# Final Exam May 11, 2015.

## - Duration 3h. One Cheat Sheet allowed (A4, manuscript, double sided). -

**- Exercise 1 -**
Solve the following linear program by the two-phase simplex algorithm.

$$\begin{array}{lrrl}
\text{Maximize} & 3x_1 & +x_2 & \\
\text{Subject to} & x_1 & -x_2 & \leq -1 \\
 & -x_1 & -x_2 & \leq -3 \\
 & 2x_1 & +x_2 & \leq 4 \\
 & x_1, & x_2 & \geq 0
\end{array}$$

Write the dual of (P) and solve it.

**- Exercise 2 -**
We want to approximate a set of $n$ points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in the plane by a line. The least square method consists of minimizing $(y_1 - ax_1 - b)^2 + (y_2 - ax_2 - b)^2 + \cdots + (y_n - ax_n - b)^2$. Propose an efficient way of minimizing $|y_1 - ax_1 - b| + |y_2 - ax_2 - b| + \cdots + |y_n - ax_n - b|$.

**- Exercise 3 -**
Show using total unimodularity that when the capacities of the arcs of a network are integer valued, the maximum $s,t$-flow is integer valued. Provide a detailed description of the problem and a complete proof.

**- Exercise 4 -** In the MAX-2-SAT problem we are given clauses $C_1, \ldots, C_m$ of size 1 or 2 with respective positive weights $w_1, \ldots, w_m$. The goal is to set the boolean variables $x_1, \ldots, x_n$ in order to satisfy a subset of these clauses with maximum total weight.

a. Propose a fractional relaxation of this problem.

b. Show that randomized rounding applied to this relaxation gives a 3/4-approximation of MAX-2-SAT.

**- Exercise 5 -** The goal of this exercise is to show that every 0/1 matrix $A$ with at most $t$ entries 1 per column has discrepancy at most $2t$, i.e. there is a -1/1 vector $y$ satisfying $-2t.\mathbf{1} \leq A.y \leq 2t.\mathbf{1}$. The idea is to iterate fractional relaxations of the problem by considering the system (S) $Ay = 0$ where $-\mathbf{1} \leq y \leq \mathbf{1}$ instead of $y$ is a -1/1 vector.

a. Show that if we delete all rows of $A$ with at most $t$ entries 1, we get an $m \times n$ matrix where $n > m$. We still call $A$ this matrix.

b. Show that when $n > m$, there exists a solution $y$ of (S) such that one coordinate $y_j$ satisfies $y_j = -1$ or $y_j = 1$.

c. * Deduce an algorithm for finding a vector $y$ satisfying $-2t.\mathbf{1} \leq A.y \leq 2t.\mathbf{1}$. Hint : when reaching $y_j = -1$ or $y_j = 1$, consider that this coordinate is fixed. Also if a row has at most $t$ entries 1 corresponding to non-fixed variables, remove it from the problem.

**- Exercise 6 -** Let $G$ be a graph on vertex set $V = \{1, \dots, n\}$ and edge set $E$. A *clique* in $G$ is a subset $K \subseteq V$ such that $ij \in E$ for all distinct $i, j$ in $K$. The maximum size of a clique in $G$ is denoted by $\omega(G)$. A *stable set* in $G$ is a subset $S \subseteq V$ such that $ij \notin E$ for all distinct $i, j$ in $S$. The *chromatic number* of $G$ is the minimum number of stable sets covering the vertices of $G$. We denote it by $\chi(G)$. For instance $\chi(C_5) = 3$, where $C_5$ is the cycle of length 5.

a. Propose a (natural) fractional relaxation $\chi_f$ of $\chi$.

b. Compute $\chi_f(C_5)$.

c. Show that $\omega(G) \leq \chi_f(G) \leq \chi(G)$ for all graphs.

d. Describe the dual notion of the fractional chromatic number. Does it correspond to some fractional relaxation of some known parameter?

e. * Discuss the largest possible gap between $\chi_f$ and $\chi$.