathrm{H}^1 i^!K$.

Proof. We have a commutative diagram with exact rows and injective columns :

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & i_*{}^p\mathrm{H}^{-1}i^*K & \longrightarrow & j_!j^*K & \xrightarrow{\mathrm{adj}} & K & & \\
 & & \Big\downarrow{\scriptstyle u} & & \Big\downarrow{\scriptstyle \delta\oplus\mathrm{adj}} & & \Big\downarrow{\scriptstyle 0\oplus\mathrm{id}} & & \\
0 & \longrightarrow & i_*\Psi_f^u j^*K & \longrightarrow & \Xi_f j^*K \oplus K & \xrightarrow{v} & j_*j^*K \oplus K & \longrightarrow & 0
\end{array}
$$

where v is the map $\begin{pmatrix} \varepsilon & -\mathrm{adj} \\ 0 & \mathrm{id} \end{pmatrix}$. This induces an injective map $u : i_*{}^p\mathrm{H}^{-1}i^*K \to i_*\Psi_f^u j^*K$, and an map $\mathrm{Coker}(u) \to \mathrm{Coker}(\delta \oplus \mathrm{adj})$, which is also injective by the snake lemma. But we have seen that the map $i_*\Psi_f^u j^*K \to \Xi_f j^*K \oplus K \to \mathrm{Coker}(\delta \oplus \mathrm{adj})$ sends $i_*\Psi_f^u j^*K$ to $i_*\Phi_f^u K$ and induces the map can, so we get a factorization $i_*\Psi_f^u j^*K \twoheadrightarrow \mathrm{Coker}(u) \hookrightarrow i_*\Phi_f^u K$ of $i_*\,\mathrm{can}$, hence an isomorphism $\mathrm{Ker}(i_*\,\mathrm{can}) \simeq \mathrm{Im}(u) \simeq i_*{}^p\mathrm{H}^{-1}i^*K$.

We denote the maps in $C^\bullet(K)$ by d^{-1} and d^0. As $\varepsilon : \Xi_f j^*K \to j_*j^*K$ is surjective, applying the snake lemma to the following diagram with exact rows

$$
\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & \Xi_f j^*K \oplus K & = & \Xi_f j^*K \oplus K & \longrightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow{\scriptstyle \varepsilon+\mathrm{id}_K} & & \Big\downarrow{\scriptstyle d^0} & & \\
0 & \longrightarrow & K & \xrightarrow[(0,\mathrm{id}_K)]{} & j_*j^*K \oplus K & \xrightarrow[pr_1]{} & j_*j^*K & \longrightarrow & 0
\end{array}
$$

gives an exact sequence

$$0 \to i_*\Psi_f^u j^*K \to \mathrm{Ker}(d^0) \to K \to 0.$$

Moreover, the composition $i_*\Psi_f^u j^*K \to \mathrm{Ker}(d^0) \to i_*\Phi_f^u K$ is the map $i_*\,\mathrm{can}$, by definition of can.

Consider the commutative diagram with exact columns :

$$\begin{array}{ccccc}
0 & & & & \\
\downarrow & & & & \\
j_!j^*K & = & j_!j^*K & & \\
{\scriptstyle d^0}\downarrow & & \downarrow{\scriptstyle \mathrm{adj}} & & \\
\mathrm{Ker}(d^0) & \longrightarrow & K & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
i_*\Phi_f^u K & \overset{v}{\dashrightarrow} & i_*{}^p\mathrm{H}^0 i^*K & & \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

It induces a surjection $v : i_*\Phi_f^u K \to i_*{}^p\mathrm{H}^0 i^*K$, and so the map

$$i_*\Psi_f^u j^*K = \mathrm{Ker}(\mathrm{Ker}(d^0) \to K) \to \mathrm{Ker}(v)$$

is an isomorphism, and it identifies $i_*{}^p\mathrm{H}^0 i^*K$ with $i_* \,\mathrm{Coker(can)}$.

□

7 The functor Ω_f

This is inspired by a similar construction of Morihiko Saito in the case of mixed Hodge modules.

Definition 7.1 Let $\Omega_f : \mathrm{Perv}(X) \to \mathrm{Perv}(X)$ be the functor sending K to $\mathrm{Ker}(\varepsilon + \mathrm{adj} : \Xi_f j^*K \oplus K \to j_*j^*K)$.

As $\varepsilon + \mathrm{adj}$ is surjective, we can also think of Ω_f as the functor $\mathrm{Perv}(X) \to D^b\,\mathrm{Perv}(X)$ sending K to the complex $\Xi_f j^*K \oplus K \to j_*j^*K$. In particular, we have functorial morphisms $\Omega_f \to i_*\Phi_f^u$ and $\Omega_f \to C^\bullet$, where $C^\bullet : \mathrm{Perv}(X) \to D^b\,\mathrm{Perv}(X)$ is the functor $K \longmapsto C^\bullet(K)$.

By definition of Φ_f^u and by the calculations of the previous section, we get two functorial exact sequences

$$0 \to j_!j^* \overset{\delta-\mathrm{adj}}{\to} \Omega_f \to i_*\Phi_f^u \to 0$$

and

$$0 \to i_*\Psi_f^u j^* \to \Omega_f \to \mathrm{id}_{\mathrm{Perv}(X)} \to 0.$$

The first exact sequence gives an isomorphism $i^*\Omega_f \overset{\sim}{\to} \Phi_f^u$, which, combined with the second exact sequence, gives a quasi-isomorphism $(\Psi_f^u j^* \overset{\mathrm{can}}{\to} \Phi_f^u) \to i^*$ (the source is seen as a complex concentrated in degrees -1 and 0).

