Exam 4/29/2013

- Duration 3h. No notes and electronic devices allowed. Exercises are in somewhat increasing difficulty order. -

- Exercise 1 - Let (P) be the following linear program :

$$\begin{array}{rrrrrcr}
\text{Maximize} & 3x_1 & -2x_2 & +2x_3 & -4x_4 & & \\
\text{Subject to} & 2x_1 & +x_2 & +2x_3 & -x_4 & \leq & -2 \\
 & & -x_2 & -x_3 & & \leq & -2 \\
 & & & & x_1, x_2, x_3, x_4 & \geq & 0
\end{array}$$

a. Solve (P) with the two phase simplex algorithm.

b. Write the dual (D) of (P).

c. Solve (D) geometrically. Is the solution compatible with the one given by the last dictionary of (P) ?

d. Certify the optimality of your solution of (P) via a linear sum of constraints.

- Exercise 2 - The goal is to modify the entries of some array T in order that the new array T' is sorted in increasing order. For instance, if $T = [3, 1, 2, 5]$, one can increase the second entry by 2 and the third by 1 to obtain $T_1' = [3, 3, 3, 5]$. We can also decrease by 3/2 the first one and increase by 1/2 the second to obtain $T_2' = [1.5, 1.5, 2, 5]$. The goal is minimize the total cost added or substracted to T. For instance T_1' costs 3 whereas T_2' costs 2.

a. Compute an optimal solution T' when $T = [2, 1, 4, 3]$, and when $T = [4, 3, 2, 1]$.

b. Model this problem by a linear program (P). We assume here that T has 5 entries $[a_1, a_2, a_3, a_4, a_5]$.

c. Can the simplex algorithm applied to the program (P) corresponding to the array $[3, 1, 2, 5, 6]$ return the optimal solution $[1.5, 1.5, 2, 5, 6]$?

- Exercise 3 - We consider two power plants, each of them producing 800 megawatt. These plants are connected to three cities which respective power demands are 700, 400 and 500 megawatt. Each plant can freely split its supply to any of the cities.

The respective transportation costs (per megawatt) in the network are :

	Plant 1	Plant 2
City 1	20	25
City 2	15	10
City 3	10	15

The problem is to minimize the total cost while providing power supply to the cities.

a. Model by a linear program.

b. Propose a (handmade) optimal solution of this problem.

c. Show the optimality of your solution via a dual certificate.

- Exercise 4 - In the MAX-2-SAT problem we are given clauses $C_1, \dots, C_m$ of size 1 or 2 with respective positive weights $w_1, \dots, w_m$. The goal is to set the boolean variables $x_1, \dots, x_n$ in order to satisfy a subset of these clauses with maximum total weight.

a. Propose a fractional relaxation of this problem.

b. Show that randomized rounding applied to this relaxation gives a 3/4-approximation of MAX-2-SAT.

- Exercise 5 - Let M be some real $n \times m$ matrix. We denote the column vectors of M by $\{V_1, \dots, V_m\}$. We assume here that every vector V_i is non null. A subset I of $\{1, \dots, m\}$ is *independent* if the set S_I of vectors $\{V_i : i \in I\}$ forms an independent set of vectors, i.e. the rank of S_I is $|I|$. To every independent set I, we associate a point x_I of $\mathbb{R}^m$ by letting $x_I := (x_i)_{1 \leq i \leq m}$ where $x_i = 1$ if $i \in I$, and $x_i = 0$ otherwise. The *independence polytope* IS of M is the convex hull of the x_I, for all independent sets I (including the empty set). Our goal is to describe IS by a set of constraints. A linear inequality which is satisfied by all points $(x_1, \dots, x_m)$ of IS is called a *valid inequality.* Each inequality corresponds to a halfspace. We want to find a *defining* set of valid inequalities, i.e. such that the intersection of their corresponding halfspaces is exactly IS.

a. Find a defining set of valid inequalities of IS for the matrix $\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

b. The *rank* of a nonempty subset A of $\{1, \dots, m\}$ is the rank of $\{V_i : i \in A\}$. We denote it by $rk(A)$. The inequality associated to A is $\sum_{i \in A} x_i \leq rk(A)$. Show that this inequality is valid for IS.

c. Let us call IS' the intersection of all inequalities associated to non empty subsets A and the nonnegativity constraints. Show that the integer points of IS' are precisely the points x_I for independent sets I.

d. We now consider a vertex $x = (x_1, \dots, x_m)$ of IS'. Our goal is to show that it is integer. A set A is *tight* if $\sum_{i \in A} x_i = rk(A)$. Show that if x_i is non integer, then i belongs to a tight set.

e. Show that, if non empty, the intersection of two tight sets is also tight. Hint : Use submodularity of the rank function, i.e. $rk(A) + rk(B) \geq rk(A \cup B) + rk(A \cap B)$.

f. Conclude by considering a tight set of minimum size containing a (supposedly) non integer coordinate of x.

- Exercise 6 - *In this exercise, we assume that our graphs always have at least one edge.* Let G be a graph on vertex set $V = \{1, \dots, n\}$ and edge set E. A *clique* in G is a subset $K \subseteq V$ such that $ij \in E$ for all distinct i, j in K.

The maximum size of a clique in G is denoted by $\omega(G)$. A *stable set* in G is a subset $S \subseteq V$ such that $ij \notin E$ for all distinct i, j in S. The maximum size of a stable set in G is denoted by $\alpha(G)$. Finally, the *chromatic number* of G is the minimum number of stable sets in which the vertices of G can be partitioned. We denote it by $\chi(G)$. The goal of this exercise is to give some relaxations of the chromatic number.

a. The fractional relaxation of χ is called the *fractional chromatic number* and is denoted by χ_f. This is the minimum weight one can distribute on some stable sets of G in such a way that for every vertex v, the sum of the weights of the stable sets containing v is at least 1. Show that for every graph G, we have $\omega(G) \leq \chi_f(G) \leq \chi(G)$.

b. Describe the dual notion of the fractional chromatic number.

c. Compute the fractional chromatic number of C_5, the cycle of length 5.

d. The second notion is the *vector chromatic number*, denoted by χ_v. For this, we associate to every vertex i of G a unit vector V_i, trying to minimize the maximum for every edge ij of $V_i.V_j$. In other words we consider the minimum c such that $V_i.V_j \leq c$ for all edge ij of G. The value $\chi_v(G)$ is then equal to $1 - 1/c$. Show that for every graph G, we have $\omega(G) \leq \chi_v(G) \leq \chi(G)$.

e. Show that the vector chromatic number of C_5 is at most $\sqrt{5}$. *Hint : the value* $\cos(4\pi/5) = -(1+\sqrt{5})/4$ *is generously provided.*

f. The third notion is Lovász' *theta function*, denoted by χ_l. For this, we maximize c for which there exists a way to associate to every vertex i of G a unit vector V_i, and some other unit vector V_0 such that $V_0.V_i \geq c$ for all vertices i, and $V_i.V_j = 0$ for every edge ij of G. The value $\chi_l(G)$ is then equal to $1/c^2$. Show that for every graph G, we have $\omega(G) \leq \chi_l(G) \leq \chi(G)$.

g. Show that $\chi_l(C_5)$ is at most $\sqrt{5}$.

h. What are the relationships between χ_f, χ_v and χ_l ?