Final Exam May 5, 2014.

- Duration 2h. No notes and electronic devices allowed. Exercises are ordered in somewhat increasing difficulty order. -

- Exercise 1 - We are given the following linear program P :

$$\begin{array}{lrrrrl} \text{Minimize} & 9x_1 & +3x_2 & +2x_3 & +2x_4 & \\ \text{Subject to} & x_1 & +x_2 & +x_3 & +x_4 & \geq 1 \\ & 3x_1 & -x_2 & & +2x_4 & \geq 1 \\ & x_1 \geq 0, & x_2 \geq 0, & x_3 \geq 0, & x_4 \leq 0 & \end{array}$$

a. Solve the dual of P graphically.

b. Use complementary slackness to find an optimal solution of P.

- Exercise 2 - Use LP-duality to prove Farkas Lemma, i.e. if A is a real $m \times n$ matrix and c be a (column) vector of $\mathbb{R}^n$, then exactly one of the following is true :

a. There exists x such that $Ax \geq 0$ and $c^T x < 0$.

b. There exists $y \geq 0$ such that $A^T y = c$.

- Exercise 3 - We are given a family of n axis-parallel rectangles R_i in the plane, where $i = 1, \ldots, n$. Each rectangle R_i is characterized by a 4-uple of reals (a_i, b_i, c_i, d_i), where $a_i \leq b_i$ and $c_i \leq d_i$, such that the four corners of R_i are respectively the points $(a_i, c_i); (a_i, d_i); (b_i, c_i); (b_i, d_i)$.

A *rectangle stabbing* is a set of vertical and horizontal lines such that each rectangle is traversed by at least one line. The goal is to find a rectangle stabbing with a minimum number of lines. Formally we want to minimize $j + k$ such that there exists real coordinates $x_1, \ldots, x_j$ and $y_1, \ldots, y_k$ for which :

For every rectangle R_i, there exists some x_ℓ such that $a_i \leq x_\ell \leq b_i$ or there exists some y_ℓ such that $c_i \leq y_\ell \leq d_i$.

a. Show that we can restrict to a finite set of possible choices for the coordinates of the lines in the rectangle stabbing problem.

b. Propose an LP-relaxation of the problem.

c. We denote by OPT the optimal value of the rectangle stabbing problem, and by OPT^* the optimal value of your relaxation. Show an example where OPT^* is strictly less than OPT. Try to make OPT/OPT^* as large as possible.

d. ** (if you have finished and get bored) What about the computational complexity of the rectangle stabbing problem ?

- Exercise 4 - We are given a directed graph $D = (V, A)$ with one source s and one sink t.

a. Propose a linear program P which corresponds to the fractional relaxation of the shortest directed path problem from s to t.

b. Write the dual D of P.

c. Propose a sensible interpretation of the integer solutions of D (hence providing lower bounds for the shortest path problem).

- Exercise 5 - Alice and Bob are playing a generalization of the Paper/Scissor/Stone game with n objects instead of three. Each game issue is either a 1 point victory for Alice or for Bob, or in the case in which they have chosen the same object, the result is a draw, valued 0.

Formally, they play on a $n \times n$ skew-symmetric matrix A (i.e. $a_{ij} = -a_{ji}$, or $A^T = -A$) which entries are 0 on the diagonal and $+1$ or -1 otherwise. Alice chooses a column j, Bob chooses a row i, and the payoff for Alice is a_{ij}.

In the case of the usual Paper/Scissor/Stone game, the optimal mixed strategy is $(1/3, 1/3, 1/3)$. This optimal strategy is unique, and the goal of this exercice is to show that every generalized Paper/Scissor/Stone game has a unique strategy.

In the context of symmetric games, we recall that x is an *optimal mixed strategy* if $A.x \geq 0$, where x is a probability distribution, i.e. $x \geq 0$ and $\mathbf{1}^T.x = 1$.

The questions in this exercise are not in increasing order of difficulty, feel free to admit the result of a question and go to the next one.

a. Show that if the i^{th} coordinate of an optimal mixed strategy x is non zero, then the i^{th} coordinate of $A.x$ is equal to 0.

b. Show that if n is odd, the determinant of A is 0. *Hint : compute the determinant of A^T in two ways.*

c. Show that if n is even, the determinant of A is odd. *Hint : Show that changing the sign of the entries of A does not change the parity of the determinant. Then change all the -1 of A into +1 to conclude.*

d. Show that the set of solutions of $A.x = 0$ is at most 1-dimensional.

e. Assume now that x and y are optimal strategies for Alice. For every α with $0 < \alpha < 1$, we let $z_\alpha = \alpha x + (1 - \alpha)y$. Show that z_α is an optimal strategy.

f. Let S be the set of coordinates in which z_α is non zero. Let A' be the submatrix of A restricted to the entries a_{ij} with $i, j \in S$. Similarly, let z'_α be the vector z_α restricted to its non zero coordinates. Show that $A'.z'_\alpha = 0$.

g. Conclude that Alice has a unique optimal strategy x, which moreover has an odd number of non zero coordinates.