Corollary 7.2 *The functor $\Phi_f j_! : \mathrm{Perv}(U) \to \mathrm{Perv}(Y)$ is canonically isomorphic to Ψ_f^u, and this induces a functorial exact sequence*

$$0 \to \Psi_f^u \to \Phi_f^u \Xi_f \to \Psi_f^u(-1) \to 0.$$

Proof. We get the isomorphism by applying the quasi-isomorphism $(\Psi_f^u j^* \stackrel{\mathrm{can}}{\to} \Phi_f^u) \to i^*$ above to $j_!$, and the exact sequence by applying Φ_f^u to the exact sequence $0 \to j_! \to \Xi_f \to i_* \Psi_f^u(-1) \to 0$ of proposition 5.1.

□

8 Gluing

We keep the situation of section 6. We consider the category $GD(X)$ of gluing data of X, whose objects are quadruples (K_U, K_Y, u, v), with :

- $K_U \in \mathrm{Perv}(U)$;
- $K_Y \in \mathrm{Perv}(Y)$;
- $u : \Psi_f^u K_U \to K_Y$;
- $v : K_Y \to \Psi_f^u K_U(-1)$,

such that $vu = N : \Psi_f^u K_U \to \Psi_f^u K_U(-1)$. Morphisms from (K_U, K_Y, u, v) to (K'_U, K'_Y, u', v') are couples of morphisms $(K_U \to K'_U, K_Y \to K'_Y)$ that make the obvious diagrams commute.

We define a functor $F : \mathrm{Perv}(X) \to GD(X)$ by sending $K \in \mathrm{Perv}(X)$ to $(j^*K, \Phi_f^u K, \mathrm{can}, \mathrm{var})$. We define a functor $G : GD(X) \to \mathrm{Perv}(X)$ by sending $c = (K_U, K_V, u, v)$ to $\mathrm{H}^0(D^\bullet(c))$, where $D^\bullet(c)$ is the complex

$$i_* \Psi_f^u K_U \stackrel{\alpha}{\to} \Xi_f K_U \oplus i_* K_Y \stackrel{\beta}{\to} i_* \Psi_f^u K_U(-1)$$

with $i_* \Psi_f^u K_U$ in degree -1, α equal to the sum of the canonical injection $i_* \Psi_f^u K_U \to \Xi_f K_U$ and of u, and β equal to the difference of the canonical surjection $\Xi_f K_U \to i_* \Psi_f^u K_U(-1)$ and of v. Note that α is injective and β is surjective.

We can now state the main theorem of these note.

Theorem 8.1 *The functors F and G are equivalences quasi-inverse to each other.*

Proof. We show that $F \circ G \simeq \mathrm{id}_{GD(X)}$. Let $c = (K_U, K_Y, u, v)$ be an object of $GD(X)$. Write $K = G(c)$ and $(L_U, L_Y, w, x) = F(K)$. Remember also the exact sequence

$$0 \to \Psi_f^u K_U \stackrel{a}{\to} \Phi_f^u \Xi_f K_U \stackrel{b}{\to} \Psi_f^u K_U(-1) \to 0$$

of corollary 7.2. We have

$$j^* D^\bullet(c) = (0 \to K_U \oplus 0 \to 0),$$

so we get a canonical isomorphism $L_U = K_U$. On the other hand,

$$\Phi_f^u D^\bullet(c) = (\Psi_f^u K_U \overset{a+u}{\to} \Phi_f^u \Xi_f K_U \oplus K_Y \overset{b-v}{\to} \Psi_f^u K_U(-1)),$$

so $L_Y = \Phi_f^u \mathrm{H}^0 D^\bullet(c) = \mathrm{H}^0 \Phi_f^u D^\bullet(c) = K_Y$, and this identifies u and w (resp. v and x).

Now let's show that $G \circ F \simeq \mathrm{id}_{\mathrm{Perv}(X)}$. Let $c = F(K)$ and $L = G(c)$. Then L is $H^0(D^\bullet(c))$, with

$$D^\bullet(c) = (i_* \Psi_f^u j^* K \overset{\alpha}{\to} \Xi_f j^* K \oplus i_* \Phi_f K \overset{\beta}{\to} i_* \Psi_f^u j^* K(-1)).$$

Applying the snake lemma to the commutative diagram (with exact rows)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_f K & \longrightarrow & \Xi_f j^* K \oplus \Omega_f K & \xrightarrow{pr_1} & \Xi_f j^* K & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\gamma} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Ker}\beta & \longrightarrow & \Xi_f j^* K \oplus i_* \Phi_f^u K & \xrightarrow[\beta]{} & i_* \Psi_f^u j^* K(-1) & \longrightarrow & 0 \end{array}$$

where γ is induced by the middle vertical map, we get an exact sequence

$$0 \to \operatorname{Ker}(\gamma) \to j_! j^* K \overset{\mathrm{id}}{\to} j_! j^* K \to \operatorname{Coker}(\gamma) \to 0.$$

Hence $\gamma : \Omega_f K \to \operatorname{Ker}(\beta)$ is an isomorphism. By the second exact sequence of section 7, the cokernel of $\alpha : i_* \Psi_f^u j^* K \to \operatorname{Ker}(\beta)$ is thus identified with K, and this gives the desired isomorphism $K = L$.

□

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