Exam January 28, 2016.

- Duration 3h. One Cheat Sheet allowed (A4, manuscript, double sided). -

- Exercise 1 - Let (P) be the following linear program:

$$\begin{array}{rrrrrcr}
\text{Maximize} & 3x_1 & -2x_2 & +2x_3 & -4x_4 & & \\
\text{Subject to} & 2x_1 & +x_2 & +2x_3 & -x_4 & \leq & -2 \\
 & & -x_2 & -x_3 & & \leq & -2 \\
 & & & & x_1, x_2, x_3, x_4 & \geq & 0
\end{array}$$

a. Solve (P) with the two phase simplex algorithm.

b. Write the dual (D) of (P).

c. Solve (D) geometrically. Is the solution compatible with the one given by the last dictionary of (P)?

d. Certify the optimality of your solution of (P) via a linear sum of constraints.

- Exercise 2 - Is $x_1 = 2/13$, $x_2 = 0$, $x_3 = 8/13$, $x_4 = 0$ an optimal solution of the following linear program? Justify.

$$\begin{array}{rrrrrcl}
\text{Maximize} & x_1 & +x_2 & +x_3 & +x_4 & & \\
\text{Subjectto} & 5x_1 & +6x_2 & +2x_3 & +3x_4 & \leq & 2 \\
 & x_1 & +3x_2 & +3x_3 & +5x_4 & \leq & 2 \\
 & 2x_1 & +6x_2 & +4x_3 & +2x_4 & \leq & 3 \\
 & 6x_1 & +5x_2 & +4x_3 & +x_4 & \leq & 6 \\
 & & & & x_1, x_2, x_3, x_4 & \geq & 0
\end{array}$$

- Exercise 3 - Let S be a set of size n and $T_1, \ldots, T_m$ be some subsets of S. We want to distribute some non negative weights on the elements of S (all weights summing to some positive value W) in such a way that all T_i's receive the same total weight. We allow fractional values.

a. Propose an instance where such a distribution does not exist.

b. When the distribution does not exist, use duality to provide a non existence certificate.

- Exercise 4 - Let $G = (V, E)$ be a graph. We assume that $V = \{1, \ldots, n\}$. A *vertex cover* is a subset C of V such that every edge xy of E satisfies $x \in C$ or $y \in C$. When the size of C is minimum, we speak of *minimum vertex cover*. The usual relaxation of vertex cover consists of the polyhedron P which points $(x_1, \ldots, x_n)$ satisfy $x_i \geq 0$ for all $i = 1, \ldots, n$, and $x_i + x_j \geq 1$ for every edge $ij \in E$.

a. Recall the definition of a vertex of a polyhedron, and provide an easy certificate of the fact that a given point in a polyhedron is not a vertex.

b. Let $x = (x_1, \ldots, x_n)$ be a vertex of P. Let $I^- = \{i \mid 0 < x_i < 1/2\}$ and $I^+ = \{i \mid 1/2 < x_i < 1\}$. Let $y = (y_1, \ldots, y_n)$ such that $y_i = -1$ when $i \in I^-$, $y_i = 1$ when $i \in I^+$, and $y_i = 0$ otherwise. Show that there is some $\varepsilon > 0$ such that $x + \varepsilon y$ is a point of P.

c. Show that if x is a vertex of P, then both I^- and I^+ are empty. Conclude that the coordinates of x are either 0,1/2 or 1.

d. Based on the previous result, propose a polytime 2-approximation algorithm of minimum cost vertex cover, when the input consists of a graph $G = (V, E)$ and a non negative cost function c on V. (we only consider the cost function c in this question)

e. Assume now that $x = (x_1, \ldots, x_n)$ is a vertex of the polyhedron P which minimizes the sum $x_1 + x_2 + \cdots + x_n$. We partition V into three sets $V_0 = \{i \mid x_i = 0\}$, $V_1 = \{i \mid x_i = 1\}$ and $V_{1/2} = \{i \mid x_i = 1/2\}$. Let $OPT_{1/2}$ be the minimum size of a vertex cover in the graph induced by G on $V_{1/2}$. Let OPT be the minimum size of a vertex cover in G. Show that $OPT \leq OPT_{1/2} + |V_1|$.

f. Let V_{OPT} be a vertex cover of G of size OPT. Show that $|V_{\mathrm{OPT}} \cap V_0| \geq |V_1 \setminus V_{\mathrm{OPT}}|$.

g. Show that $OPT = OPT_{1/2} + |V_1|$.

h. Show that the problem having as input a graph G and an integer k and output True if G has a vertex-cover of size at most k can be reduced in polynomial time to the particular case of this problem where the input G has at most $2k$ vertices.

- Exercise 5 - In the MAX-2-SAT problem we are given clauses $C_1, \ldots, C_m$ of size 1 or 2 with respective positive weights $w_1, \ldots, w_m$. The goal is to set the boolean variables $x_1, \ldots, x_n$ in order to satisfy a subset of these clauses with maximum total weight.

a. Propose a fractional relaxation of this problem.

b. Show that randomized rounding applied to this relaxation gives a 3/4-approximation of MAX-2-SAT.

- Exercise 6 - The goal of this exercise is to illustrate the Gomory cut technique, used to find optimal integer solutions to linear programs. Let (P) be the following integer program:

$$\begin{array}{rrrcl} \text{Maximize} & 4x_1 & +3x_2 & & \\ \text{Subject to} & 2x_1 & +x_2 & \leq & 11 \\ & -x_1 & +2x_2 & \leq & 6 \\ & & x_1, x_2 & \geq & 0 \text{ and integer valued} \end{array}$$

When applying simplex to the linear relaxation of (P), we obtain the following last dictionary:

$$\begin{array}{rclll} x_1 & = & \frac{16}{5} & -\frac{2}{5}x_3 & +\frac{1}{5}x_4 \\ x_2 & = & \frac{23}{5} & -\frac{1}{5}x_3 & -\frac{2}{5}x_4 \\ \hline z & = & \frac{133}{5} & -\frac{11}{5}x_3 & -\frac{2}{5}x_4 \end{array}$$

a. Show that the constraint $\frac{2}{5}x_3 + \frac{4}{5}x_4 \geq \frac{1}{5}$ is a valid inequality for (P), i.e. integer solutions of (P) satisfy it. (Hint: derive it from the first equation of the dictionary using the fact that we focus on integer solutions)

b. Derive a valid inequality from the second equation of the dictionary.

c. Propose a general method based on this example.